Sharp Benefit-to-Cost Rules for the Evolution of Cooperation on Regular Graphs

Yu-Ting Chen

Centre de Recherches Mathématiques
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Background

Evolution of cooperation

LINK?

Nature ⟷ **Theory**

Examples:
– Lions hunting,
– meerkat sentry,
– ant societies.

Perspective:
Evolutionary game theory

Well-mixed populations:
– Tractable but unrealistic,
– (may) predict defection.

References.

[1] Nowak, M. A. (2006). *Evolutionary dynamics*. Harvard University Press.

[2] Nowak, M. A., Tarnita, C. E., and Antal, T. (2010). Evolutionary dynamics in structured populations. *Phil. Trans. R. Soc. B*

Main results

Model

- Ohtsuki, Hauert, Lieberman, and Nowak [*Nature* **441** (2006)].
- Finite structured populations.
- Simple rules discovered by non-rigorous methods.

Main result

Rigorous proof (very different argument, stronger conclusion).

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Point of view (Voter model perturbations)

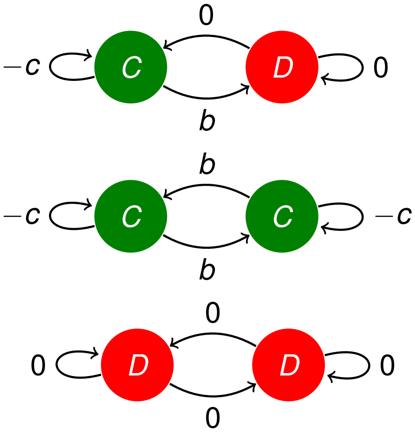
Cox, Durrett, and Perkins [*Astérisque* **349** (2013)] (on $\mathbb{Z}^d$ for $d \geq 3$).

Key Observation: The models by Ohtsuki et al. are voter model perturbations.

Machinery: for voter model perturbations on **finite** graphs.

Setup

- Cooperators (C), defectors (D).
- Benefit (b), cost (c).



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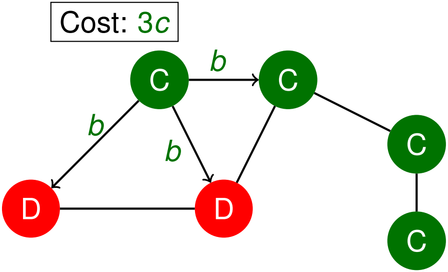

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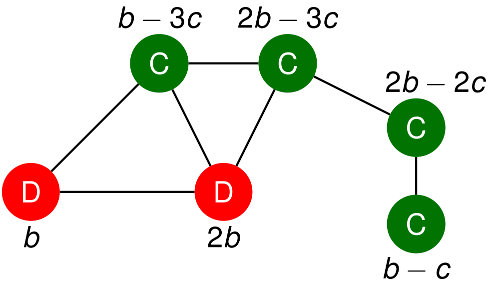

$$\textbf{fitness} = (1 - w) \times 1 + w \times \text{payoff}$$

w: intensity of selection (small).

Death-birth updating

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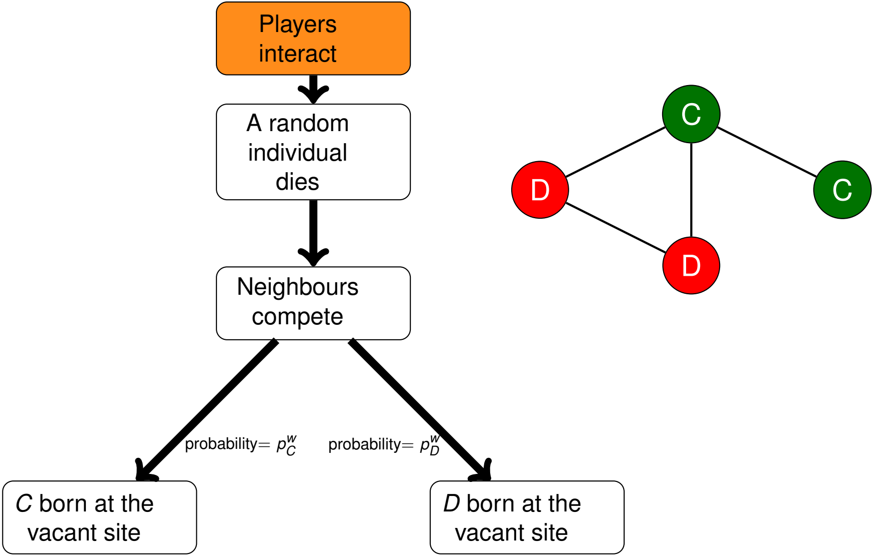

