Common access resource games with asymmetric players

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Outline of the presentation

- Motivation for heterogeneous discounting (and non-constant discounting)
- An exhaustible resource model under common access
  - The case of two-asymmetric players
  - The case of N-asymmetric players
- References

- Economic agents usually choose between profits distributed over time. To make them comparable one must discount these payments at a reference moment and the theory of the discounted utility provides one framework for evaluating such delayed payoffs.
- Preferences are time consistent if, and only if, discount functions are exponentials with a constant instantaneous time preference rate (Strotz (1956)).

$$U_t = \int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\delta(T-t)} F(x(T))$$

- However, there are some situations that can not be captured by standard discounting: impatient agents for short-run decisions (hyperbolic preferences / non-constant discounting) or situations in which the relative valuation of final function increases o decreases as we approach to the end of the planning horizon.
- This last case could be, for instance, when we want to model preferences about pensions plans or the legacy that an individual will leave to her/his descendants. Here, it could be the case that a decision maker will give more importance to her/his pension plan as she/he approaches the retirement date.

Heterogeneous (constant) discount rates

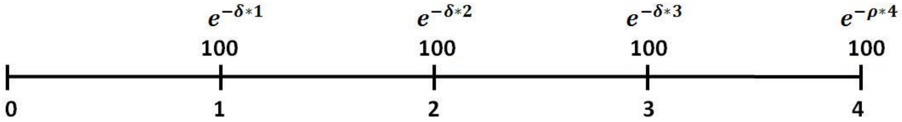
The preferences of the agent at time t take the form

$$U_t = \int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T)) \ ,$$

In this case, if $\rho > \delta$, the relative valuation of the final function $F(x(T))$ increases as we are getting closer to T.

Example

Imagine an agent who wants to value at $t = 0$ the following payments distributed along the horizon $[0, 4]$.



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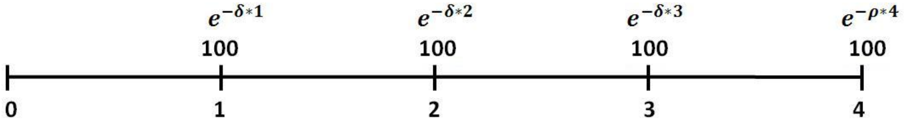
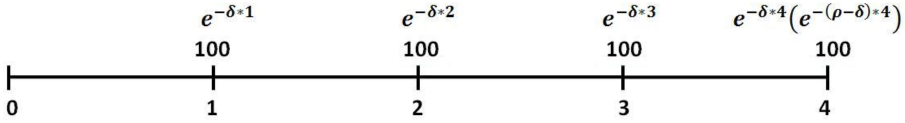
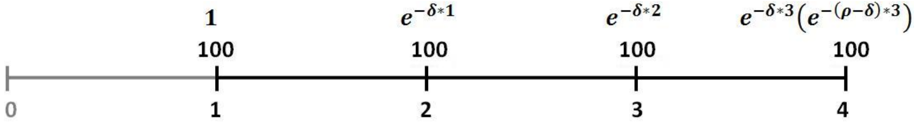


Note that we can rewrite the last discount factor term
$e^{-\rho*4} = e^{-\delta*4}(e^{-(\rho-\delta)*4})$



Example

At $t = 1$



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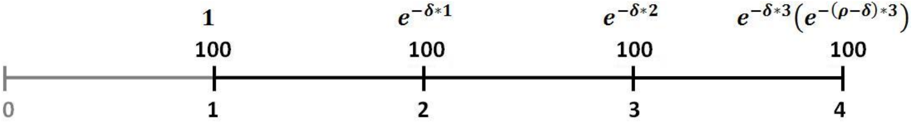
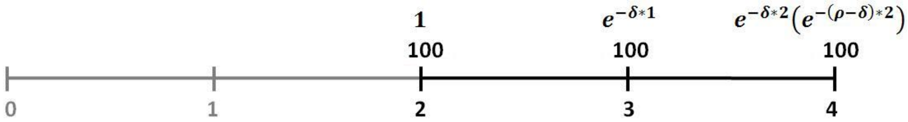


At $t = 2$



Example

At $t = 3$

Therefore, since we have assumed that $\rho > \delta$:

$$e^{-(\rho-\delta)*4} < e^{-(\rho-\delta)*3} < e^{-(\rho-\delta)*2} < e^{-(\rho-\delta)*1}$$

the relative value of the final function increases over time.

