Schramm-Loewner Evolution & Liouville Quantum Gravity

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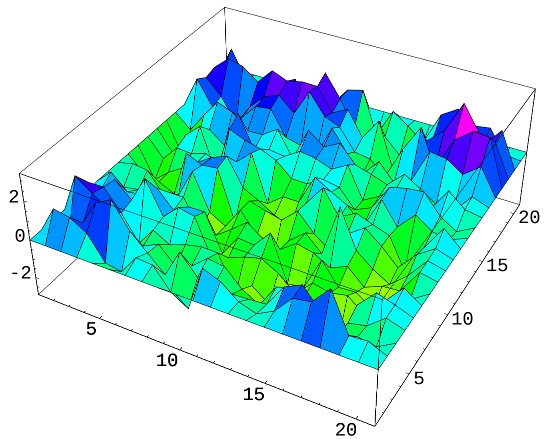
Workshop on Manifolds of Metrics

& Probabilistic Methods in Geometry & Analysis

Centre de Recherches Mathématiques, Univ. de Montréal

July 2-6, 2012

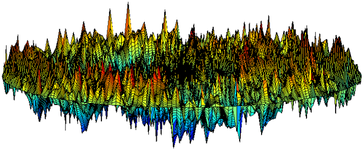
Gaussian Free Field (GFF)



Distribution h with *Gaussian weight* $\exp\left[-\frac{1}{2}(h,h)_\nabla\right]$, and **Dirichlet inner product** in domain D

$$(f_1, f_2)_\nabla := (2\pi)^{-1}\int_D \nabla f_1(z)\cdot\nabla f_2(z)\, d^2z$$
$$= \mathrm{Cov}\big((h, f_1)_\nabla, (h, f_2)_\nabla\big)$$

◊ STARRING THE GFF! *(Courtesy of N.-G. Kang)* ◊



Liouville QG

Random Measure

$$d\mu = \text{“}e^{\gamma h} d^2 z\text{”}$$

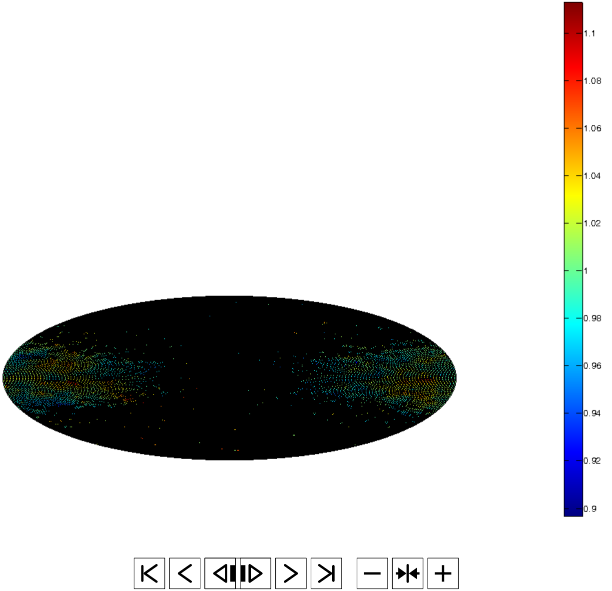
◊

The Emergence of Quantum Gravity

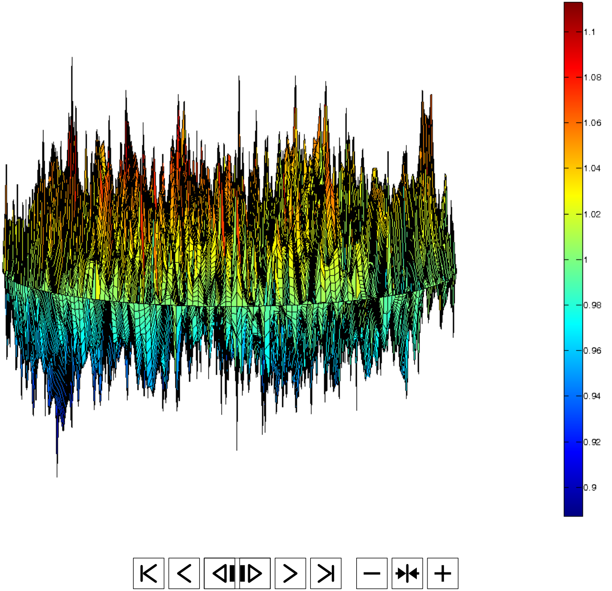
(Courtesy of N.-G. Kang)

