Liouville Quantum Gravity & KPZ

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Workshop on Manifolds of Metrics

& Probabilistic Methods in Geometry & Analysis

Centre de Recherches Mathématiques

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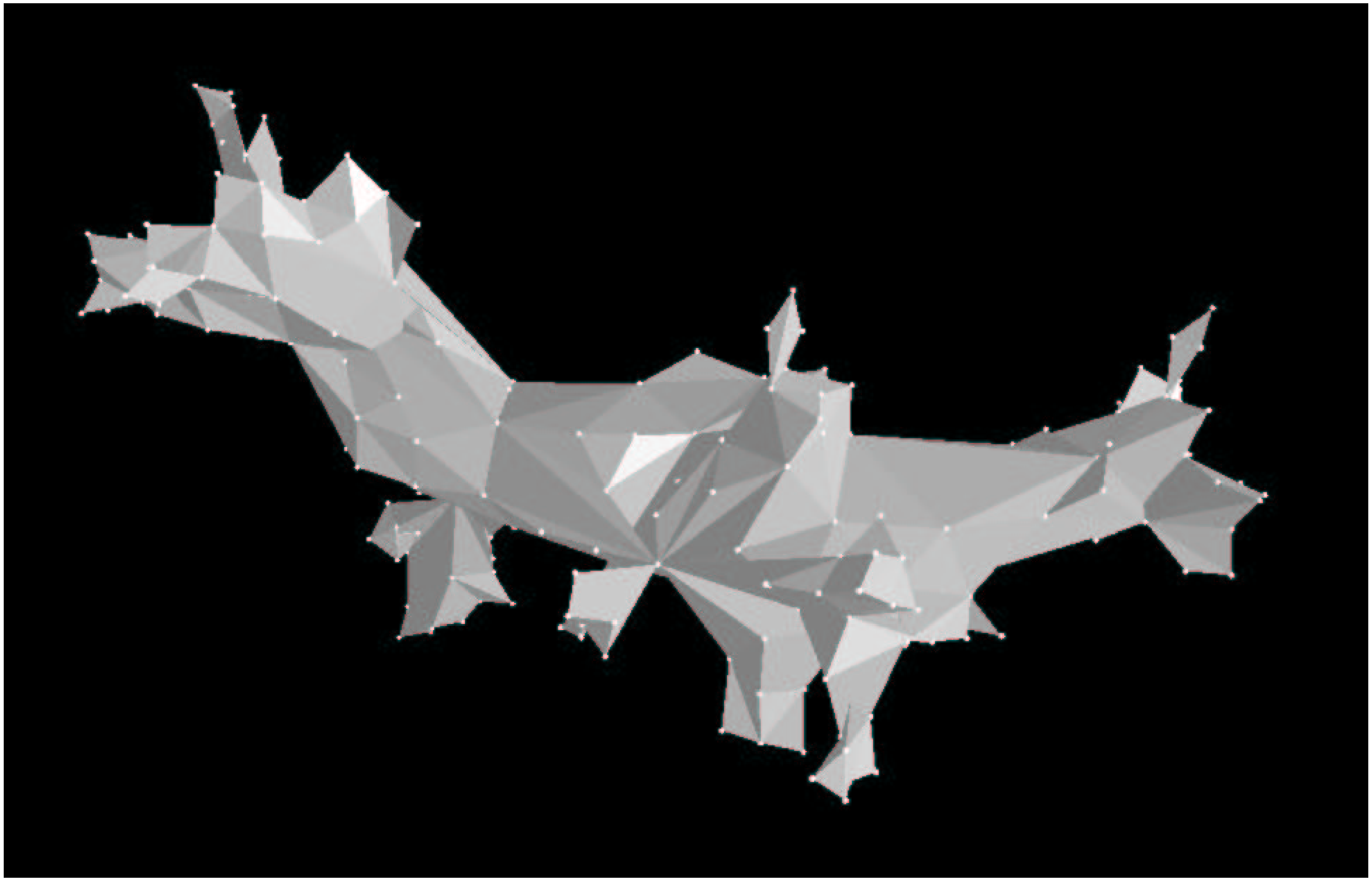
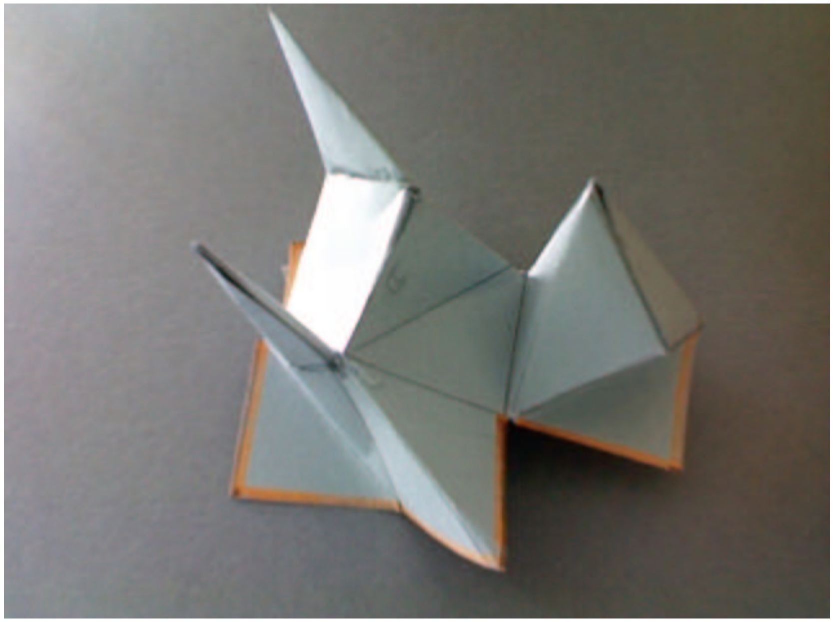
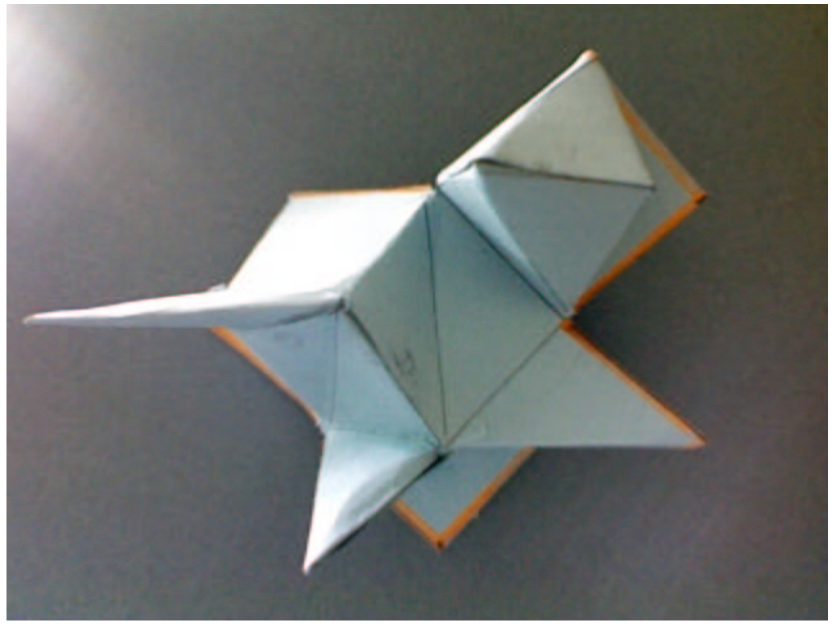
A Random Surface



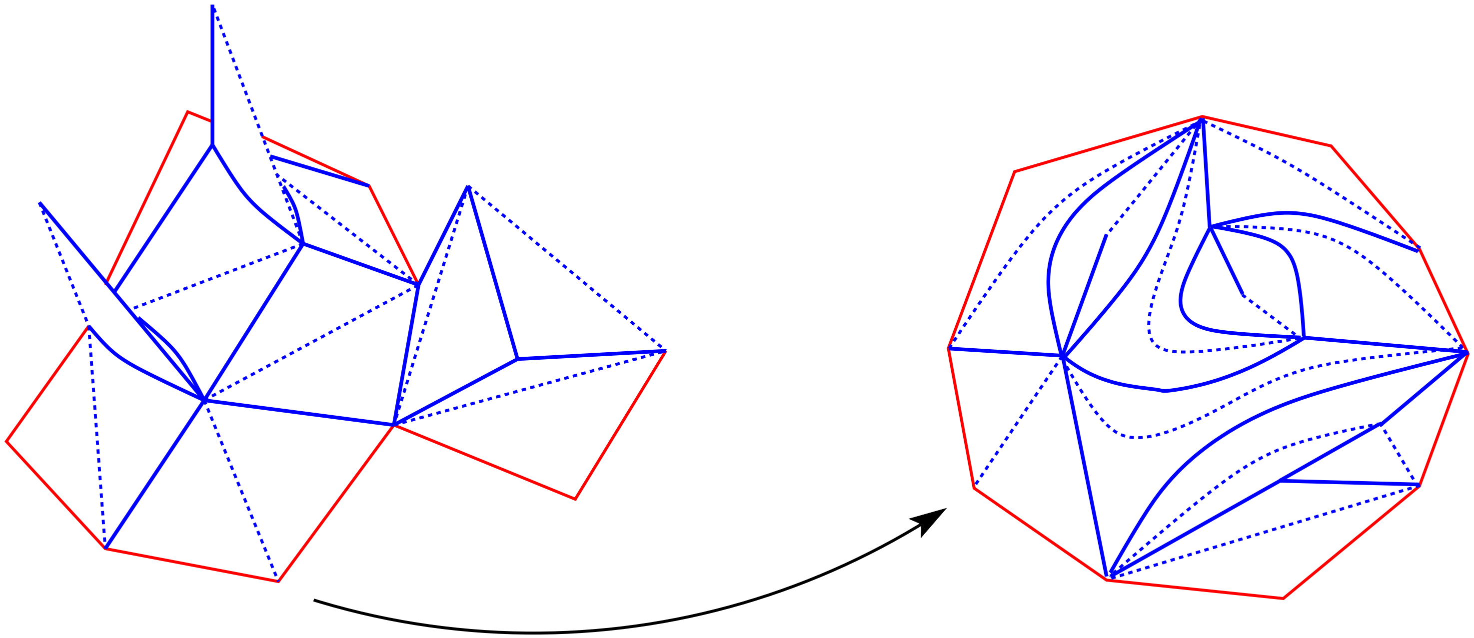
[Courtesy of G. Chapuy (2009)]