Non-constant discounting: the deterministic case

The objective of an agent at time t (*the t-agent*) is:

$$\max_{\{u(s)\}} \int_t^T \theta(s-t) L(x(s), u(s), s)\, ds + \theta(T-t) F(x(T)) \ ,$$

$$\dot{x} = f(x, u, s), \quad x(t) = x_t.$$

Exponential function with a non-increasing instantaneous discount rate $r(s)$

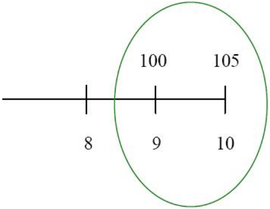
$$\theta(s-t) = e^{-\int_t^s r(\tau - t) d\tau} \left(\neq \bar{\theta}(s,t) = e^{-\int_t^s r(\tau) d\tau} \right)$$

Example

Suppose that you have to choose at $t = 0$ between two sets of two capitals placed over a time horizon $[0, 10]$, the first set is placed in the short-run (Set 1) and the second in the long-run (Set 2):

If you are more impatient for decisions in the short-run than in the long-run, then you can prefer (now) 100 EUR from Set 1 but 105 EUR from Set 2. In that case, your optimal current decision at $t = 0$ will be to choose 100 EUR at $t = 1$ and 105 EUR at $t = 10$.

But imagine that time goes by (for instance, we are now at $t = 8$) and you are offered the possibility to decide again. Now, you will prefer to receive the 100 EUR at $t = 9$ rather that 105 EUR at $t = 10$ (since now you are at evaluating both payments from moment $t = 8$) rather than to wait a year and receive your original decision.



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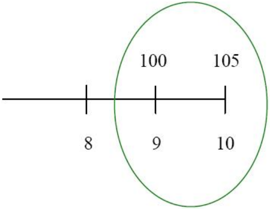

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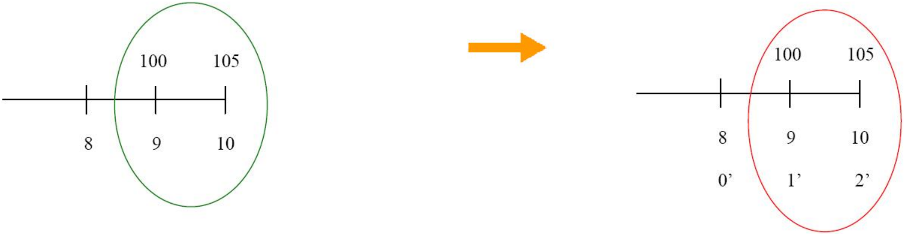


Thus, our original plan is time inconsistent since, if we can re-optimize at later dates from the beginning, we will want to change our past decisions.

The relevant effect of heterogeneous discounting (and also non-constant discounting) is that preferences change with time (in a similar way than with non-constant discounting). In this sense, an agent making a decision in time t has different preferences compared with those in time t'. Therefore, we can consider an agent who decides at different times as different agents. An agent making decisions in time t is usually called the t-agent.

Deterministic case with heterogeneous constant discount rates

The objective of the agent at time t (*t-agent*) is:

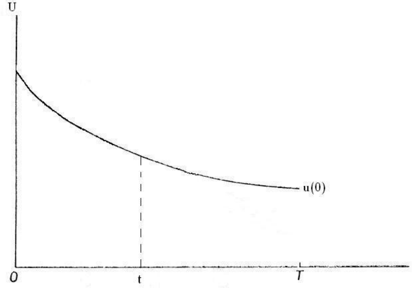
$$\max_{\{u(s)\}} \int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T)) \,,$$

$$\dot{x} = f(x, u, s), \quad x(t) = x_t.$$

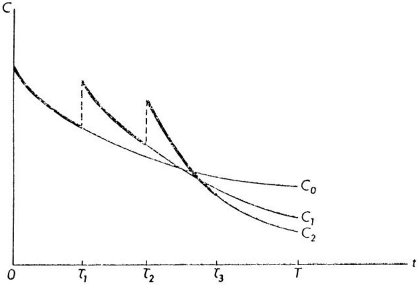
Agents' strategies:

- **Commitment solution**
- **Naive solution**
- **Sophisticated solution**

Commitment solution: The decision maker commits himself not to change the decisions initially taken and solves a standard optimal control problem over the horizon planning $[0, T]$.



Naive solution: The decision maker takes decisions without taking into account that his preferences will change in the near future. Then, naive t-agent solves a standard optimal control problem over the horizon planning $[t, T]$, but at $t + \epsilon$ he will change his decision rule by solving again a standard optimal control problem over $[t + \epsilon, T]$.



Sophisticated solution: The agent recognizes that he is unable to precommit his future behavior beyond the next instant and adopts a strategy of consistent planning by restricting his present behavior to his optimal future behavior.