◊

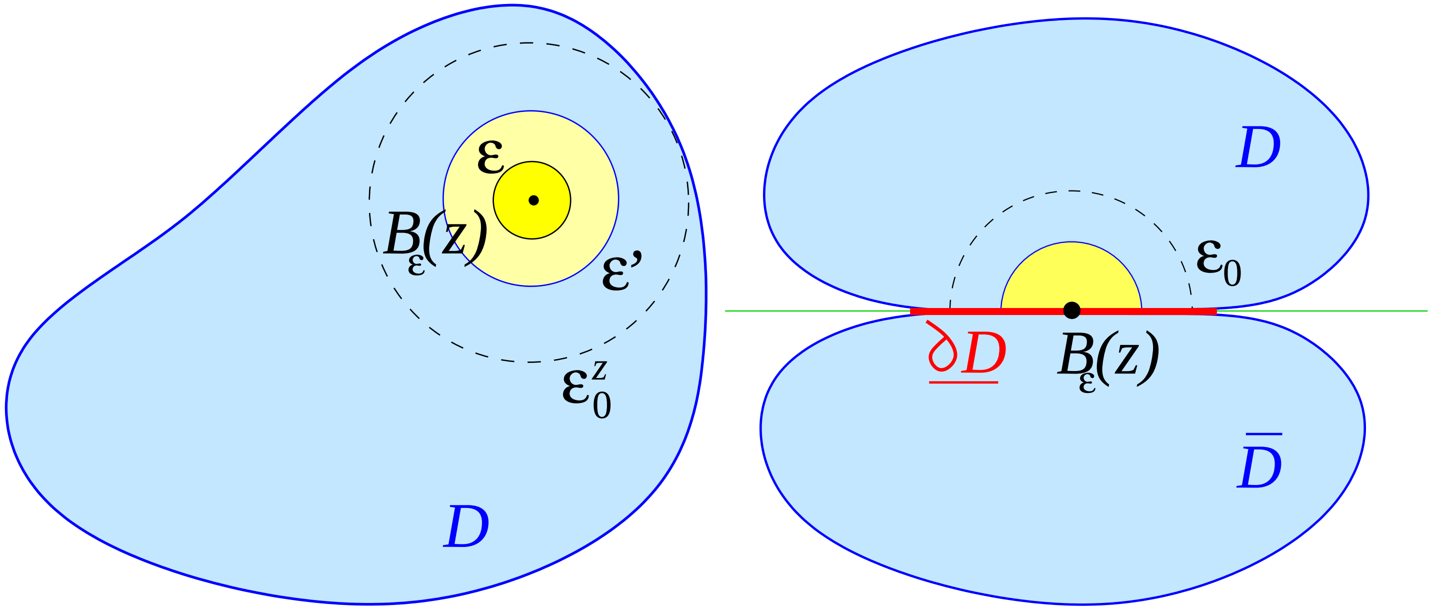
gamma = 0.02



gamma = 0.02



Bulk & Boundary Liouville Quantum Gravity



- *Circle averages $h_\varepsilon(z)$, $z \in D$ (Dirichlet)*
- *GFF with free boundary conditions on ∂D*
- *Half-circle averages $\hat{h}_\varepsilon(z)$, $z \in \partial D$.*

Quantum Area Measure

$$d\mu_\varepsilon := \exp\left[\gamma h_\varepsilon(z)\right] \varepsilon^{\gamma^2/2} d^2 z$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$*, to a random measure, denoted by* $e^{\gamma h(z)} d^2 z$*.*

(Høegh-Krohn, '71)

Quantum Boundary Measure

$$d\hat{\mu}_\varepsilon := \exp\left[\frac{\gamma}{2} \hat{h}_\varepsilon(z)\right] \varepsilon^{\gamma^2/4} dz$$

converges, as $\varepsilon \to 0$ *and for* $\gamma < 2$*, to a boundary random measure, denoted by* $e^{(\gamma/2)\, h(z)} dz$*.*

Critical Quantum Area Measure ($\gamma = 2$)

For $\gamma = 2$, the derivative martingale

$$d\mu'_\varepsilon := -\frac{\partial}{\partial\gamma}\left[\exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2}\right]_{\gamma=2} d^2z$$

and the renormalized one

$$d\mu_\varepsilon := \sqrt{\log(1/\varepsilon)}\left[\exp\left[\gamma h_\varepsilon(z)\right]\varepsilon^{\gamma^2/2}\right]_{\gamma=2} d^2z$$

converge, as $\varepsilon \to 0$, to equivalent (up to constant factor) positive non-atomic random measures.

(D., Rhodes, Sheffield, Vargas, arXiv:1206.1671v2)

Dual Quantum Gravity *(D. & Sheffield '09, Barral & al. '12)*

The construction of the singular quantum measures $\mu_{\gamma>2}$ uses the dual measure $\mu_{\gamma'<2}$ and the additional randomness of sets of point masses $\{z_k\}$, where *finite* amounts $\mu_\gamma(z_k)$ of quantum area are *localized*, and such that $\{z_k, \mu_\gamma(z_k)\}$ is distributed as a *Poissonian point process* of intensity $\mu_{\gamma'<2} \times \Lambda^\nu$, where $\Lambda^\nu(d\eta) := \eta^{-\nu-1} d\eta$, with $\nu := 4/\gamma^2 \in (0,1)$.
The Laplace transform of the quantum area $\mu_{\gamma>2}(\mathcal{D})$ of any subdomain $\mathcal{D} \subset D$ is then, given $\mu_{\gamma'<2}(\mathcal{D})$,

$$\mathbb{E} \exp[-\lambda \mu_{\gamma>2}(\mathcal{D})] = \exp[\Gamma(-\nu) \lambda^\nu \mu_{\gamma'<2}(\mathcal{D})],$$

for any $\lambda > 0$, and with $\Gamma(-\nu) = -\Gamma(1-\nu)/\nu < 0$ the Euler Γ-function.