A Random Quadrangulation





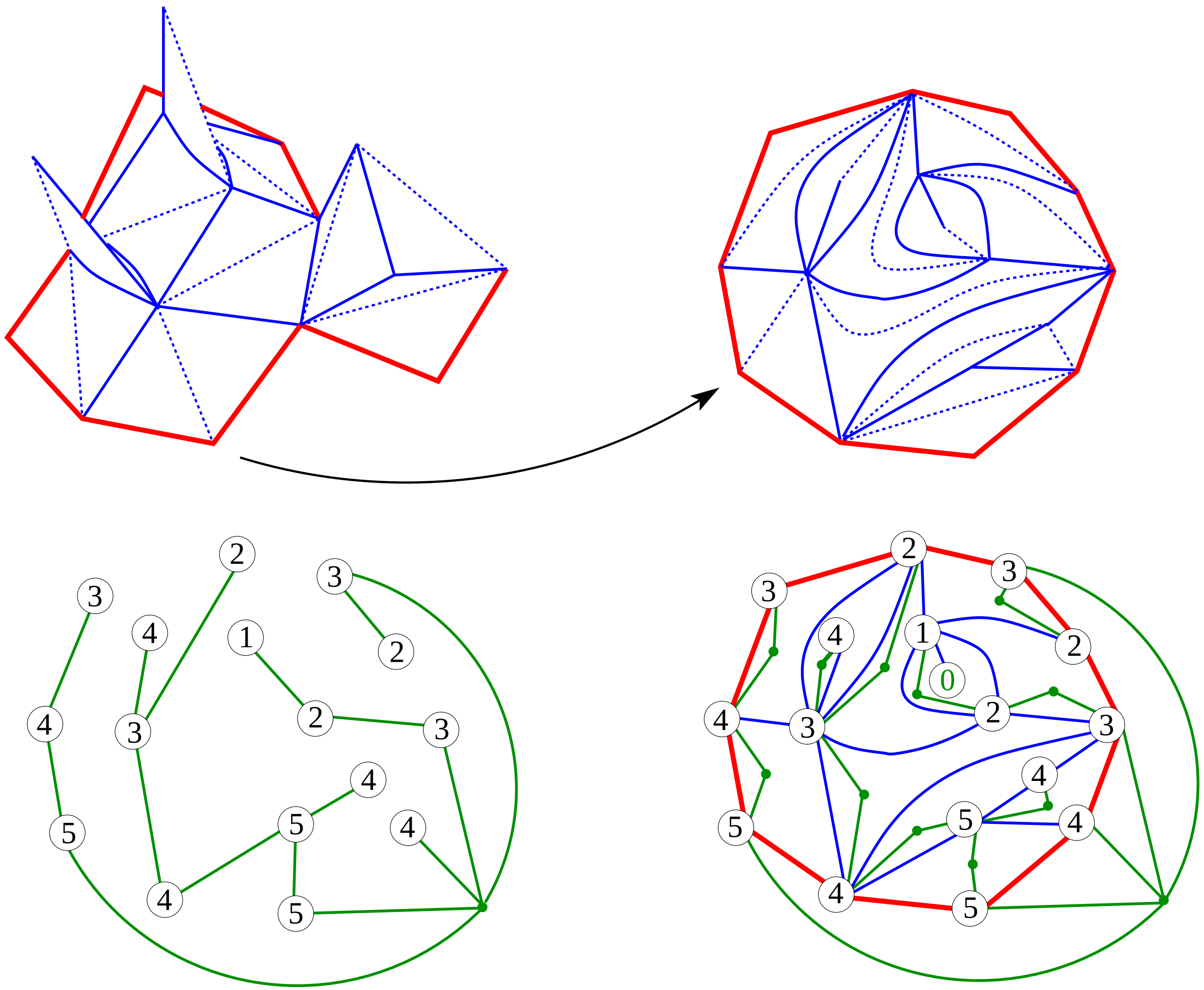
Random Quadrangulation & Random Planar Map



Random Matrices *BIPZ '78; Ambjørn, Durhuus, Fröhlich, Jonsson '83-85; David '85; Boulatov, Kazakov, Kostov, Migdal '85...*

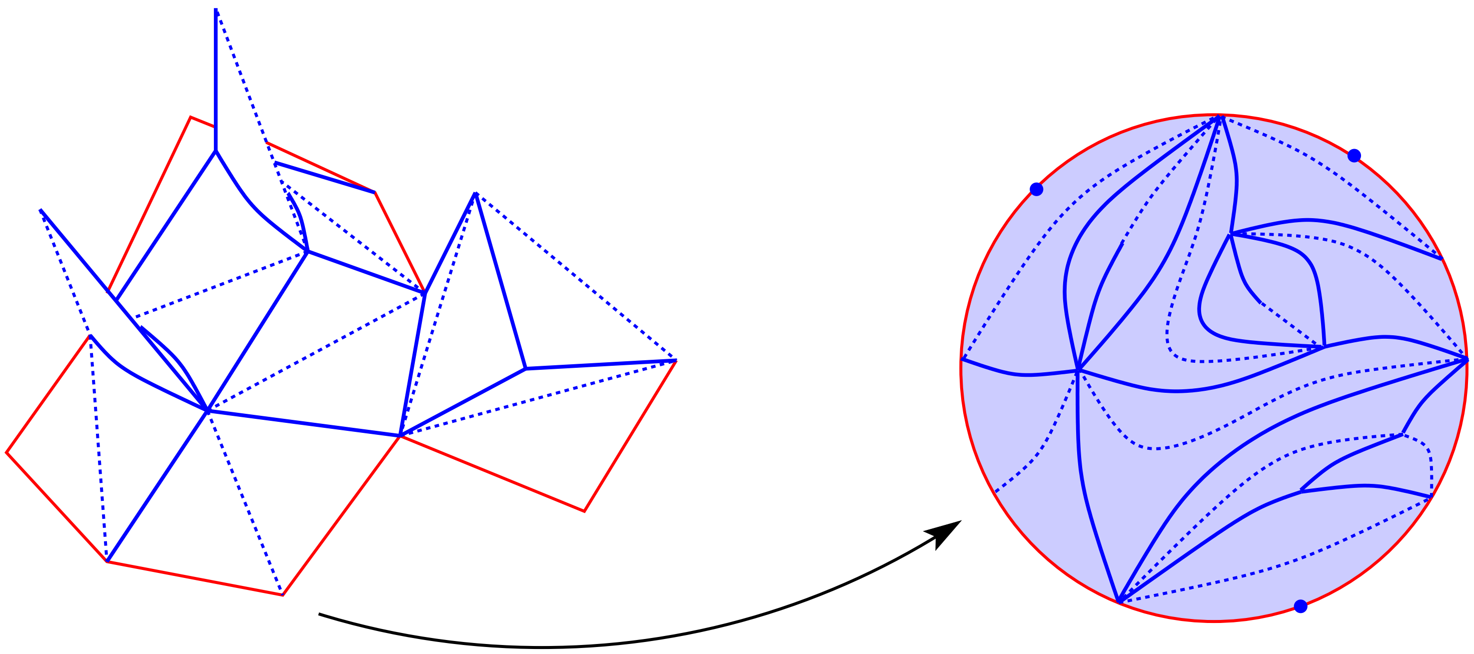
Bijective Combinatorics *Cori, Vauquelin '81; Schaeffer '97; Angel, Schramm '03; Bouttier, Di Francesco, Guitter '04; Le Gall, Miermont...*

Random Quadrangulations & Schaeffer Bijection



Courtesy of E. Guitter (2009)