Then, in order to obtain a time consistent strategy, we must derive the corresponding Dynamic Programming Equation. This can be done in two (essentially equivalent) different ways:

1. To discretize the problem, find the corresponding DPE in discrete time, and take the formal continuous time limit. (Karp (JET 2007). Hyperbolic discouting).
2. To follow a variational approach (Ekeland and Pirvu (2008), Hyperbolic discounting, or Marín-Solano and Patxot (OCAM 2011), Heterogeneous discounting in a deterministic setting).

Dynamic programming equation (DPE):the deterministic case in continuous time

Let $V(x,t)$ be the value function for the sophisticated t-agent and assume that it is continuously differentiable in (x,t). Then $V(x,t)$ satisfies the dynamic programming equation:

$$\rho V(x,t) + K(x,t) - V_t(x,t) =$$

$$= \max_{\{u\}} \{L(x,u,t) + V_x(x,t) f(x,u,t)\} \ ,$$

$$V(x,T) = F(x) \ , \tag{1}$$

where

$$K(x,t) = \int_t^T e^{-\delta(s-t)} \left[\delta - \rho\right] L^*(x,s) ds \ .$$

and $L^*(x,s) = L(x, u^*(x,s), s)$.

Finally, we can now differentiate $K(x,t)$ with respect to t, to obtain the "auxiliary DPE"

$$\delta K(x,t) - \nabla_t K(x,t) = (\delta - \rho) U(x, \phi(x,t), t)$$

$$+ \nabla_x K(x,t) \cdot f(x, \phi(x,t), t)$$

together with

$$F(x,T) = 0, \tag{2}$$

so that, we can characterize the time consistent solution as the solution of the system of PDE (1)-(2).

Cooperative solutions in a two-player differential game

$$J_C\left(u(\cdot)\right) = \int_t^T e^{-r_1(s-t)} L_1\left(x(s), u_1(s), u_2(s), s\right) ds + \int_t^T e^{-r_2(s-t)} L_2\left(x(s), u_1(s), u_2(s), s\right) ds$$

subject to:

$$\dot{x}(s) = f(x(s), u_1(s), u_2(s), s), \qquad x(t) = x_t.$$

- **Case 1:** $L_1 = L_2 = L$ and $r_1 = r_2 = r$
- **Case 2:** $L_1 \neq L_2$ and $r_1 = r_2 = r$
- **Case 3:** $L_1 = L_2 = L$ and $r_1 \neq r_2$
- **Case 4:** $L_1 \neq L_2$ and $r_1 \neq r_2$

Cases 1 and 2 can be solved by means of PMP or HJB.
Case 3 can be solved as a model with non-constant discounting (Karp (2007) or Marín-Solano and Navas (2009)). Note that in this case:

$$J_C\left(u(\cdot)\right) = \int_t^T \theta(s-t) L\left(x(s), u_1(s), u_2(s), s\right) ds$$

where

$$\theta(s-t) = e^{-r_1(s-t)} + e^{-r_2(s-t)} = e^{-\int_t^s \bar{r}(\tau - t) d\tau}$$

where the instantaneous time preference rate $\bar{r}$ is a (non-constant) function of its argument:

$$\bar{r}(\tau) = -\frac{\theta'(\tau)}{\theta(\tau)} = \frac{r_1 e^{-r_1\tau} + r_2 e^{-r_2\tau}}{e^{-r_1\tau} + e^{-r_2\tau}}$$

Finally, **Case 4** can be transformed into a problem with non-homogeneous discounting:

All optimal control problem can be stated in three different (but equivalent) ways: Functional objective given

1. integral form (Lagrange problem)
2. integral and terminal value term (Bolza problem)
3. only terminal value term (Mayer problem)

Then, the problem

$$\max J_C\left(u(\cdot)\right) = \max \int_t^T e^{-r_1(s-t)} L_1\left(x(s), u_1(s), u_2(s), s\right) ds + \int_t^T e^{-r_2(s-t)} L_2\left(x(s), u_1(s), u_2(s), s\right) ds$$

subject to:

$$\dot{x}(s) = f(x(s), u_1(s), u_2(s), s), \qquad x(t) = x_t.$$

can be transformed into

$$\max J_C(u(\cdot)) = \max \int_t^T e^{-r_1(s-t)} L_1(x(s), u_1(s), u_2(s), s)\, ds + e^{-r_2(T-t)} Y(T)$$

subject to:

$$\dot{x}(s) = f(x(s), u_1(s), u_2(s), s), \qquad x(t) = x_t.$$
$$\dot{Y}(s) = r_2 Y(s) + L_2(x(s), u_1(s), u_2(s), s),$$

i.e., we have rewritten the functional objective for one of the players in the Mayer form, and therefore, transformed the cooperative problem into a problem with integral and terminal value term, but with different time preferences rates.