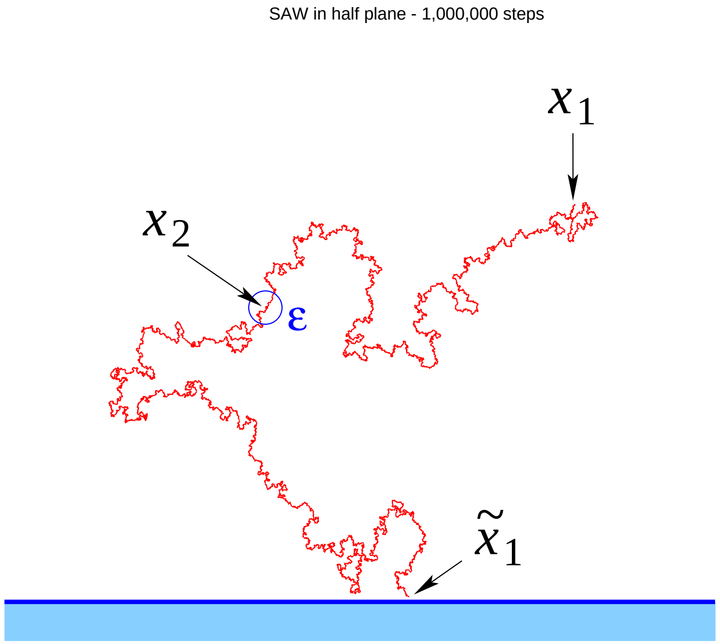
Dual Quantum Gravity

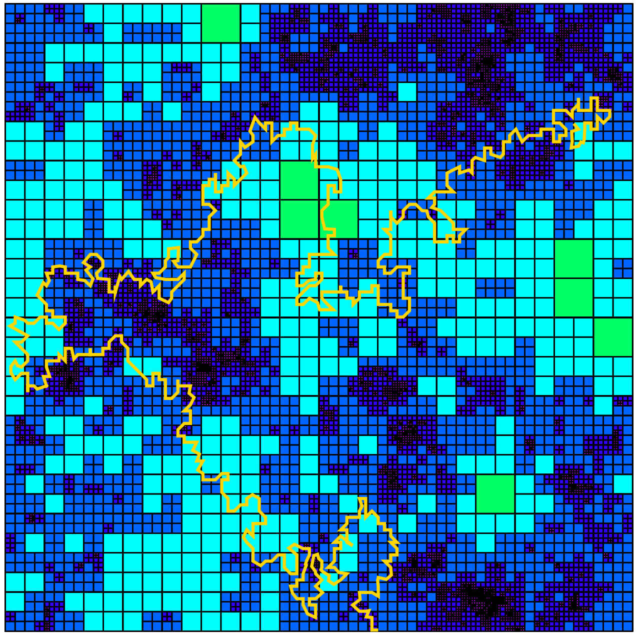
The dual cosmological constant associated with $\mu_{\gamma'}$ is

$$\lambda' := -\Gamma(-\nu)\lambda^{\nu}, \; \nu = 4/\gamma^2.$$

In other words, given $\mu_{\gamma'}$, the typical size of $\mu_{\gamma>2}(\mathcal{D})$ is $\mu_{\gamma'}(\mathcal{D})^{\gamma^2/4}$. The relation above is the probabilistic formulation of the *Legendre transform* relating the free energies of dual Liouville theories *(Klebanov & Hashimoto '95).*

Scaling Exponents of (Random) Fractals





Probabilities & Hausdorff Dimensions (e.g., SLE$_\kappa$)

$$\mathbb{P} \asymp \varepsilon^{2x}, \quad \tilde{\mathbb{P}} \asymp \varepsilon^{\tilde{x}}, \qquad d = 2 - 2x \qquad (= 1 + \kappa/8)$$

Quantum case: $\mathbf{P} \asymp \delta^{\Delta}, \quad \tilde{\mathbf{P}} \asymp \tilde{\delta}^{\tilde{\Delta}}$

Knizhnik, Polyakov, Zamolodchikov ’88

x and Δ ($\tilde{x}$ and $\tilde{\Delta}$) are related by the **KPZ formula**

$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\Delta^2$$

KPZ is a Theorem *[D. & Sheffield, ’08]*

PRL **102**, *150603 (2009) & Invent. Math.* **185**, *333 (2011)*

Kazakov ’86; D. & Kostov ’88 [Random matrices]
David; Distler & Kawai ’88 [Liouville field theory]
Benjamini & Schramm ’08; Rhodes & Vargas ’11 [Math]
David & Bauer ’09

Euclidean & Quantum Fractal Measures

- Rescale a d-dimensional fractal $X \subset \mathcal{D} \subset \mathbb{C}$ via the map $z \to \psi(z) = bz$, $b \in \mathbb{C}$ (so that the Euclidean area of the domain $\mathcal{D}$ is multiplied by $|b|^2$); then the *d-dimensional Euclidean fractal measure* of X is multiplied by $|b|^d = |b|^{2-2x}$, where x (the *Euclidean scaling weight*) is defined by $d := 2 - 2x \, (\leqslant 2)$.
- If X is a *fractal subset* of a *random surface* $\mathcal{S} := (\mathcal{D}, h)$, and we rescale $\mathcal{S}$ so that its quantum area increases by a factor of $|b|^2$, then the *quantum fractal measure* $\mathcal{Q}(X, h)$ of X is multiplied by $|b|^{2-2\Delta}$, where Δ is the analogous *quantum scaling weight*.

• $\mathcal{Q}\big(\psi(X,h)\big) = \mathcal{Q}(X,h)$ whenever ψ is conformal and

$$\begin{aligned}\psi(\mathcal{D},h) &:= \big(\psi(\mathcal{D}), h\circ\psi^{-1} - Q\log|\psi'|\big)\\ Q &:= \frac{\gamma}{2}+\frac{2}{\gamma}.\end{aligned}$$

The pair $\mathcal{S} = (\mathcal{D},h)$ describes the same Liouville quantum surface (up to coordinate change) as the conformally transformed pair $\psi(\mathcal{D},h)$.