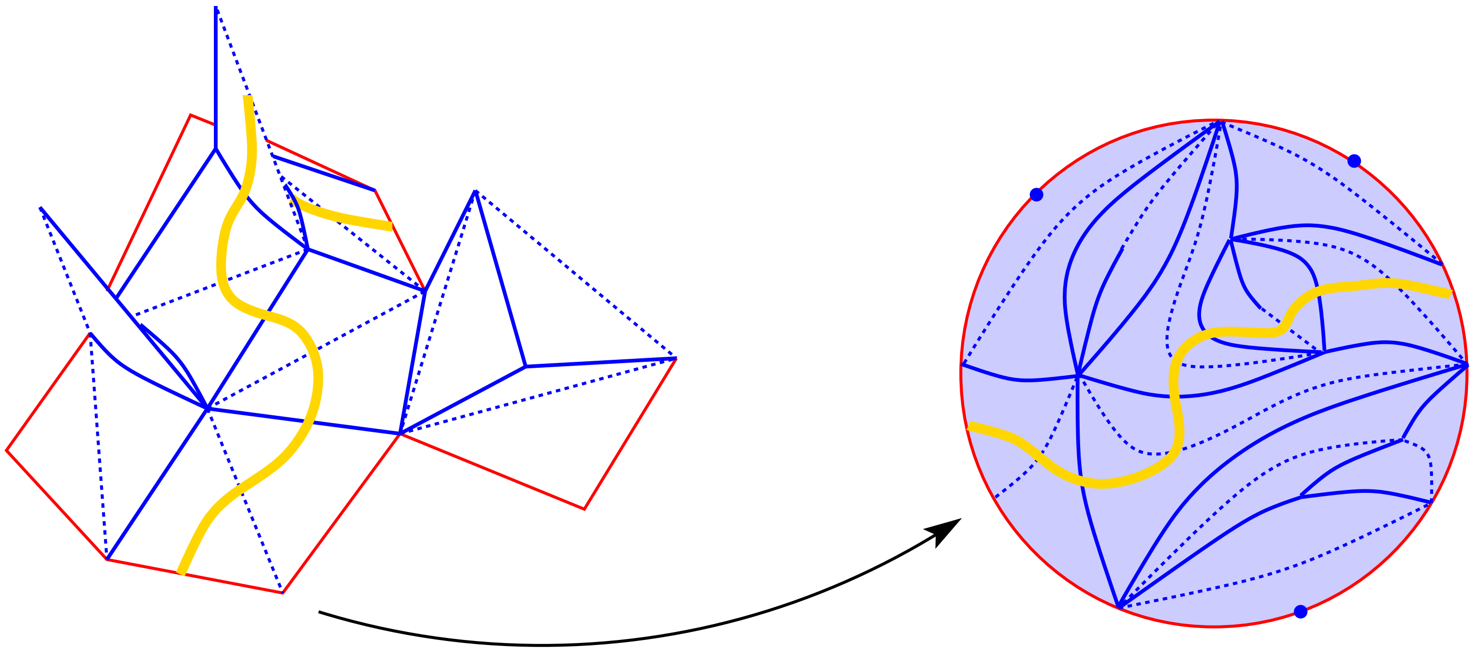
Random Quadrangulation & Conformal Map to $\mathbb{D}$



In the continuum scaling limit: **Liouville Quantum Gravity**
A.M. Polyakov '81

Correlation Functions *Seiberg, '90; Goulian, Li '91; Ginsparg, Moore '93; Dorn, Otto '94; Takhtajan '95; Teschner '95; Zamolodchikov^2 '96; Fateev-ZZ '00; Ponsot, Teschner '02; Kostov, Ponsot, Serban '04...*

Random Quadrangulation & Random Sets & Paths



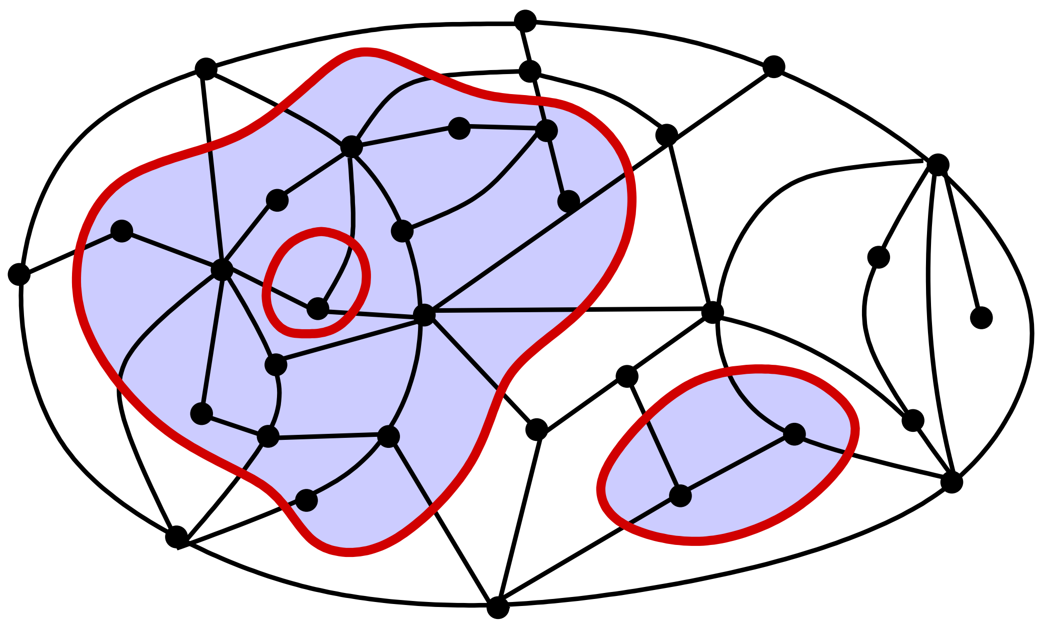
Ising, SAW, $O(N)$ & Potts models: **Random Matrix Models**

Kazakov '86; D. & Kostov '88; Kostov; Daul; Eynard, Zinn-Justin^2...

Bijective Combinatorics *Chassaing & Schaeffer '02;*

Bousquet-Mélou & Schaeffer '02; BDFG '02; Bernardi & B.-M. '09...

Continuum: **Liouville Gravity & Conformal Field Theory**



Quadrangulation with a loop model *(courtesy of E. Guitter).*

Liouville Field Theory (POLYAKOV '81)

$$S_{\mathcal{L}} = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a h\partial_b h + Q\hat{R}h + \lambda e^{\gamma h}\right) [+\mathrm{CFT}]$$

Background metric $\hat{g}$ & curvature $\hat{R}$

Quantum random metric: $g_{ab} = e^{\gamma h}\hat{g}_{ab}$

Quantum area: $\mathcal{A} = \int d^2z\sqrt{\hat{g}}\,e^{\gamma h}$

Conformal invariance & CFT central charge c

$$Q = \frac{2}{\gamma} + \frac{\gamma}{2} = \sqrt{\frac{25-c}{6}}, \quad c \leqslant 1$$

$$\gamma = \frac{1}{\sqrt{6}}\left(\sqrt{25-c} - \sqrt{1-c}\right) = \sqrt{\kappa \wedge 16/\kappa} \leqslant 2 \ (\mathrm{SLE}_\kappa)$$

Gaussian Free Field

Pour distinguer les choses les plus simples de celles qui sont compliquées et pour les chercher avec ordre, il faut, dans chaque série de choses où nous avons déduit directement quelques vérités d'autres vérités, voir quelle est la chose la plus simple, et comment toutes les autres en sont plus, ou moins, ou également éloignées.

René Descartes, Règles pour la direction de l'esprit (1628-1629).

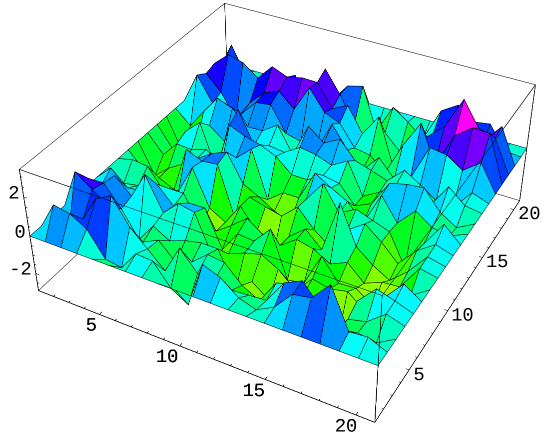
GAUSSIAN FREE FIELD

In order to separate out what is quite simple from what is complex,
and to arrange these matters methodically, we ought, in the case of every series
in which we have deduced certain facts the one from the other, to notice
which fact is simple, and to mark the interval, greater, less, or equal,
which separates all the others from this.

RENÉ DESCARTES, Rules for the Direction of the Mind, VI (1628-1629).

A natural GFF?

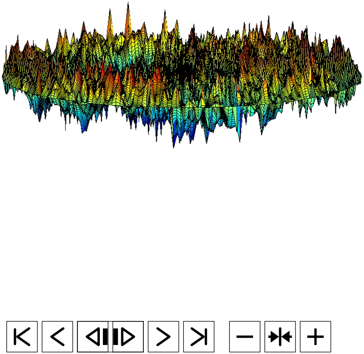
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$\begin{aligned}(f_1, f_2)_\nabla \quad &:= \quad (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)\,d^2z \\ &= \quad \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)\end{aligned}$$

♢ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ♢



Liouville QG

Random Measure

$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

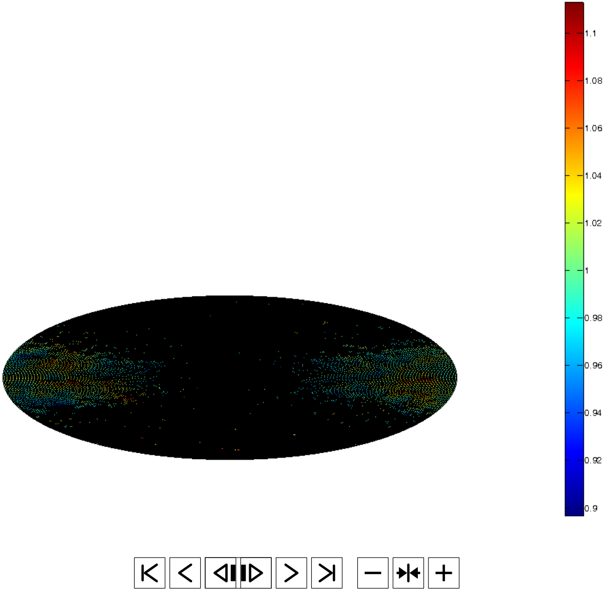
◊

The Emergence of Quantum Gravity

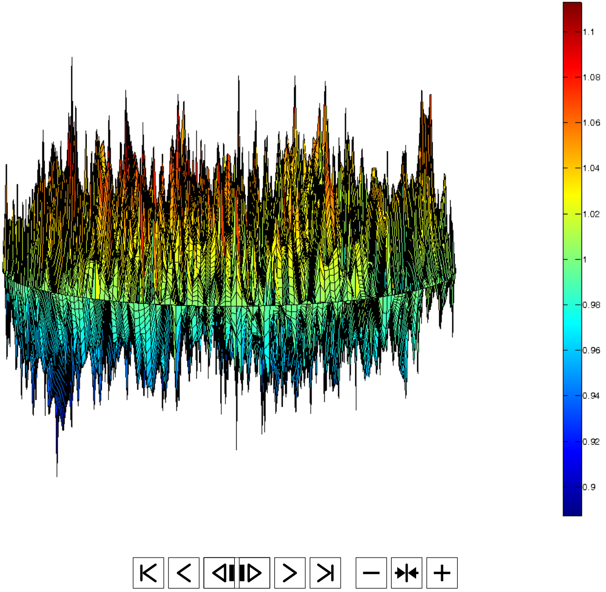
(Courtesy of N.-G. Kang)