An exhaustible resource model under common access: the case of two-asymmetric players

Consider the following model of a common property non-renewable resource extraction where the objective for the coalition is to maximize

$$\int_0^T \ln\left(c_1(s)\right) e^{-r_1 s} ds + \int_0^T \ln\left(c_2(s)\right) e^{-r_2 s} ds$$

subject to

$$\dot{x}(t) = -c_1(t) - c_2(t), \quad x(0) = x_0, \quad x(T) = 0.$$

Precommitment (at $t = 0$) solution (PMP):

$$c_m^0(s) = \frac{e^{-r_m s}}{\sum_{i=1}^{2} \frac{1-e^{-r_i T}}{r_i}} x_0 = \frac{e^{-r_m s}}{\sum_{i=1}^{2} \frac{e^{-r_i s}-e^{-r_i T}}{r_i}} x_s \ ,$$

The precommitment solution, which is optimal from the viewpoint of the 0-coalition (we can associate it with the existence of some binding agreement), is not longer optimal if players in the coalition can recalculate the cooperative solution at some instant $t \in (0, T]$. Note that the maximum of

$$\int_t^T \ln(c_1(s))\, e^{-r_1(s-t)} ds + \int_t^T \ln(c_2(s))\, e^{-r_2(s-t)} ds,$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s), \quad x(t) = x_t, \quad x(T) = 0$$

is given by

$$c_m^t(s) = \frac{e^{-r_m(s-t)}}{\sum_{i=1}^2 \frac{1-e^{-r_i(T-t)}}{r_i}} x_t \ , \quad s \in [t, T] \ .$$

This solution differs from that calculated at $t = 0$. For instance, $c_1^t(t) = c_2^t(t)$, whereas $c_1^0(t) \neq c_2^0(t)$ for every $t > 0$.

In general, if players in the coalition can continuously re-calculate the "cooperative" solution, they will follow what we call the (time inconsistent) **naive** decision rule $c_m^N(t)$. In this case $c_m^t(s)$ in is followed only at the time $s = t$ at which the agents of the t-coalition have calculated the extraction rate, so that the actual extraction rate becomes

$$c_m^N(t) = c_m^t(t) = \frac{1}{\sum_{i=1}^{2} \frac{1-e^{-r_i(T-t)}}{r_i}} x_t.$$

- Note that the precommitment and naive solutions do not coincide unless $r_1 = r_2$. In fact, $c_1^P(t) \neq c_2^P(t)$, for every $t \in (0, T]$, whereas $c_1^N(t) = c_2^N(t)$ for every $t \in [0, T]$.

How to obtain a time-consistent solution?

In order to determine a time-consistent equilibrium, we first reformulate our problem by rewriting the payoff of player 2 in the Mayer form. The objective functional becomes now

$$\int_t^T e^{-r_1(s-t)} \ln\left(c_1(s)\right) ds + e^{-r_2(T-t)} y(T)$$

subject to

$$\dot{x}(s) = -c_1(s) - c_2(s) \ , \quad \dot{y}(s) = r_2 y(s) + \ln\left(c_2(s)\right)$$

with $x(T) = 0$.

We can now use the DPE introduced above:

$$r_2 W(x,y,t) + K(x,y,t) - W_t(x,y,t)$$
$$= \max_{\{c_1,c_2\}} \left\{ \ln c_1 + W_x(x,y,t)(-c_1 - c_2) + W_y(x,y,t)\,(r_2 y + \ln(c_2)) \right\},$$

where $K(x,y,t) = (r_1 - r_2) \int_t^T e^{-r_1(s-t)} \ln(c_1^*, s) ds$.

For this particular problem, the solution obtained for the naive coalition is a time-consistent policy. This feature, also arising in non-constant discounting models (see Pollak (1968) and Marín-Solano and Navas (2009)), is a consequence of using logarithmic utility functions, and it no longer holds when more general utility functions are considered.

An exhaustible resource model under common access: the case of N-asymmetric players

Next, we extend the two-player case analyzed above. Then, consider the case of N players who decide to form a coalition seeking for a time-consistent solution maximizing

$$J\left(c(\cdot)\right) = \sum_{m=1}^{N} \lambda_m \int_t^T e^{-r_m(s-t)} U^m\left(x(s), c(s), s\right) ds \tag{3}$$

subject to

$$\dot{x}(s) = f(x(s), c(s), s), \qquad x(t) = x_t \ . \tag{4}$$

Note that in this case we cannot use the above approach of transforming the Lagrange problem into a Bolza problem!