• The *Knizhnik, Polyakov, Zamolodchikov (KPZ)* relation

$$x = (\gamma^2/4)\Delta^2 + \big(1-\gamma^2/4\big)\,\Delta$$

is then equivalent to

$$d = \alpha Q - \alpha^2/2,$$

where $d = 2-2x$ and $\alpha := \gamma(1-\Delta)$.

SLE - GFF (QG) Coupling

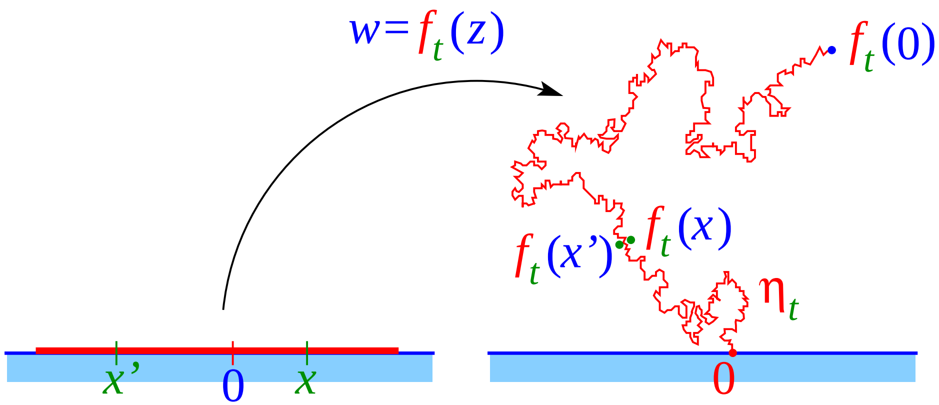
Dubédat, 2009

Sheffield, arXiv:1012.4797

D. & Sheffield, PRL **107***, 131305 (2011), arXiv:1012.4800*

Miller & Sheffield, arXiv:1201.14896-97-98

“*Zipping-up*” SLE Map



Let f_t be the (reverse) SLE_κ conformal map

$$z \in \mathbb{H} \to w = f_t(z) \in \mathbb{H} \setminus \eta_t,$$

with trace η_t and tip $f_t(0)$ [$t = 0$, $f_0(z) = z$]. It satisfies the stochastic differential equation (B_t *standard Brownian motion*)

$$d\, f_t(z) = -2dt / f_t(z) - \sqrt{\kappa}\, dB_t.$$

(Reverse) SLE Martingale

Real stochastic process in the upper-half plane:

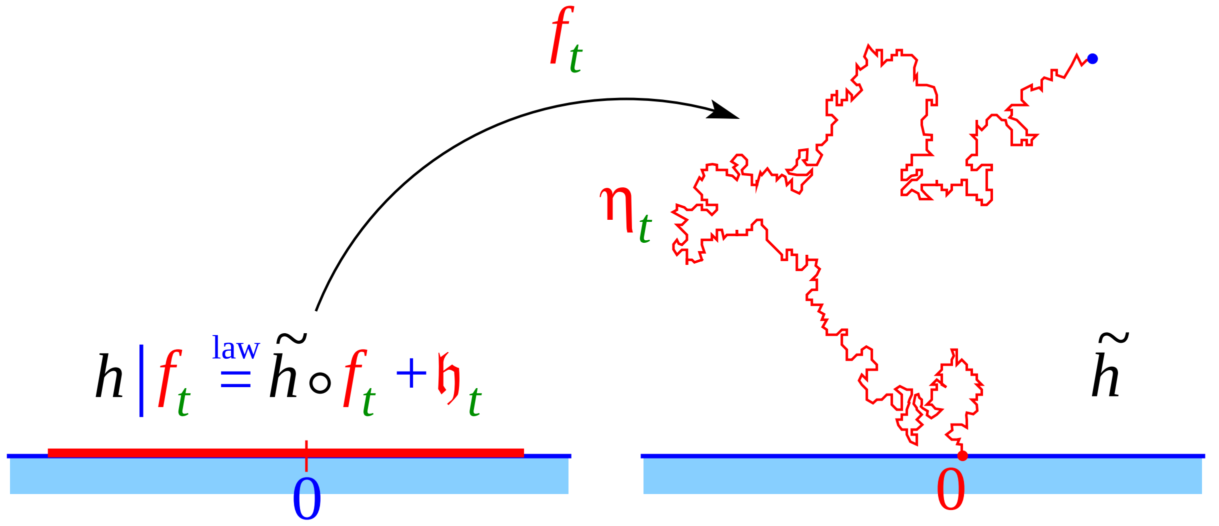
$$\begin{aligned} \mathfrak{h}_0(z) &:= \frac{2}{\sqrt{\kappa}} \log|z|, \\ \mathfrak{h}_t(z) &:= \mathfrak{h}_0 \circ f_t(z) + Q \log|f_t'(z)|. \end{aligned}$$

This process $\mathfrak{h}_t(z)$ is a *martingale* (so that $\mathbb{E}\mathfrak{h}_t(z) = \mathfrak{h}_0(z)$) for the particular choice:

$$Q = \sqrt{\kappa}/2 + 2/\sqrt{\kappa},$$

for which $d\mathfrak{h}_t(z) = -\Re[2/f_t(z)]dB_t$.