◊

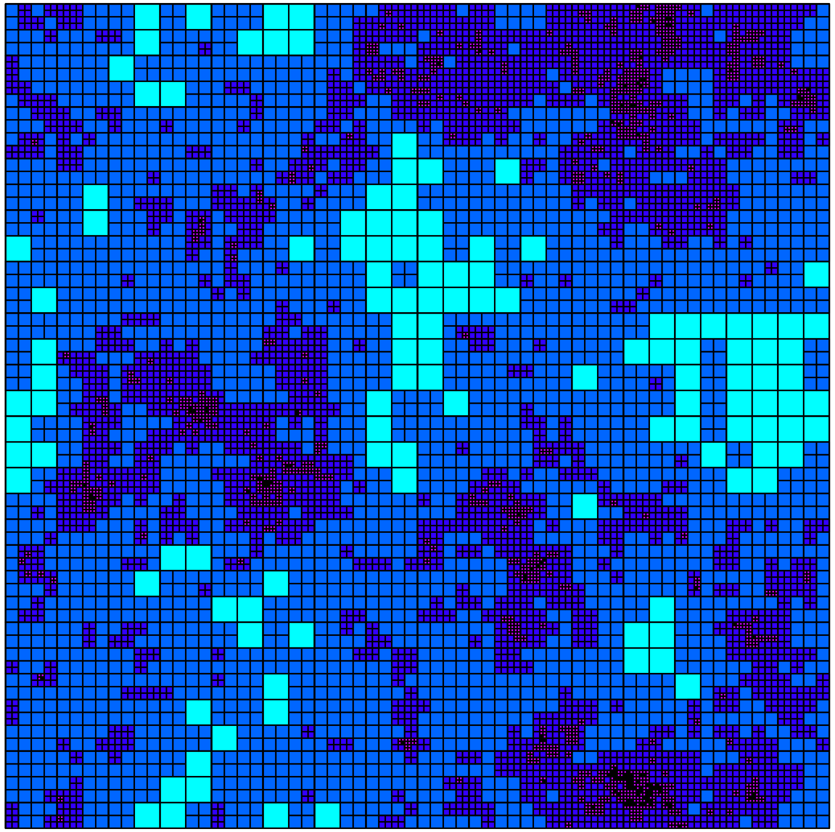
gamma = 0.02



gamma = 0.02

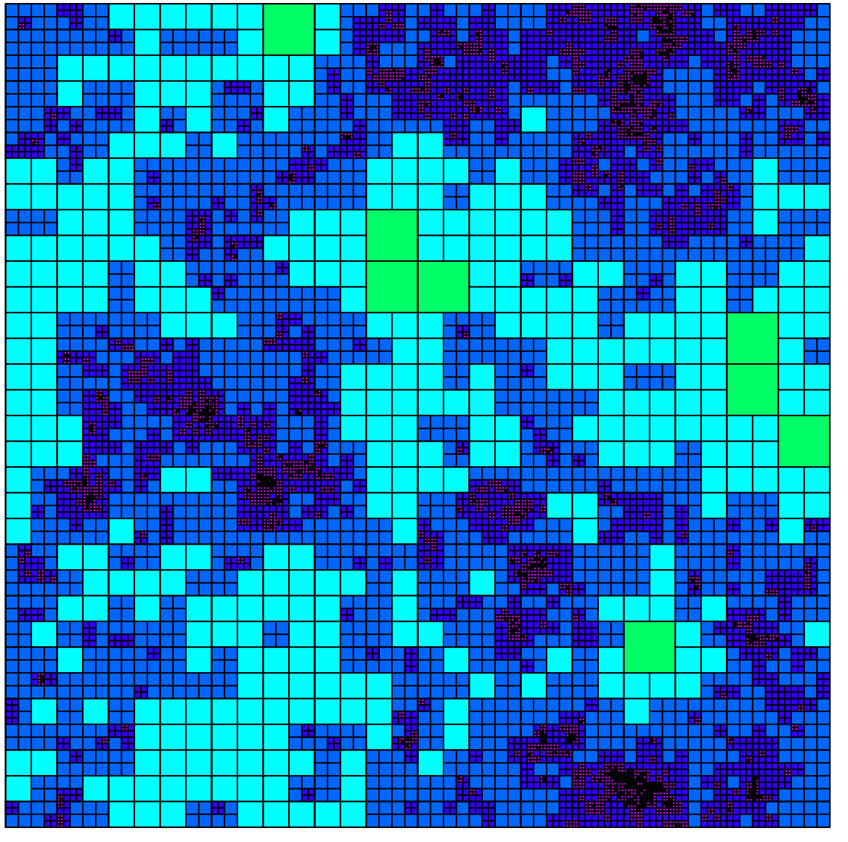


Discrete Quantum Gravity Measure ($\gamma = 1$)



Random measure $d\mu = e^{\gamma h} d^2 z$, $\gamma = 1$ *with h discrete GFF on a fine torus lattice. Euclidean squares of similar quantum area* $\leqslant \delta$ ($= 2^{-12} \times$ *total area).*

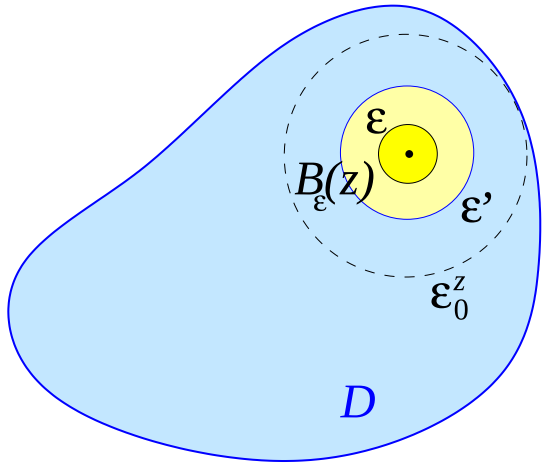
Discrete Quantum Gravity Measure ($\gamma = 3/2$)



Euclidean squares of similar quantum area δ

Regularization: Circular Average of the GFF

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

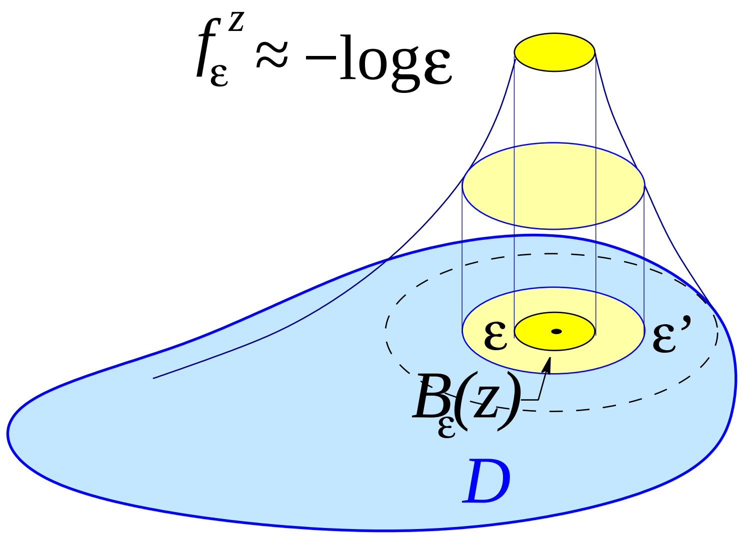


$$(h,\rho) := \int_D h(y)\rho(y)d^2y$$

$$h_\varepsilon(z) := (h,\rho_\varepsilon^z) = (h, f_\varepsilon^z)_\nabla$$

$\rho_\varepsilon^z(\cdot)$ *uniform Dirac dist. of mass* 1 *on circle* $\partial B_\varepsilon(z)$

GFF Circular Average & Logarithmic Potential



$$h_\varepsilon(z) \; := \; (h, f_\varepsilon^z)_\nabla$$

$$f_\varepsilon^z(\cdot) \; := \; -\log(|\cdot - z| \vee \varepsilon) + G_z(\cdot)$$

$G_z(\cdot)$ *harmonic extension of* $\log|\cdot - z|$ *in* D

- Regularization

$h_\varepsilon(z)$ **mean value of** h **on circle** $\partial B_\varepsilon(z)$

- Variance

$$\mathrm{Var}\, h_\varepsilon(z) = (f_\varepsilon^z, f_\varepsilon^z)_\nabla = f_\varepsilon^z(z) = \log[C(z,D)/\varepsilon]$$

$C(z,D)$ *conformal radius of* D *viewed from* z

$h_\varepsilon(z)$ **Gaussian random variable**

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\gamma^2 \mathrm{Var}\, h_\varepsilon(z)/2} = \left(\frac{C(z,D)}{\varepsilon}\right)^{\gamma^2/2} \qquad \square$$

Stochastic Quantum Area

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges to a random measure as $\varepsilon \to 0$ *for*

$$\gamma < 2$$

(Høegh-Krohn, '71)

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$, *to a random measure, denoted by* $e^{\gamma h(z)} d^2 z$.

(Høegh-Krohn, '71)

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$, *to a boundary random measure, denoted by* $e^{(\gamma/2)\,h(z)} dz$.