In order to solve the N player case, we discretize (3-4):

$$\max_{\{c_1,\dots,c_n\}} V_j = \sum_{m=1}^{N} V_j^m = \sum_{i=0}^{n-j-1} \sum_{m=1}^{N} \lambda_m e^{-r_m(i\epsilon)} U^m(x_{(i+j)}, c_{(i+j)}, (i+j)\epsilon)\epsilon$$

subject to $x_{i+1} = x_i + f(x_i, c_i, i\epsilon)\epsilon$, $i = j, \dots, n-1$, x_j given .
In this case, the can obtain the following dynamic programming algorithm:

$$V_j^*(x_j, j\epsilon) = \max_{\{c_j\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x_j, c_j, j\epsilon)\epsilon \right.$$
$$+ \sum_{k=1}^{n-j-1} \sum_{m=1}^{N} \lambda_m \left(1 - e^{r_m \epsilon}\right) e^{-r_m k\epsilon} \bar{U}^m_{(j+k)}(x_{(j+k)}, (j+k)\epsilon)\epsilon$$
$$\left. + V^*_{(j+1)}(x_{(j+1)}, (j+1)\epsilon) \right\},$$

DPE for the N player case (II)

with $x_{(j+1)} = x_j + f(x_j, c_j, j\epsilon)\epsilon, \quad , j = 0, \ldots, n-1$, and $V_n^* = 0$.

We then define the value function for Problem 3-4 as the solution to the DPE obtained by taking the formal continuous time limit when $\epsilon \to 0$ of the DPE obtained from the discrete approximation to the problem, assuming that the limit exists and that the solution is of class C^1 in all their arguments. Proceeding in this way, it can be easily proved that:

If $W^m(x,t)$, $m = 1, \ldots, N$, is a set of continuously differentiable functions in (x,t) satisfying the DPE

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \nabla_t W^m(x,t) = \max_{\{c\}} \left\{ \sum_{m=1}^{N} \lambda_m U^m(x,c,t) + \sum_{m=1}^{N} \nabla_x W^m(x,t) \cdot f(x,c,t) \right\} \quad (5)$$

with $W^m(x,T) = 0$, for every $m = 1, \ldots, N$, and

$$W^m(x,t) = \lambda_m \int_t^T e^{-r_m(s-t)} U(x(s), \phi(x(s),s), s)\, ds\ , \quad (6)$$

where, $c^*(t) = \phi(x(t),t)$ is the maximizer of the right hand term in Equation (5), then $W(x,t) = \sum_{m=1}^{N} W^m(x,t)$ is the value function of the whole coalition, the decision rule $c^* = \phi(x,t)$ is the (time-consistent) MPE, and $W^m(x,t)$, for $m = 1, \ldots, N$, is the value function of player m in the cooperative problem (3-4).

Remark

Note that, throughout the equilibrium rule $c^* = \phi(x,t)$, for every player m, $W^m(x,t)$ is a solution to the partial differential equation

$$r_m W^m(x,t) - \nabla_t W^m(x,t)$$

$$= \lambda_m U^m(x, \phi(x,t), t) + \nabla_x W^m(x,t) \cdot f(x, \phi(x,t), t) \ , \qquad (7)$$

for $m = 1, \ldots, N$, with $W^m(x,T) = 0$. Hence, we can compute the value function by first determining the decision rule solving the right hand term in Eq. (5) as a function of $\nabla_x W^m(x,t)$, $m = 1, \ldots, N$, and then substituting the decision rule into the system of N partial differential equations (7).

An exhaustible resource model under common access: the case of N-asymmetric players

Now we can extend the results for the non-renewable resource model in Section 2 to the general case of N asymmetric players. If $\lambda_1 = \cdots = \lambda_N = 1$, we must solve

$$\max_{\{c_1,\ldots,c_n\}} \sum_{m=1}^{N} \int_t^T e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m} - 1}{1-\sigma_m}\, ds \tag{8}$$

subject to

$$\dot{x}(s) = -\sum_{m=1}^{N} c_m(s), \qquad x(t) = x_t, \quad x(T) = 0\,. \tag{9}$$

Precommitment and naive solutions

For $m = 1, \ldots, N$, the precommitment and naive solutions are:

$$c_m^P(t) = \frac{e^{-\gamma_m t}}{\sum_{i=1}^{N} \frac{1}{\gamma_i}\left(e^{-\gamma_i t} - e^{-\gamma_i T}\right)} x_t$$

and

$$c_m^N(t) = \frac{1}{\sum_{i=1}^{N} \frac{1}{\gamma_i}\left(1 - e^{-\gamma_i (T-t)}\right)} x_t \ ,$$

respectively, where $\gamma_m = \frac{r_m}{\sigma_m}$.