SLE–GFF Coupling

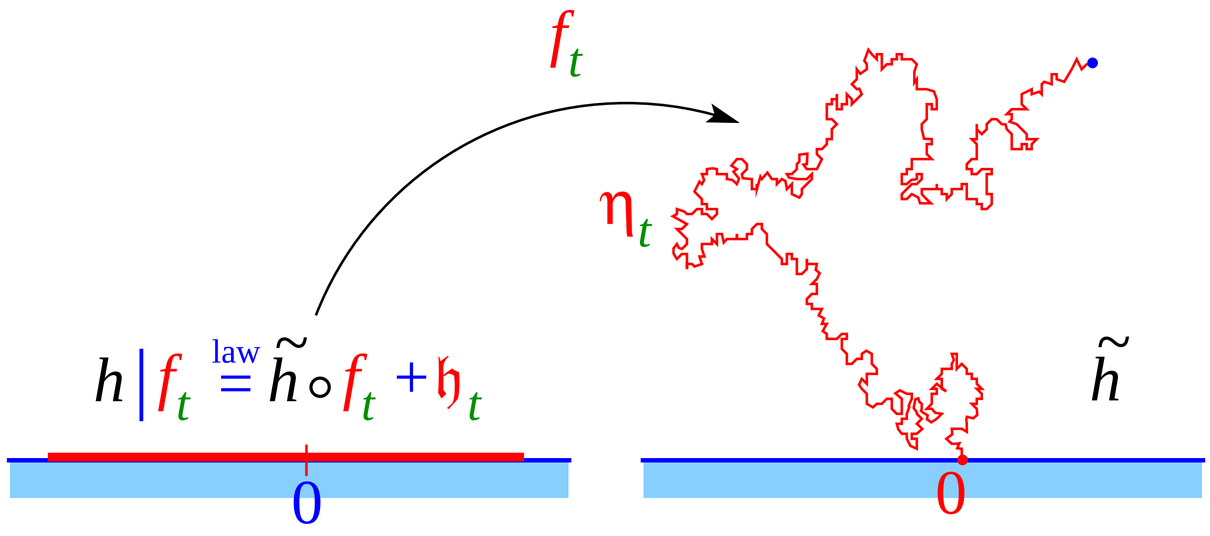


Define $h := \tilde{h} + \mathfrak{h}_0$, sum of the GFF $\tilde{h}$ on $\mathbb{H}$ with *free boundary conditions* on $\mathbb{R}$, and of the deterministic function $\mathfrak{h}_0$. Given f_t, the conditional law of h (denoted by $h|\, f_t$) is

$$h(z)|\, f_t \overset{(\text{law})}{=} \tilde{h}\circ f_t(z) + \mathfrak{h}_t(z),$$

where $\tilde{h}\circ f_t$ is the pullback of the free boundary GFF $\tilde{h}$.

SLE–GFF Coupling



$$h(z)\,|\,f_t \overset{\text{(law)}}{=} \tilde{h} \circ f_t(z) + \mathfrak{h}_t(z)$$

To sample h, one can first sample the B_t process (which determines f_t), then sample independently the f.b.c. GFF $\tilde{h}$ and take the above sum *[Sheffield, 2010]*.
The conditional expectation w.r.t. $\tilde{h}$ is the *martingale* $\mathbb{E}\left[h(z)\,|\,f_t\right] = \mathfrak{h}_t(z)$.

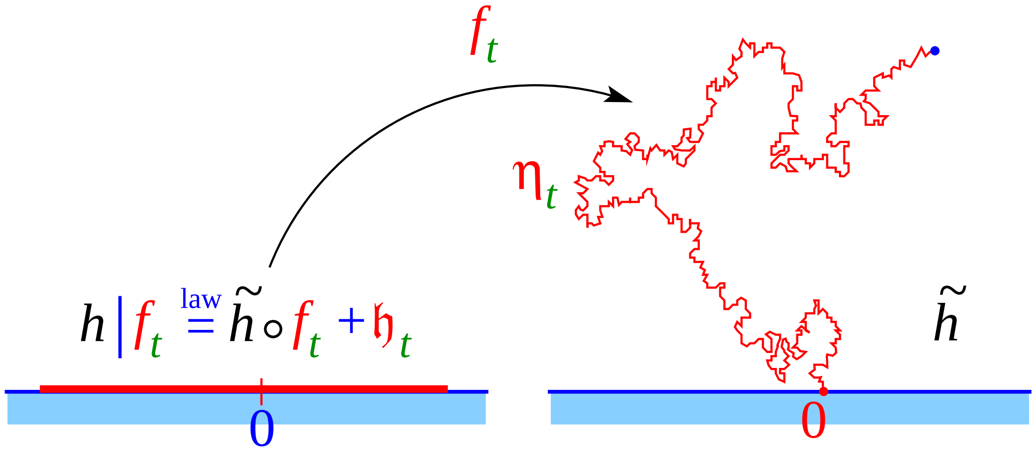
Neumann Green's Function

In $\mathbb{H}$: $G_0(y,z) := -\log(|y-z||y-\bar{z}|)$; define the *time-dependent* $G_t(y,z) := G_0\big(f_t(y), f_t(z)\big)$, i.e., G_0 taken at image points under f_t. One has:
$dG_t(y,z) + d\langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle = 0$ (*Hadamard's formula*).