CRITICAL QUANTUM AREA MEASURE ($\gamma = 2$)

$$\lim_{\varepsilon\to 0}\left[\exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2}\right]_{\gamma=2} d^2z = 0;$$

the derivative martingale

$$d\mu'_\varepsilon := -\frac{\partial}{\partial\gamma}\left[\exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2}\right]_{\gamma=2} d^2z$$

and the renormalized one

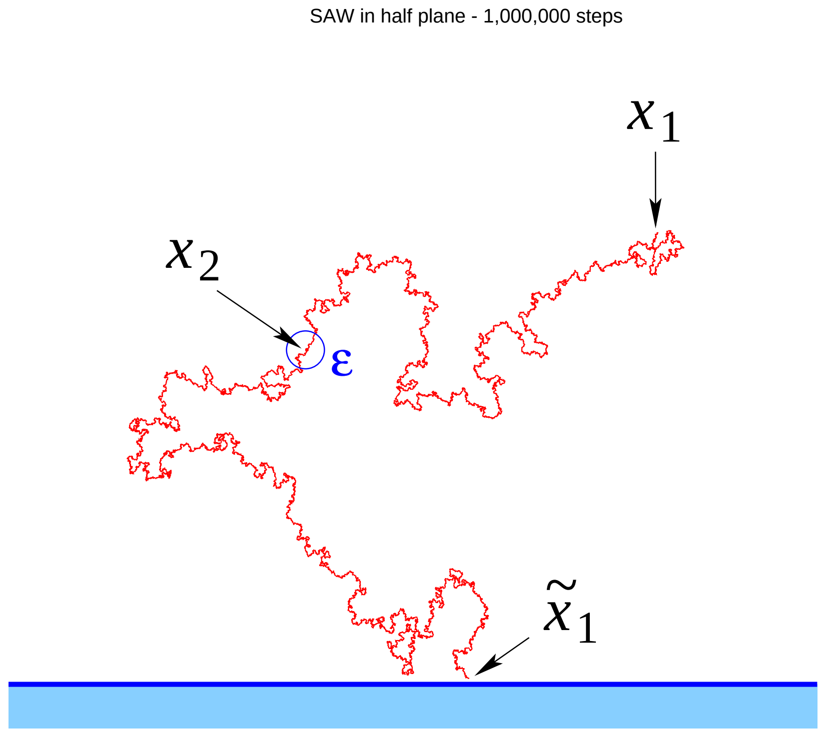
$$d\mu_\varepsilon := \sqrt{\log(1/\varepsilon)}\left[\exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2}\right]_{\gamma=2} d^2z$$

converge, as $\varepsilon \to 0$, to equivalent (up to constant factor) positive non-atomic random measures.
(D., Rhodes, Sheffield, Vargas, arXiv:1206.1671v2)

KPZ RELATION

Knizhnik, Polyakov, Zamolodchikov ’88

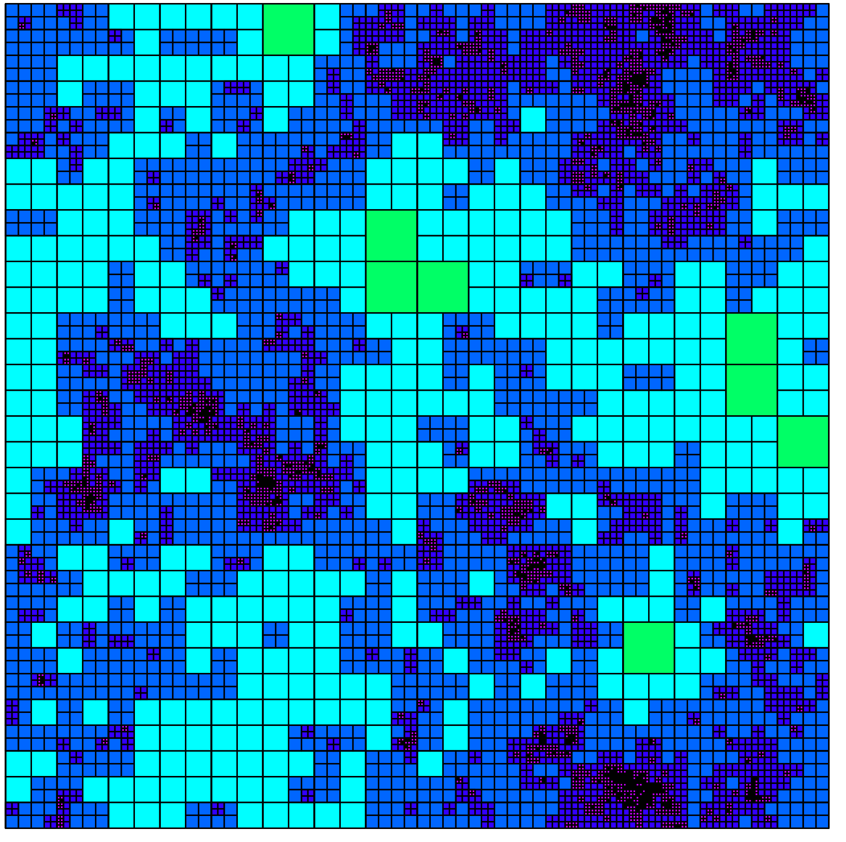
Scaling Exponents of (Random) Fractals in $\mathbb{H}$



Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

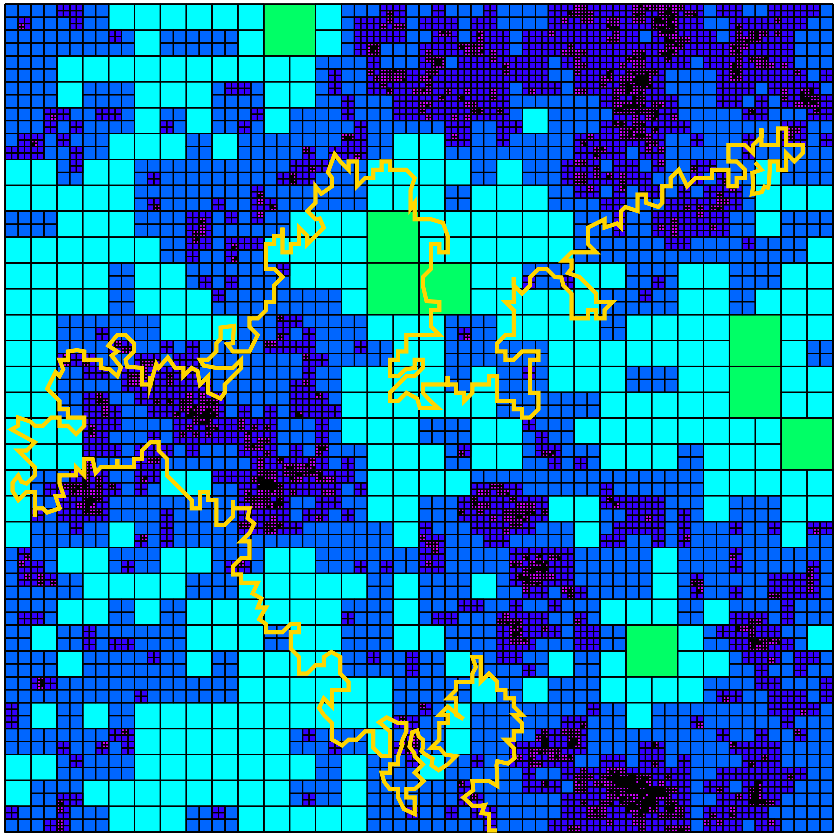
$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad d = 2 - 2x_2 \qquad (= 1 + \kappa/8)$$

Discrete Quantum Gravity Measure $(\gamma = 3/2)$



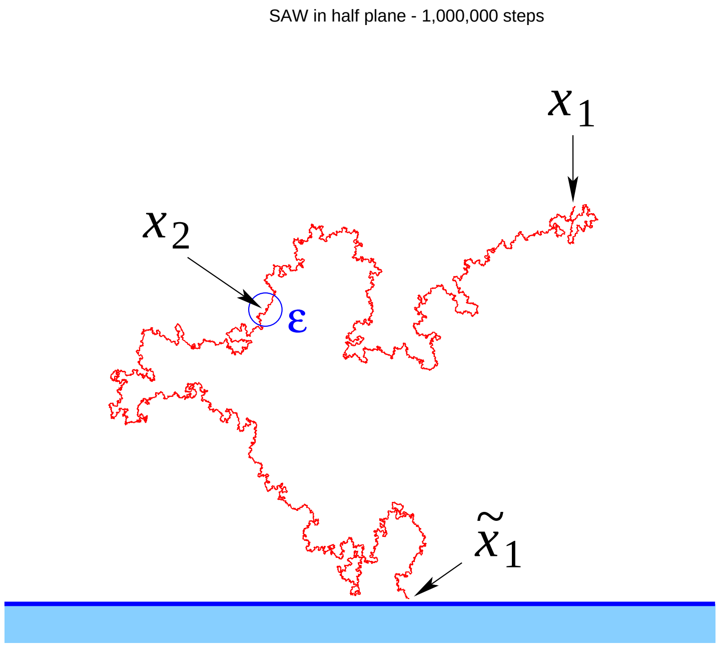
Euclidean squares of similar quantum area δ