- Note that in the naive case the extraction rates of all agents coincide.

Time consistent solutions

Now, we have to solve:

$$\sum_{m=1}^{N} r_m W^m(x,t) - \sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial t}$$
$$= \max_{c_1,\ldots,c_N} \left\{ \sum_{m=1}^{N} \frac{c_m(s)^{1-\sigma_m} - 1}{1-\sigma_m} + \left(\sum_{m=1}^{N} \frac{\partial W^m(x,t)}{\partial x} \right) \left(-\sum_{m=1}^{n} c_m(s) \right) \right\} .$$

where $c_m^S(t) = \left(\sum_{j=1}^{N} \frac{\partial W^j(x,t)}{\partial x} \right)^{-\frac{1}{\sigma_m}}$, for $m = 1, \ldots, N$.

- The extraction rates of agents m and m' coincide ($c_m^S = c_{m'}^S$) if, and only if, $\sigma_m = \sigma_{m'}$. Thus, if there are two players m and m' such that $\sigma_m \neq \sigma_{m'}$ (hence $c_m^S \neq c_{m'}^S$), the naive solution is always time-inconsistent.

In order to compute the actual decision rule we can solve the system of N coupled partial differential equations:

$$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t}$$
$$= \frac{1}{1-\sigma_m}\left[\left(\sum_{j=1}^{N} \frac{\partial W^j(x,t)}{\partial x}\right)^{\frac{\sigma_m - 1}{\sigma_m}} - 1\right]$$
$$- \frac{\partial W^m(x,t)}{\partial x} \sum_{j=1}^{N} \left(\sum_{i=1}^{N} \frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma_j}},$$

for $m = 1, \ldots, N$

In the particular case that $\sigma_1 = \cdots = \sigma_N = \sigma$, the above system simplifies to

$r_m W^m(x,t) - \frac{\partial W^m(x,t)}{\partial t} =$

$\frac{1}{1-\sigma}\left[\left(\sum_{j=1}^{N} \frac{\partial W^j(x,t)}{\partial x}\right)^{1-\frac{1}{\sigma}} - 1\right] - N \frac{\partial W^m(x,t)}{\partial x}\left(\sum_{i=1}^{N} \frac{\partial W^i(x,t)}{\partial x}\right)^{-\frac{1}{\sigma}},$

$m = 1, \ldots, N.$

By guessing $W^m(x,t) = A^m(t)\frac{x^{1-\sigma}-1}{1-\sigma} + B^m(t)$, $m = 1, \ldots, N$, with $A^m(t) > 0$ for every $t \in [0,T)$, and substituting in the system of DPE, we find that the functions $A^m(t)$ are the solution to the following system of ordinary differential equations

$$\dot{A}^m - r_m A^m = N(1-\sigma)A^m \left(\sum_{j=1}^{N} A^j\right)^{-\frac{1}{\sigma}} - \left(\sum_{j=1}^{N} A^j\right)^{1-\frac{1}{\sigma}},$$

for $j = 1, \ldots, N.$

Some results

- For the case of logarithmic utility functions ($\sigma = 1$), the naive solution also is time consistent in the case of N asymmetric players. On the contrary, if $\sigma \neq 1$, the naive solution is not longer time consistent.

- In the time-consistent solution, the extraction rates of two agents coincide if, and only if, they have the same marginal elasticity σ.

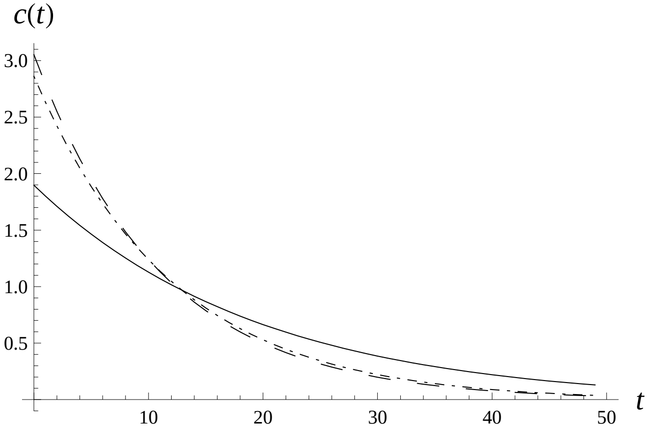
- If $U^m(c_m) = U(c_m)$, i.e., all the agents have the same utility function (in the isoelastic case, $\sigma_1 = \cdots = \sigma_N = \sigma$), along the equilibrium rule all players extract the resource at the same rate and problem becomes equivalent to the problem of a representative agent using the discount function $\sum_{m=1}^{N} e^{-r_m(s-t)}$ (this is the case of non-constant discounting!). On the contrary, if there two agents m and m' with different marginal utilities ($\sigma_m \neq \sigma_{m'}$), the problem cannot be simplified to a non-constant discounting problem.