Green's function: $G_0(y,z) = \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)]$, thus $G_t(y,z) = \mathrm{Cov}[\tilde{h}\circ f_t(y), \tilde{h}\circ f_t(z)]$. *The random distribution $\tilde{h}\circ f_t$ and the set of (time changed) Brownian motions $\mathfrak{h}_t$ are Gaussian processes, whose respective covariance G_t and covariation $\langle \mathfrak{h}_t, \mathfrak{h}_t\rangle$ thus add to constant covariance G_0:*

$$\begin{aligned} G_t(y,z) + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle &= G_0(y,z) \\ \mathrm{Cov}[\tilde{h}\circ f_t(y), \tilde{h}\circ f_t(z)] + \langle \mathfrak{h}_t(y), \mathfrak{h}_t(z)\rangle &= \mathrm{Cov}[\tilde{h}(y), \tilde{h}(z)] \\ &= \mathrm{Cov}[h(y), h(z)] \quad \square \end{aligned}$$

Liouville Invariance



Recall: $h := \tilde{h} + \mathfrak{h}_0$, and $\mathfrak{h}_t := \mathfrak{h}_0 \circ f_t + Q\log|f_t'|$. Thus $\tilde{h}\circ f_t + \mathfrak{h}_t = h\circ f_t + Q\log|f_t'|$. For $Q = \gamma/2 + 2/\gamma$, this is the transformation law of the GFF h under the conformal map f_t^{-1}. *The pair* $(\mathbb{H}, \tilde{h}\circ f_t + \mathfrak{h}_t) = f_t^{-1}(\mathbb{H}\setminus\eta_t, h)$ *describes the same random surface as the pair* $(\mathbb{H}\setminus\eta_t, h)$.

Liouville Quantum Measure

$$\left(e^{\gamma h(z)} \,|\, f_t\right) d^2z \overset{(\text{law})}{=} e^{\gamma h(w)}\, d^2w \quad \text{(conformal invariance)}$$

for $d = 2 = \gamma Q - \gamma^2/2$, *i.e.*, $Q = \gamma/2 + 2/\gamma = \sqrt{\kappa}/2 + 2/\sqrt{\kappa}$

$$\Longrightarrow \ \gamma = \sqrt{\kappa} \wedge (4/\sqrt{\kappa}), \ \ \gamma' = 4/\gamma$$

• $\gamma \leqslant 2$: *KPZ prediction* $\gamma = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6}$ for the *central charge* $c = \frac{1}{4}(6-\kappa)(6-16/\kappa) \leqslant 1$ of the SLE's CFT coupled to gravity.

• $\gamma' = 4/\gamma > 2$: *Duality* property of Liouville quantum gravity; the quantum measure develops atoms with localized area.

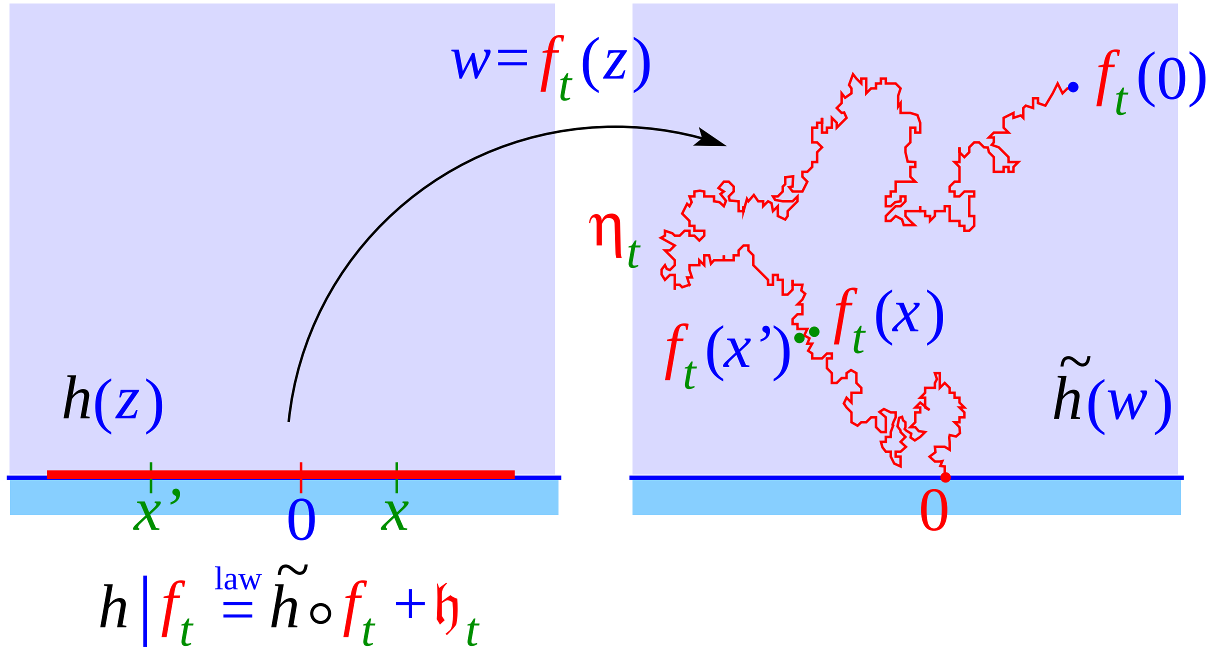
Conformally welding two γ-Liouville quantum surfaces produces SLE$_\kappa$.