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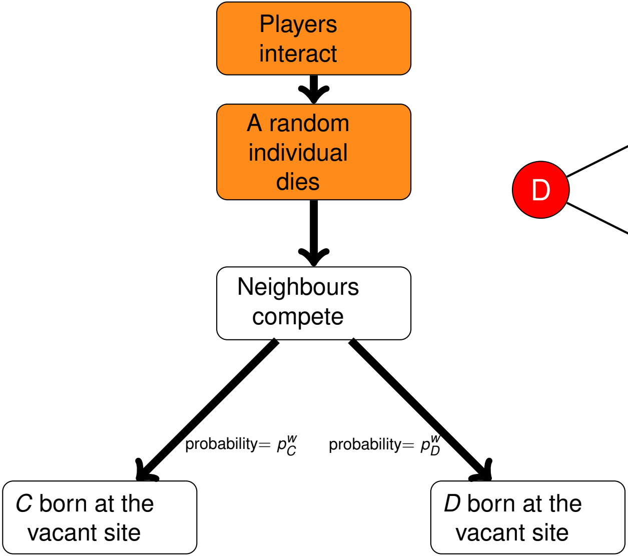

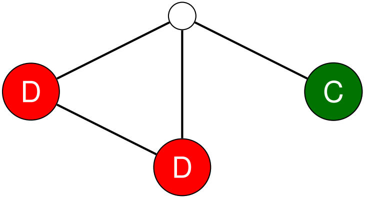

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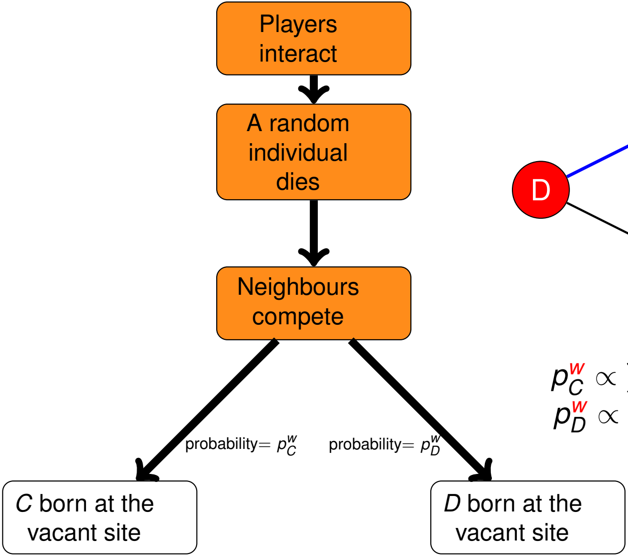

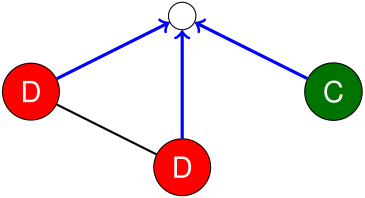

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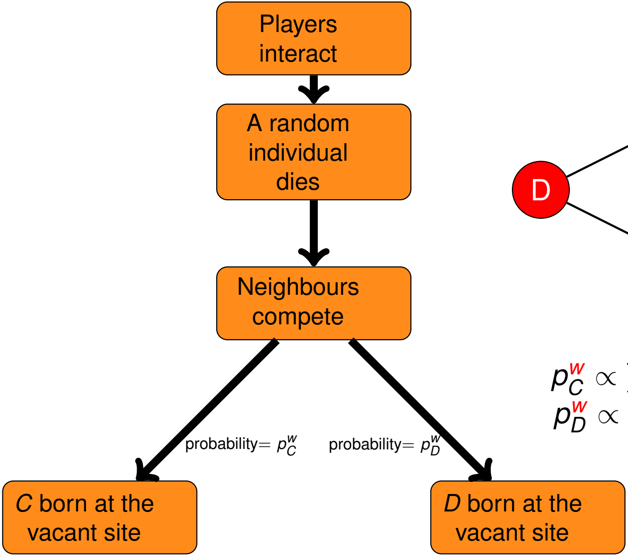

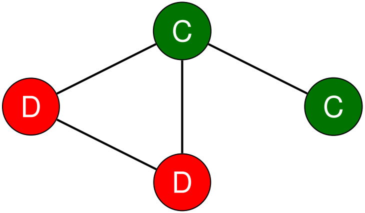

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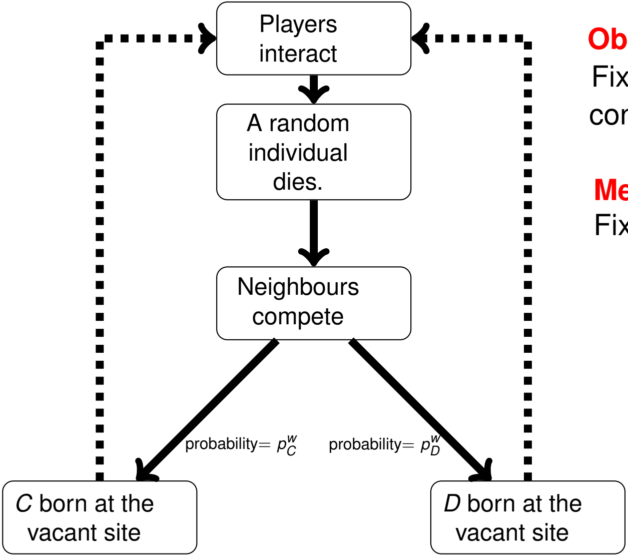

Observation:
Fixation in finite connected networks.