Numerical illustration

We consider as a baseline case the following:

- $N = 3$
- Time preference rates: $r_1 = 0.03$, $r_2 = 0.06$ and $r_3 = 0.09$.
- Initial stock of the resource of $x_0 = 100$
- Time horizon from $t_0 = 0$ to $T = 50$ periods.
- Utilities from consumption are assumed to be of the iso-elastic type with equal intertemporal elasticity of substitution $(1/\sigma)$ for all three players in the coalition.

Next two Figures show the individual extraction rate for every agent in the coalition under the assumption of cooperation for the naive (dot dashed line) and the sophisticated solutions (dashed line), with $\sigma = 0.6$ (Figure 1) and $\sigma = 2$ (Figure 2). In both graphs, the solid line shows the extraction rate for logarithmic utilities.



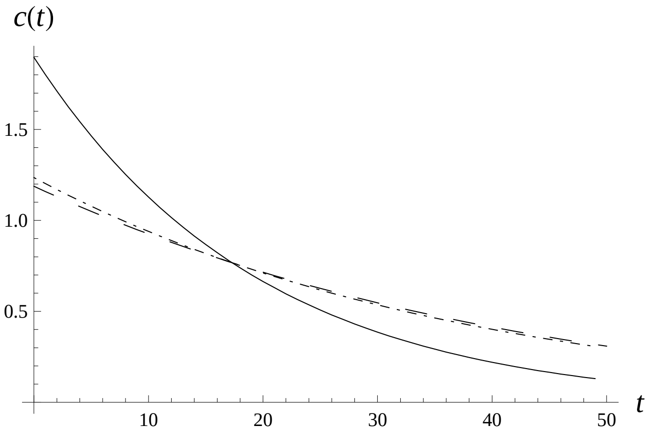


Figure: Extraction rates for naive and sophisticated agents ($\sigma = 2$) and logarithmic case.

Some comments

- Unless $\sigma = 1$ (logarithmic utilities), the time-consistent and naive solutions do not coincide, as expected. For $\sigma = 0.6$, the time-consistent agents' extraction rate is higher at initial periods compared with naive agents, this behavior being reversed for $\sigma = 2$.
- It can be observed that the equilibrium appears to be more sensitive to the value of σ than to the behavior (naive or time-consistent) of the t-coalitions. In addition, higher values of σ lead agents to smooth their extraction rate path along the time horizon.

Precommitment vs. sophisticated solution

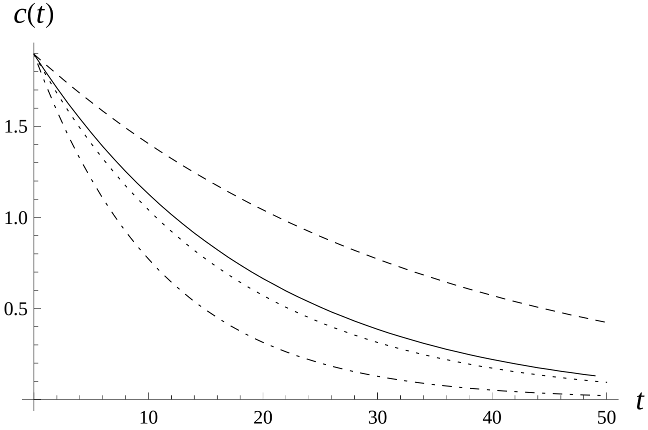


Figure: Extraction rates for sophisticated agents in the coalition (solid line) and individual extraction rates under precommitment at $t = 0$ (dashed, dotted and dot dashed lines correspond to players 1, 2 and 3, respectively). Logarithmic utility.

An extension: infinite planning horizon

Let's consider now the problem of

$$\max_{\{c_1,\ldots,c_n\}} \sum_{m=1}^{N} \int_t^{\infty} e^{-r_m(s-t)} \frac{(c_m(s))^{1-\sigma_m} - 1}{1-\sigma_m} \, ds \ ,$$

subject to

$$\dot{x}(s) = g(x) - \sum_{m=1}^{N} c_m(s) \ , \qquad x(t) = x_t \ ,$$

where $c_m(t)$ is the harvest rate of agent m, for $m = 1, \ldots, N$, and $g(x)$ is the natural growth function of the resource stock x.