Expected Liouville Quantum Area

For $\gamma = \sqrt{\kappa \wedge 16/\kappa}$

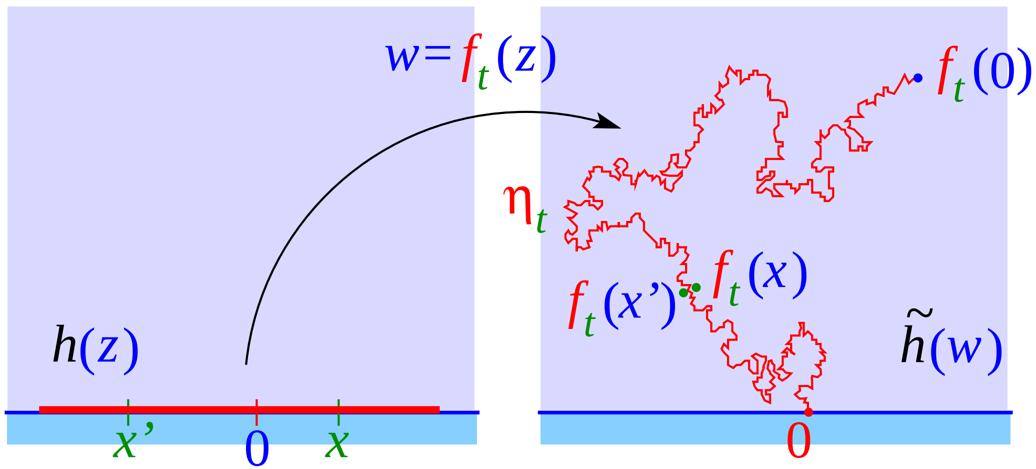
$$\begin{aligned} d\mathcal{A} &:= d^2 z\, \mathbb{E}[e^{\gamma h(z)} \,|\, f_t] = d^2 w\, \mathbb{E}[e^{\gamma h(w)}] \\ &= d^2 w\, |w|^{2-\kappa/2} (\sin\varphi)^{-\kappa/2}, \kappa \leqslant 4 \\ &= d^2 w (\sin\varphi)^{-8/\kappa}, \kappa \geqslant 4 \\ &\quad (\varphi := \arg w) \end{aligned}$$

Conformal Welding



Conformal welding: the *quantum boundary lengths* of any pair of real segments $[0, x]$ and $[x', 0]$ such that $f_t(x) = f_t(x')$ on the SLE trace are *a.s. equal* for $h = \tilde{h} + \mathfrak{h}_0$ *[Sheffield, 2010].*

Liouville Quantum Gravity & SLE



$$h|f_t \overset{\text{law}}{=} \tilde{h}\circ f_t + \mathfrak{h}_t$$

- *Conformally welding two γ-Liouville quantum boundaries yields SLE_κ for*

$$\gamma = \sqrt{\kappa} \wedge (4/\sqrt{\kappa}) = \left(\sqrt{25-c} - \sqrt{1-c}\right)/\sqrt{6} < 2 \quad \text{(KPZ II)}$$

- *Exponential martingales yield SLE quantum measures:*

$$\mathbb{E}\left[h|f_t\right] = \mathfrak{h}_t, \quad \mathbb{E}\left(e^{\alpha h}|f_t\right) = \exp\left[\alpha\mathfrak{h}_t - (\alpha^2/2)\langle\mathfrak{h}_t, \mathfrak{h}_t\rangle\right]$$

[D. & Sheffield, PRL ***107****, 131305 (2011)]*

SLE Exponential Martingales & KPZ

$$\mathcal{M}_t^{\alpha}(z) := \mathbb{E}\left(e^{\alpha h(z)} \,|\, f_t\right), \quad \alpha \in \mathbb{R}$$

$$\left(e^{\alpha h(z)} \,|\, f_t\right) d^2 z \overset{\text{(law)}}{=} \left|f_t'(z)\right|^{d-2} e^{\alpha h(w)} d^2 w$$

$$d := \alpha Q - \alpha^2/2 \quad \text{(KPZ)}$$

where $w = f_t(z)$, $d^2 w = |f_t'(z)|^2 d^2 z$.

SLE Exponential Martingales

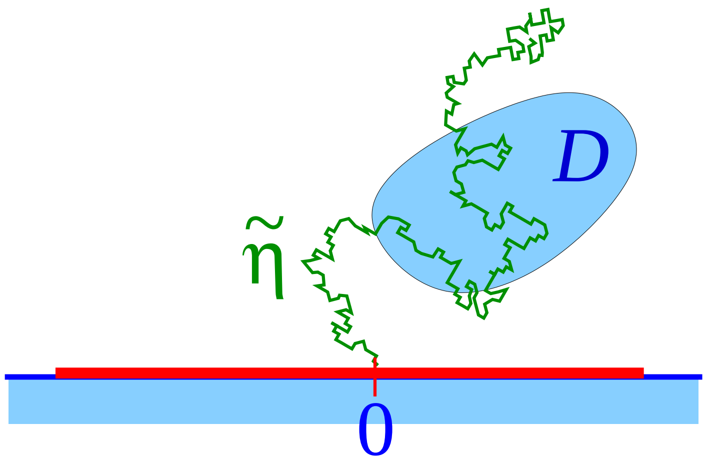
- Conditional expectation w.r.t. GFF h: $\mathbb{E}\left[h(z)\,|\,f_t\right] = \mathfrak{h}_t(z)$.
- Conditional expectations of exponentials:

$$\begin{aligned}
\mathcal{M}_t^{\alpha}(z) &:= \mathbb{E}\left(e^{\alpha h(z)}\,|\,f_t\right), \;\; \alpha \in \mathbb{R} \\
&= \exp\left[\alpha \mathfrak{h}_t(z) - (\alpha^2/2) C_t(z)\right] \\
&= \left|f_t'(z)\right|^d |w|^{2\alpha/\sqrt{\kappa}} (\Im w)^{-\alpha^2/2}; \; d := \alpha Q - \alpha^2/2 \\
C_t(z) &:= \langle \mathfrak{h}_t(z), \mathfrak{h}_t(z) \rangle = \log[\Im f_t(z) | f_t'(z)||]
\end{aligned}$$

where $w = f_t(z)$; $\mathcal{M}_t^{\alpha}(z)$ *is an exponential martingale with respect to the Brownian motion driving the SLE process:*

$$\mathbb{E}\mathcal{M}_t^{\alpha}(z) = \mathcal{M}_0^{\alpha}(z) = |z|^{2\alpha/\sqrt{\kappa}} (\Im z)^{-\alpha^2/2}.$$