Measurement:
Fixation probabilities.

The (b, c, k)-rule

Discovery (Ohtsuki et al. (2006))

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Original non-rigorous proof:

1. A variety of finite graphs.
2. Pair approximation, diffusion approximation.

Motivating methods

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$$p^w(x,\eta) = \underbrace{p^0(x,\eta)}_{\text{voter model}} + w h_{1-\eta(x)}(x,\eta) + w^2 g_w(x,\eta).$$

(a) Condition for nontrivial "equilibria": $\frac{\partial u}{\partial t} = \frac{\sigma^2 \Delta}{2} u + \boldsymbol{f(u)}$.
(b) Reaction function $\boldsymbol{f(u)}$ involves

$$D(x,\eta) = \begin{cases} h_1(x,\eta), & \eta(x) = 0, \\ -h_0(x,\eta), & \eta(x) = 1. \end{cases}$$

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3. **Mayer and Montroll (1941)** Expansion of Boltzmann factors:

$$\prod_i e^{-\beta u_i} = 1 + \sum_i \left(e^{-\beta u_i} - 1\right) + \text{higher order terms}$$

by $\quad e^{-\beta u_i} = 1 + (e^{-\beta u_i} - 1) = \text{neutral model} + \text{perturbation}.$

Main result – voter model perturbations

For VMP's on **finite graphs** with flip rates:

$$p^w(x,\eta) = p^0(x,\eta) + w h_{1-\eta(x)}(x,\eta) + w^2 g_w(x,\eta)$$

and absorbing states $\{\text{all } C\}$ and $\{\text{all } D\}$ (subject to mild conditions), we have

$$\begin{aligned} \mathbb{P}^w(C\text{'s fixate}) =& \mathbb{P}^0(C\text{'s fixate}) \\ & + w\, \mathbb{E}^{\mathbb{P}^0} \int_0^\infty \overline{D}(\xi_s)ds + \mathcal{O}(w^2), \quad \text{as } w \longrightarrow 0+. \end{aligned}$$

Here, $\mathbb{P}^w$ and $\mathbb{P}^0$ are subject to the same initial condition.

Starting point for proof:

(a) $P^w = P^0 + K^w =$ voter model $+$ perturbation.

(b) Apply "Meyer's expansion" to n-step transition probabilities:

$$\begin{aligned} P^w(\eta_0,\eta_1)\cdots P^w(\eta_{n-1},\eta_n) =& (P^0+K^w)(\eta_0,\eta_1)\cdots(P^0+K^w)(\eta_{n-1},\eta_n) \\ =& \text{voter model} + \text{1st order} + \text{higher order}. \end{aligned}$$

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How to compute coefficients:

(a) Fixation probabilities under voter model: exact solutions.

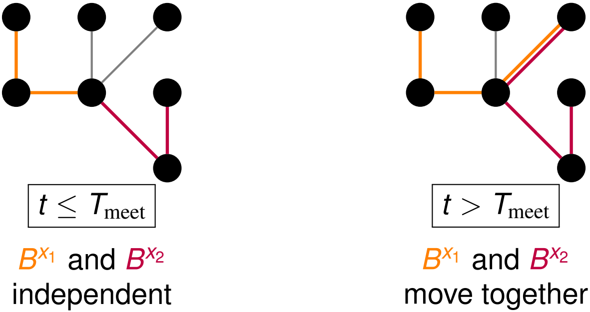
(b) Potential term: (usually) linear combinations of coalescing times for RW's by **duality**.

Duality (randomness-tranferring)

Theorem (Continuous-time setting)

1. (ξ_t) *is a* **voter model** *with initial configuration* ξ.
2. $(B^{x_1}, \cdots, B^{x_m})$ *is a system of* **coalescing random walks** *with* B^{x_i} *started at site* x_i.

Then $\mathbb{E}\left[\xi_t(x_1)\cdots\xi_t(x_m)\right] = \mathbb{E}\left[\xi(B_t^{x_1})\cdots\xi(B_t^{x_m})\right].$



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Theorem

$$\mathbb{P}^{w}(n \text{ random } C\text{'s fixate}) = \mathbb{P}^{0}(n \text{ random } C\text{'s fixate}) + w\left[\frac{kn(N-n)}{2N(N-1)}\right] \times \left[\left(\frac{b}{k}-c\right)(N-2) + b\left(\frac{2}{k}-2\right)\right] + \mathcal{O}(w^2),$$

*as $w \longrightarrow 0+$, whenever the graph is k-***regular***.*
(N is the population size.)

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Stronger conclusion

Fix degree k and (b, c). Then

the (b, c, k)-rule holds for n random C's

on any large k-regular graph and $n \in \{1, \cdots, N-1\}$, if $w \ll 1$.

THANK YOU!