In the case that both utility functions and state equation are autonomous, we concentrate in the case of state dependent value functions where the DPE is:

$$\sum_{m=1}^{N} r_m W^m = \max_{c_1,\dots,c_N} \left\{ \sum_{m=1}^{N} \frac{c_m^{1-\sigma_m} - 1}{1-\sigma_m} + \left(\sum_{j=1}^{N} W_x^j \right) \left(g(x) - \sum_{m=1}^{N} c_m \right) \right\} ,$$

hence

$$c_m^* = \phi_m(x) = \left(\sum_{j=1}^{N} W_x^j \right)^{-\frac{1}{\sigma_m}} .$$

Note that:

1. Therefore, $c_m^* = c_{m'}^*$ if, and only if, $\sigma_m = \sigma_{m'}$
2. In this model, in general, along the equilibrium rule, marginal utilities coincide, i.e., $U'(c_m^*) = U'(c_{m'}^*) = \sum_{j=1}^{N} W_x^j$, for all $m \neq m'$.

As in the finite horizon case, now we also have the set of DPEs

$$r_m W^m = \frac{(\phi_m(x))^{1-\sigma_m} - 1}{1-\sigma_m} + W_x^m \left(g(x) - \sum_{j=1}^{N} (\phi_j(x)) \right) ,$$

for all $m = 1, \ldots, N$, where $\phi_m(x)$ are given by the expression above.

Linear strategies I

Next, let us restrict our attention to the case of linear decision rules.

- Since $(c_i^*)^{-\sigma_i} = (c_j^*)^{-\sigma_j}$, for all $i, j = 1, \ldots, N$, if $c_m^* = \phi_m(x) = \alpha_m x$ then $(\alpha_i x)^{-\sigma_i} = (\alpha_j x)^{-\sigma_j}$. Therefore, no linear decision rules exist unless $\sigma_i = \sigma_j$, for all i, j.

- For $\sigma_i = \sigma_j = \sigma$, then $\alpha_i = \alpha_j$ and the DPE becomes $\sum_{m=1}^N r_m W^m = \frac{N}{1-\sigma}\left(\alpha^{1-\sigma} x^{1-\sigma} - 1\right) + \alpha^{-\sigma} x^{-\sigma}\left(g(x) - N\alpha x\right)$. This equation has a solution if $g(x) = ax$. In this case, we obtain

$$\sum_{m=1}^{N} r_m W^m(x) = \left[\frac{N\sigma}{1-\sigma}\alpha^{1-\sigma} + a\alpha^{-\sigma}\right] x^{1-\sigma} - \frac{N}{1-\sigma} ,$$

together with $\sum_{m=1}^N W_x^m(x) = \alpha^{-\sigma} x^{-\sigma}$.

Linear strategies I

If we try $W^m(x) = A^m \frac{x^{1-\sigma}-1}{1-\sigma} + B^m$, by simplifying we obtain that A^m, B^m and α are obtained by solving the equation system

$$[r_m - (1-\sigma)(a - N\alpha)]\, A^m = \alpha^{1-\sigma},$$

$$r_m A^m - (1-\sigma) r_m B^m = 1 \ \text{ and } \ \sum_{m=1}^{N} A^m = \alpha^{-\sigma} \; .$$

- In the case of logarithmic utilities (corresponding to the limit $\sigma = 1$), by trying $W^m(x) = A^m \ln x + B^m$, we can reproduce the calculations to obtain $A^m = \frac{1}{r_m}$ and $\alpha = \frac{1}{\sum_{m=1}^{N} \frac{1}{r_m}}$. If $r_1 = \cdots = r_N = r$ then

Main results:

1. In the infinite case problem, the extraction rates of two agents are equal if, and only if, they have the same marginal elasticity (equal σ). Note that agents with different discount rates harvest the resource at equal rates. This solution is different from that obtained in a noncooperative setting, or from that obtained by applying the PMP (the precommitment solution).

2. Since $c_i^{\sigma_i} = c_j^{\sigma_j}$, for every $i, j = 1, \ldots, N$, extraction / harvesting rates are higher for agents with a higher intertemporal elasticity of substitution (lower value of the parameter σ) when $c_i, c_j > 1$. This property is reversed when $c_i, c_j < 1$. Note that this property is independent on the use of different discount rates (although discount rates affect to the value of extraction / harvesting rates).

1. If there are two players with different marginal elasticities, no linear decision rules exist. This property is independent on the use of different discount rates. As a consequence, in the case of different marginal elasticities, it becomes very difficult to derive analytic solutions.

2. If the natural growth function is linear and all the agents have the same marginal elasticity σ, then the decision rules $c_m = \alpha x$ and the value functions $W^m(x) = A^m \frac{x^{1-\sigma}-1}{1-\sigma} + B^m$, $m = 1, \ldots, N$ solve our infinite time horizon problem.

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