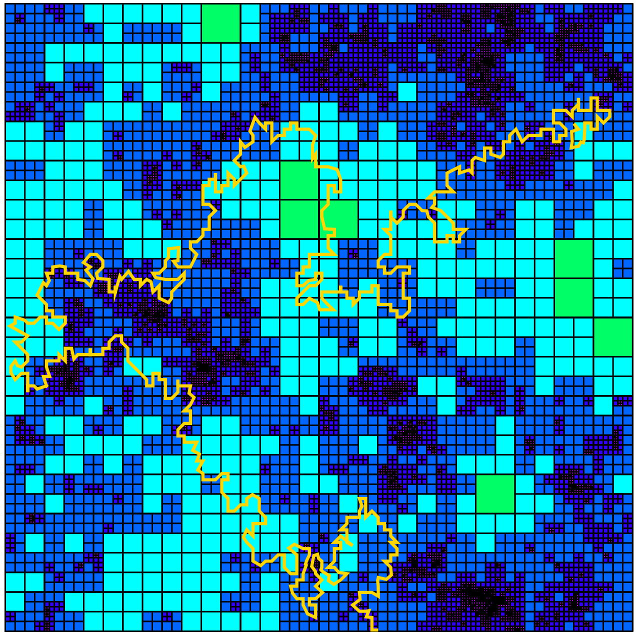
Quantum Gravity Scaling Exponents



$$\mathbb{P} \asymp \delta^{\Delta}, \qquad \tilde{\mathbb{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$$

Scaling Exponents of (Random) Fractals





Probabilities & Hausdorff Dimensions (e.g., SLE_κ)

$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad d = 2 - 2x \qquad (= 1 + \kappa/8)$$

Quantum case: $\mathbf{P} \asymp \delta^{\Delta}, \quad \tilde{\mathbf{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$

KPZ ’88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

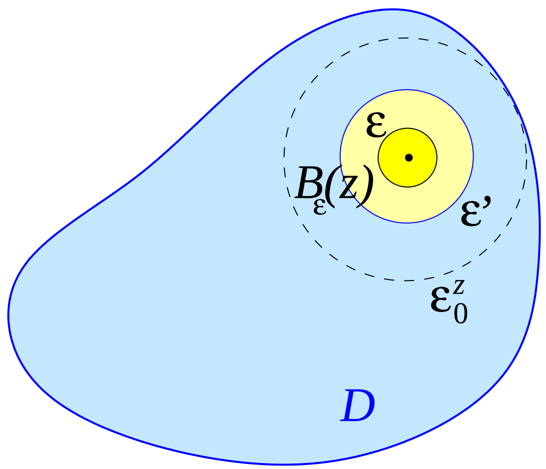
$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, ’08]*

PRL **102**, *150603 (2009) & Invent. Math.* **185**, *333 (2011)*

Kazakov ’86; D. & Kostov ’88 [Random matrices]
David; Distler & Kawai ’88 [Liouville field theory]
Benjamini & Schramm ’08; Rhodes & Vargas ’11 [Math]
David & Bauer ’09

GFF & Brownian Motion



- $h_\varepsilon(z)$ mean value of h on circle $\partial B_\varepsilon(z)$

- Define $t := -\log \varepsilon$, $\mathcal{B}_t := h_{\varepsilon = e^{-t}}(z)$; for z fixed, the law of $\mathcal{B}_t$ is **standard Brownian motion** in t

$$\mathrm{Var}\left[(h_\varepsilon - h_{\varepsilon'})(z)\right] = |\log(\varepsilon/\varepsilon')| = |t - t'| = \mathrm{Var}\left[\mathcal{B}_t - \mathcal{B}_{t'}\right] \quad \square$$

GFF Liouville Weighted Measure

$$h_\varepsilon(z) = (h, f_\varepsilon^z)_\nabla \qquad \mathrm{Var}\, h_\varepsilon(z) = (f_\varepsilon^z, f_\varepsilon^z)_\nabla$$

$$\begin{aligned}
\exp\left[-\frac{1}{2}(h,h)_\nabla + \gamma(h, f_\varepsilon^z)_\nabla\right] &= \exp\left[-\frac{1}{2}(h',h')_\nabla + \frac{\gamma^2}{2}(f_\varepsilon^z, f_\varepsilon^z)_\nabla\right] \\
&= \exp\left[-\frac{1}{2}(h',h')_\nabla\right] \mathbb{E}\, e^{\gamma h_\varepsilon(z)} \\
h &\overset{\text{(in law)}}{=} h' + \gamma f_\varepsilon^z \quad (h' \text{ standard GFF}) \quad \square \\
h_\varepsilon(z) &= \mathcal{B}_t + \gamma f_\varepsilon^z(z) \quad \bullet
\end{aligned}$$

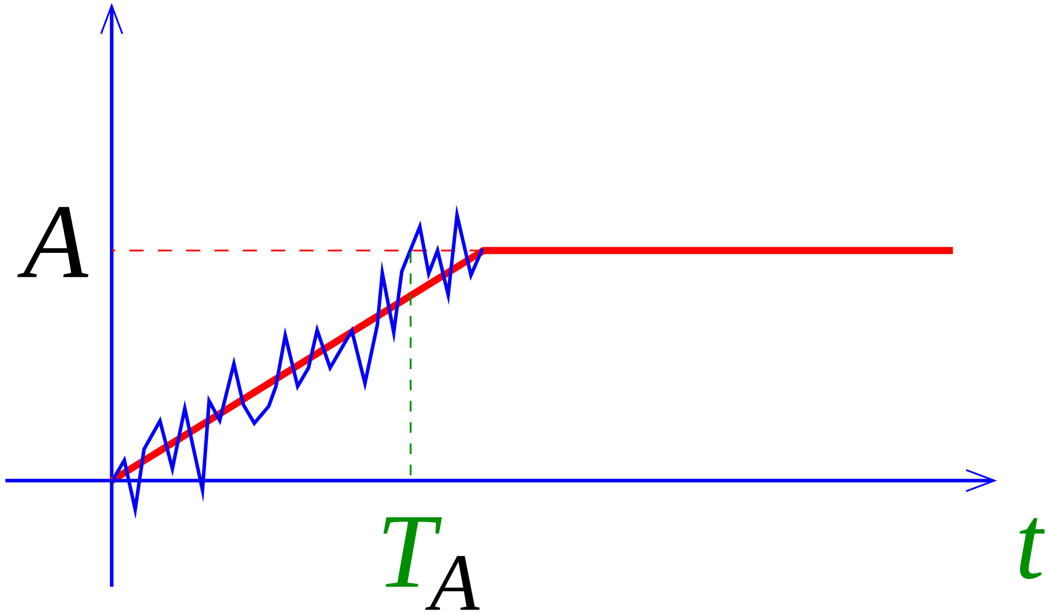
Quantum Ball & Brownian Motion

Quantum area

- $\delta := \exp\left[\gamma h_\varepsilon(z)\right] \pi \varepsilon^{2+\gamma^2/2}$

Given z, $h_\varepsilon(z)$ is standard Brownian motion $\mathcal{B}_t$, $t = -\log\varepsilon$,
plus the deterministic term: -$\gamma \log\varepsilon = \gamma t$

$$\begin{aligned} \delta &= \exp(\gamma \mathcal{B}_t - at), \qquad a := 2 - \gamma^2/2 \\ -\log\delta &= at - \gamma \mathcal{B}_t \qquad \square \qquad \text{(B. M. \& drift)} \end{aligned}$$

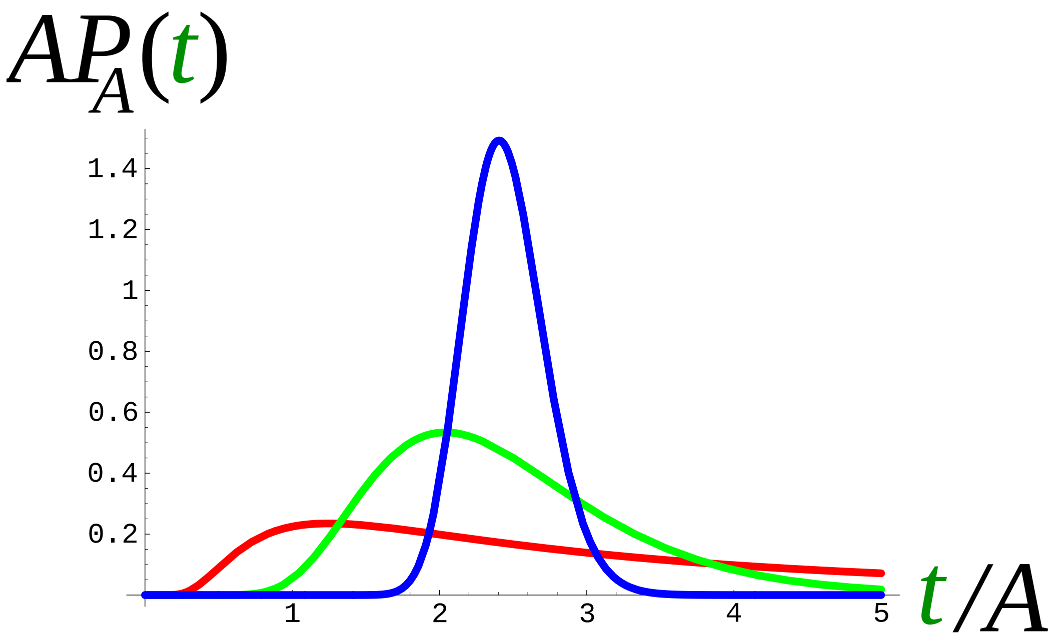


The stochastic area of ball $B_\varepsilon(z)$ equals δ at **stopping time** T_A

$$-\log \varepsilon_A = T_A \quad := \quad \inf\{t : at - \gamma \mathcal{B}_t = A\}$$

$$A := -\log\delta > 0,\; a \quad = \quad 2 - \gamma^2/2 > 0 \quad (\gamma < 2)$$

Probability Distribution $(\gamma = \sqrt{8/3})$ $[A = 2;\ 20;\ 200]$



$$P_A(t)dt \quad := \quad \mathbb{P}(T_A \in [t, t+dt])$$

$$P_A(t) \quad = \quad \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}(A - at)^2\right]$$

Euclidean scaling exponent

X a (random) fractal of *Euclidean scaling exponent x* (Hausdorff dimension $2-2x$):

$$\mathbb{P}\{B_\varepsilon(z) \cap X \neq \emptyset\} \asymp \varepsilon^{2x}$$

uniformly in z.