SLE Natural Length



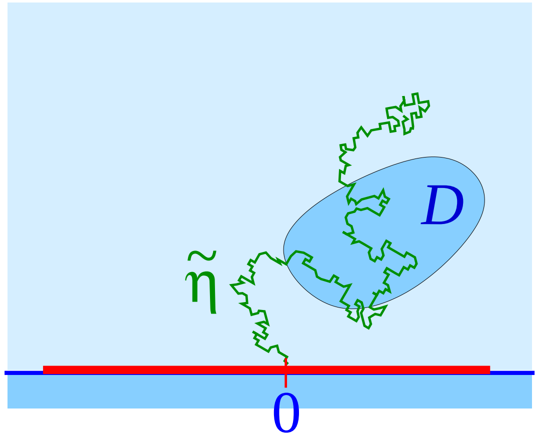
Expected (w.r.t. the $\text{SLE}_{\kappa\in[0,8)}$ law) *length* of an infinite SLE $\tilde{\eta}$ in D *(Lawler & [Sheffield, 2009; Zhou, 2010; Rezaei, 2012])*

$$\nu(D) = \int_D G(z)d^2z,$$

SLE Green's function in $\mathbb{H}$:

$$G(z) := |z|^a|\Im z|^b,\ a = 1-8/\kappa,\ b = 8/\kappa+\kappa/8-2.$$

SLE Quantum Length



$$h = \tilde{h} + \mathfrak{h}_0$$

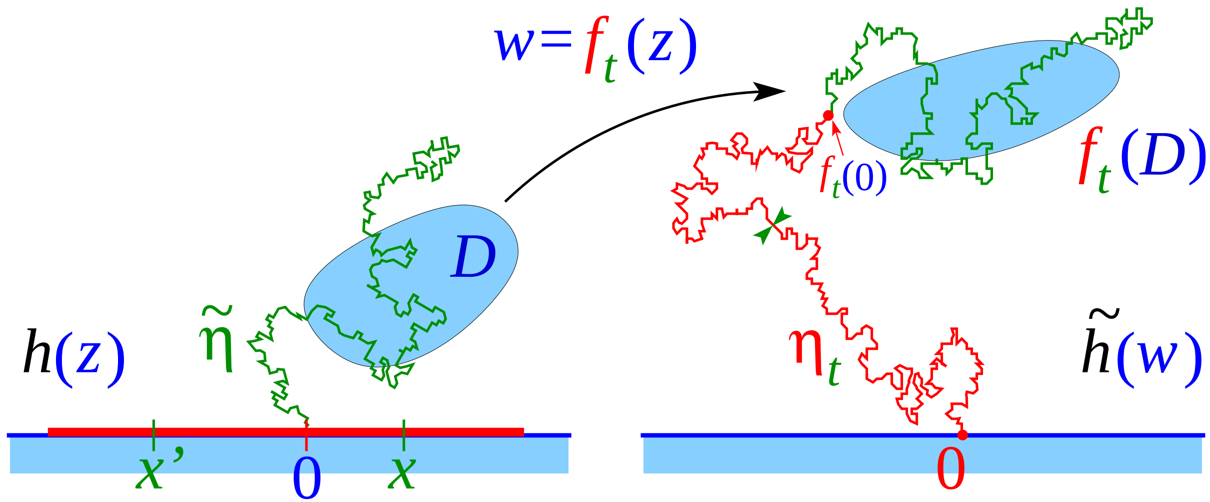
Expected (w.r.t. $\tilde{\eta}$, given h) Liouville *quantum length* ν_Q in D

$$\nu_Q(D,h) := \int_D e^{\alpha h(z)} G(z) d^2 z,$$

$\alpha = \sqrt{\kappa}/2$ $(= \gamma/2$ for $\kappa < 4)$ satisfies KPZ for the SLE Hausdorff dimension $d = 1 + \kappa/8$.

[Doob-Meyer, second moment method.]

Expected SLE Quantum Length



$$h \,|\, f_t \overset{\text{law}}{=} \tilde{h} \circ f_t + \mathfrak{h}_t$$

$$\mathbb{E}[\nu_Q(D,h) \,|\, f_t] = \int_D \mathcal{M}_t^{\alpha}(z) G(z) d^2 z$$

$$\mathbb{E}\nu_Q(D,h) = \int_D \mathcal{M}_0^{\alpha}(z) G(z) d^2 z = \int_D (\sin\vartheta)^{8/\kappa - 2} d^2 z,$$

with $\vartheta := \arg z$. It is finite for $\kappa \in [0,8)$ and coincides with the *Euclidean area* of D for $\kappa = 4$.

Perspectives

- *Scaling limits of discrete models on random planar maps*
- *Quantum wedges and cones*
- *Quantum bubbles and foam* ($\gamma\gamma' = 4$ *duality*)
- *Geodesics & random metrics*

$\diamondsuit \quad \diamondsuit \quad \diamondsuit$