Quantum scaling exponent

Quantum scaling exponent Δ of X when (h, z) and X are sampled *independently* from the quantum gravity measure and from the law of X:

$$\mathbb{E}\,\mathbb{P}\{B_{\varepsilon_A}(z) \cap X \neq \emptyset\} \asymp \mathbb{E}\left[\varepsilon_A{}^{2x}\right] \asymp \delta^{\Delta}$$

KPZ Theorem

Stochastic probability & stopping time

$$-\log \varepsilon_A = T_A = \inf\{t : at - \gamma \mathcal{B}_t = A = -\log \delta\}$$
$$\varepsilon_A{}^{2x} = \exp(-2xT_A)$$

BROWNIAN MARTINGALE & LARGE DEVIATIONS

$$\mathbb{E}\left[\varepsilon_A{}^{2x}\right] = \mathbb{E}\left[e^{-2xT_A}\right] = \exp(-\Delta A) = \delta^{\Delta}$$
$$2x = a\Delta + \frac{\gamma^2}{2}\Delta^2, \qquad a = 2 - \frac{\gamma^2}{2} \qquad \text{(KPZ)} \quad \square$$

Brownian Exponential Martingale Lemma

$T_A = -\log \varepsilon_A$ is the first time t such that

$$at - \gamma \mathcal{B}_t = A,$$

$\mathcal{B}_t$ standard Brownian motion ($\mathcal{B}_0 = 0$). Consider for any β the *Brownian exponential martingale*

$$\mathbb{E}\left[\exp(-\beta \mathcal{B}_t - \beta^2 t/2)\right] = \mathbb{E}\left[\exp(-\beta \mathcal{B}_0\right] = 1.$$

At the stopping time $t = T_A < +\infty$ *in particular:*

$$\mathbb{E}\left[\exp(-\beta \mathcal{B}_{T_A} - \beta^2 T_A/2)\right] = 1$$

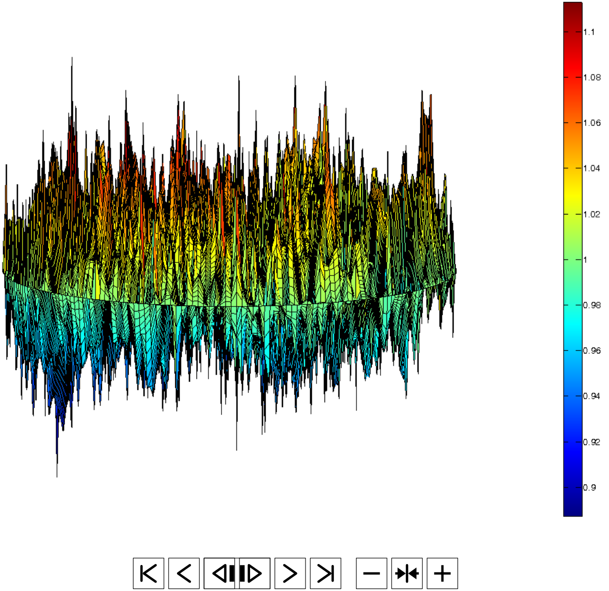
By definition $\gamma \mathcal{B}_{T_A} = aT_A - A$, whence

$$\begin{aligned} \mathbb{E}\left(\exp[-(a\beta/\gamma + \beta^2/2)T_A]\right) &= \exp(-\beta A/\gamma) \\ = \mathbb{E}\left[\exp(-2xT_A)\right] & \quad 2x := a\beta/\gamma + \beta^2/2 \\ \Delta := \beta/\gamma; & \quad = a\Delta + \frac{\gamma^2}{2}\Delta^2 \quad \square \end{aligned}$$

Liouville Quantum Duality

$$\gamma > 2, \gamma' = 4/\gamma < 2$$

gamma = 0.02



LIOUVILLE QUANTUM DUALITY

Baby-Universes: *Das, Dhar, Sengupta, Wadia '90; Jain & Mathur 92; Korchemsky '92; Alvarez-Gaumé, Barbón, Crnković '93; Durhuus '94; Ambjørn, Durhuus, Jonsson '94*

The Other Branch of Gravity, *Klebanov '95*

Dual Dimensions

$$\gamma > 2, \gamma' = 4/\gamma < 2$$

$$\Delta_\gamma - 1 := \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1)$$

D. & Sheffield, PRL **102**, *150603 (2009)*

Quantum Area Measure ($\gamma > 2$)

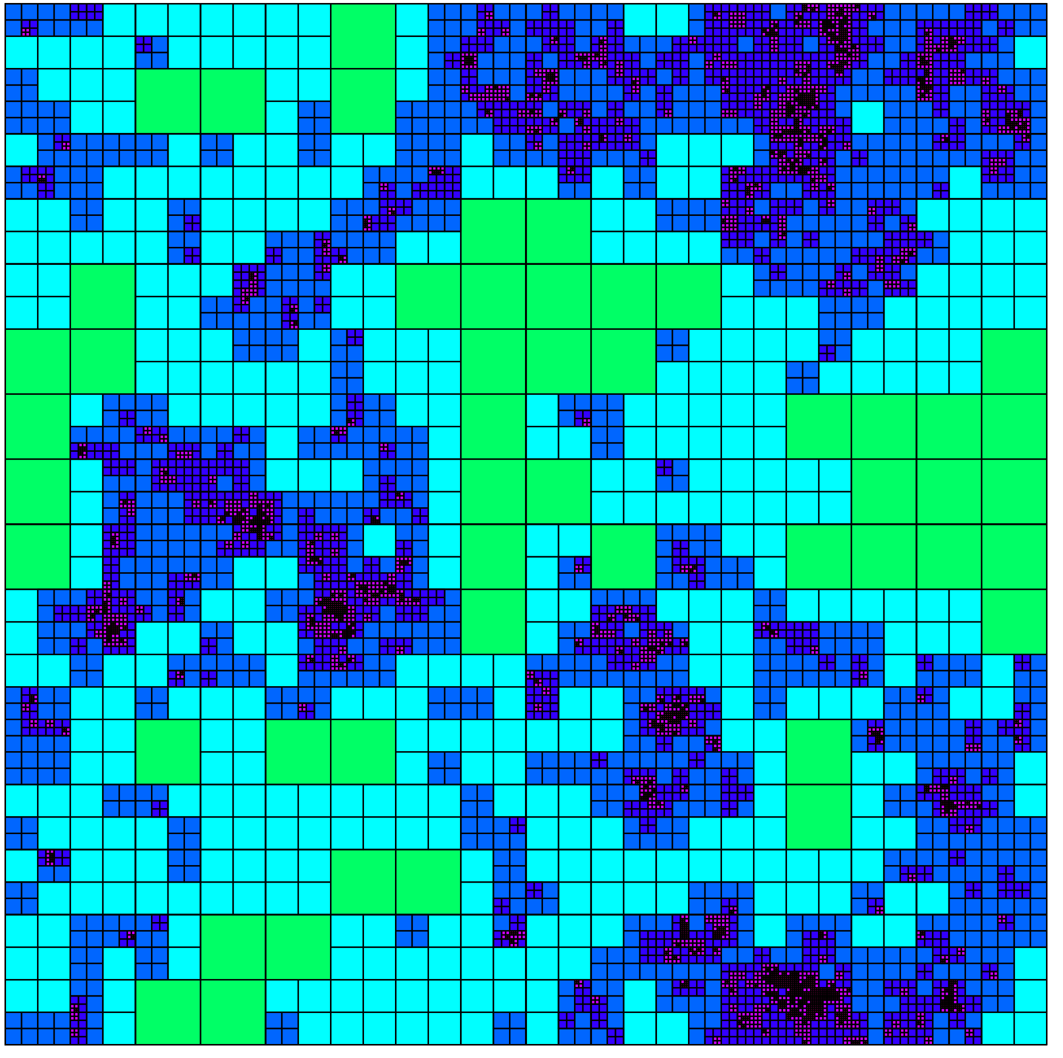
$$\lim_{\varepsilon \to 0} \left[\exp[\gamma h_\varepsilon(z)] \, \varepsilon^{\gamma^2/2} \right]_{\gamma > 2} d^2 z = 0;$$

the measure develops atoms with localized area.

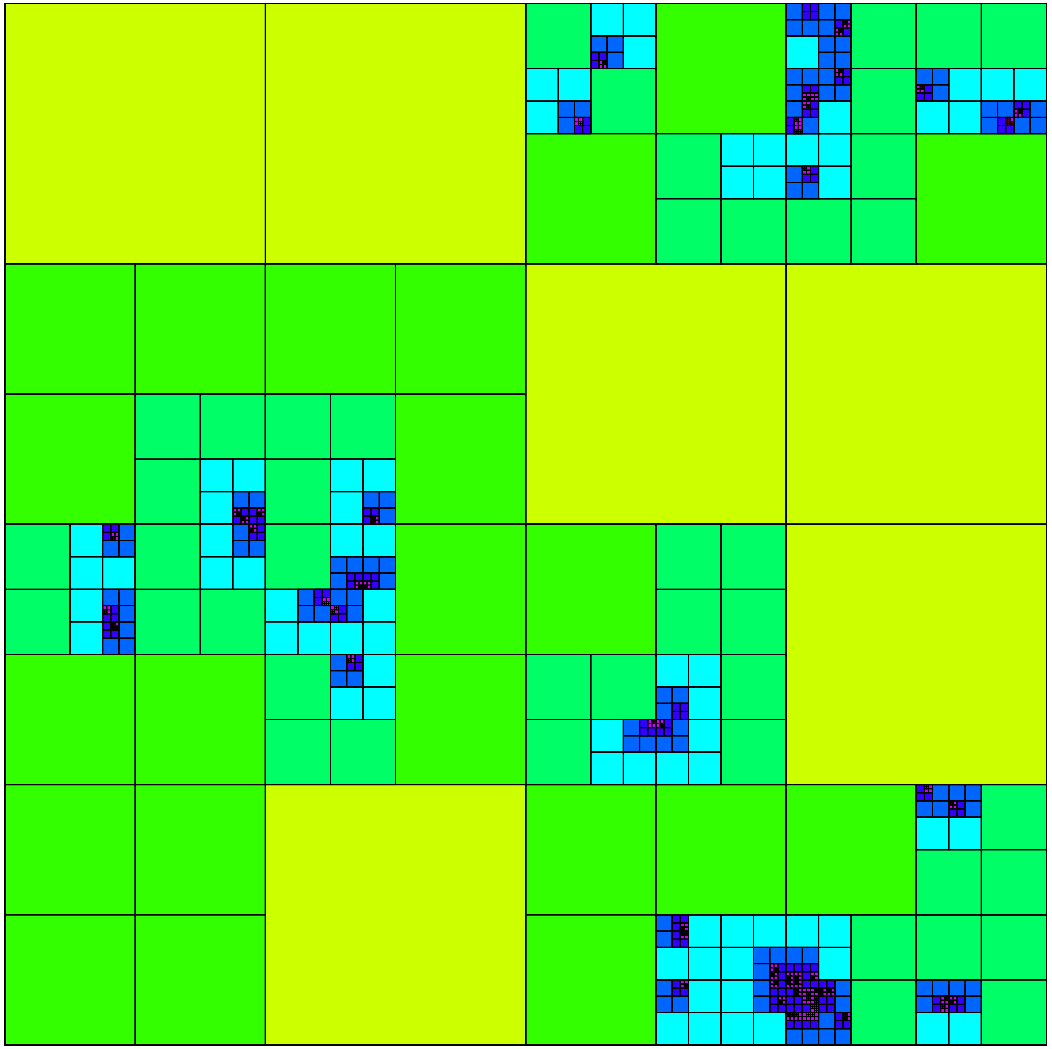
Liouville Quantum Duality

$$\gamma > 2, \gamma' = 4/\gamma < 2.$$

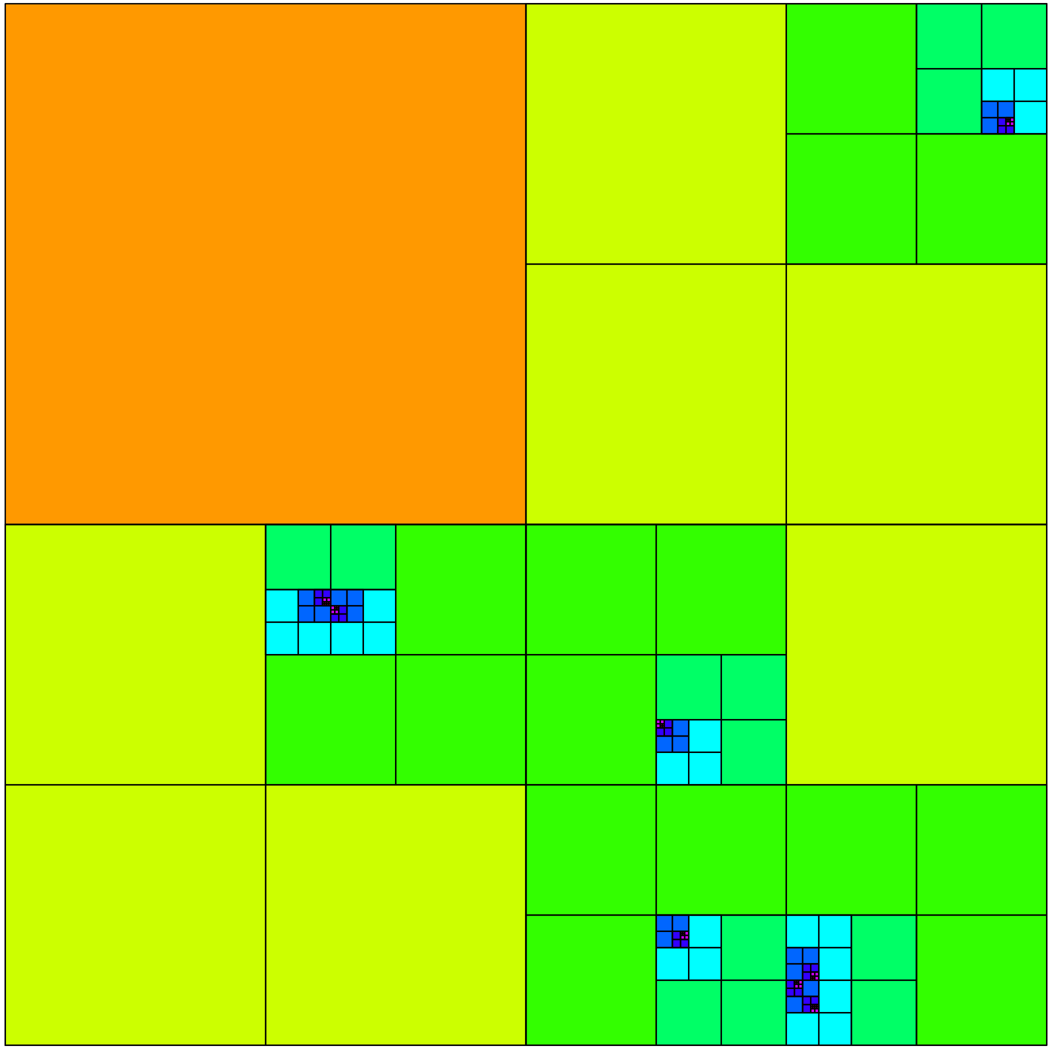
QG Measure ($\gamma = 2$)



QG Measure ($\gamma = 5$)



QG Measure ($\gamma = 10$)



Dual Quantum Gravity *(D. & Sheffield '09, Barral & al. '12)*

The construction of the singular quantum measures $\mu_{\gamma>2}$ uses the dual measure $\mu_{\gamma'<2}$ and the additional randomness of sets of point masses $\{z_k\}$, where *finite* amounts $\mu_\gamma(z_k)$ of quantum area are *localized*, and such that $\{z_k, \mu_\gamma(z_k)\}$ is distributed as a *Poissonian point process* of intensity $\mu_{\gamma'<2} \times \Lambda^\nu$, where $\Lambda^\nu(d\eta) := \eta^{-\nu-1} d\eta$, with $\nu := 4/\gamma^2 \in (0,1)$.
The Laplace transform of the quantum area $\mu_{\gamma>2}(\mathcal{D})$ of any subdomain $\mathcal{D} \subset D$ is then, given $\mu_{\gamma'<2}(\mathcal{D})$,

$$\mathbb{E} \exp[-\lambda \mu_{\gamma>2}(\mathcal{D})] = \exp[\Gamma(-\nu)\lambda^\nu \mu_{\gamma'<2}(\mathcal{D})],$$

for any $\lambda > 0$, and with $\Gamma(-\nu) = -\Gamma(1-\nu)/\nu < 0$ the Euler Γ-function.

Dual Quantum Gravity

The dual cosmological constant associated with $\mu_{\gamma'}$ is

$$\lambda' := -\Gamma(-\nu)\lambda^{\nu}, \ \nu = 4/\gamma^2.$$

In other words, given $\mu_{\gamma'}$, the typical size of $\mu_{\gamma>2}(\mathcal{D})$ is $\mu_{\gamma'}(\mathcal{D})^{\gamma^2/4}$. The relation above is the probabilistic formulation of the *Legendre transform* relating the free energies of dual Liouville theories *(Klebanov & Hashimoto '95).*

Duality: $\gamma > 2, \gamma' := 4/\gamma < 2$

γ & γ'-Quantum Balls

$$Q_{\gamma'} = Q_\gamma := \frac{2}{\gamma} + \frac{\gamma}{2}$$

$$\mu_{\gamma'}(B_\varepsilon(z)) = \varepsilon^{\gamma' Q} e^{\gamma' h_\varepsilon(z)} = \mu_\gamma(B_\varepsilon(z))^{\gamma'/\gamma} = \mu_\gamma^{4/\gamma^2}$$

$$\delta' = \delta^{4/\gamma^2}$$

Dual Dimensions

Ball covering of fractal X

$$
\begin{aligned}
N_{\gamma'}(\delta', X) &= N_{\gamma}(\delta, X) \\
\delta'^{\Delta_{\gamma'}-1} = \delta^{(4/\gamma^2)(\Delta_{\gamma'}-1)} &= \delta^{\Delta_\gamma - 1} \\
\Delta_\gamma - 1 &:= \frac{4}{\gamma^2}(\Delta_{\gamma'} - 1)
\end{aligned}
$$

"The other branch of gravity," I. Klebanov, '95

Brownian Approach to Duality

$$\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}] = \exp(-\beta_\gamma A) = \delta^{\Delta_\gamma}$$

$$\beta_\gamma(x) := (a_\gamma^2 + 4x)^{1/2} - a_\gamma, \qquad \Delta_\gamma := \beta_\gamma/\gamma$$

$$a_\gamma := \frac{2}{\gamma} - \frac{\gamma}{2} < 0$$

$$\mathbb{P}(T_A < \infty) = \mathbb{E}[1_{T_A<\infty}] = \delta^{\Delta_\gamma(0)} = \delta^{1-4/\gamma^2} = \delta/\delta',$$

$$\frac{\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}]}{\mathbb{E}[1_{T_A<\infty}]} = \delta^{\Delta_\gamma} \times \frac{\delta'}{\delta} = \delta'^{\Delta_{\gamma'}}.$$

Perspectives

- *Scaling limits of discrete models on random planar maps*
- *Quantum wedges and cones*
- *Quantum bubbles and foam ($\gamma\gamma' = 4$ duality)*
- *Geodesics & random metrics*

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