Non-linear oscillations and long-term evolution

application to planetary systems and spin-orbit problem

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Based on research works in collaboration with
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Planetary Motions, Satellite Dynamics
and Spaceship Orbits
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Aim of the work

Problems

Stability of the Solar System (secular dynamics).
Secular dynamics of exoplanetary system.
Long-time stability around a Cassini state.

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Models considered

Sun-Jupiter-Saturn-Uranus (plane).
Systems with two coplanar planets.
Spin-orbit problem: Saturn-Titan.

A common point

The analytical form of the Hamiltonian is similar to that of a Hamiltonian in the neighbourhood of an elliptic equilibrium, namely

$$H(x, y) = \frac{1}{2} \sum_{l} \omega_l (x_l^2 + y_l^2) + H_1(x, y) + H_2(x, y) + \dots \ ,$$

where H_s is a homogeneous polynomial of degree $s + 2$.

This is a perturbed system of harmonic oscillators.

Birkhoff normal form

We look for a near the identity canonical change of coordinates such that the Hamiltonian is in Birkhoff normal form up to order r, namely

$$H^{(r)}(x, y) = H_0(I) + Z_1(I) + \ldots Z_r(I) + \mathcal{R}^{(r)}_{r+1}(x, y) + \ldots \ ,$$

where $I_l = \frac{1}{2}(x_l^2 + y_l^2)$ are the actions of the system, Z_s is a homogeneous polynomial of degree $(s + 2)/2$ in I and the terms $\mathcal{R}^{(r)}_s(x, y)$ are homogeneous polynomial of degree $s + 2$ in (x, y).

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At each step one has to solve the equation

$$\{\chi^{(r+1)}, \omega\cdot I\} + \mathcal{R}^{(r)}_{r+1}(x,y) = Z_{r+1}(I)\ ,$$

provided the non-resonance condition

$$k\cdot\omega \neq 0 \qquad \text{for } 0 < |k| \leq r+3\ .$$

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$$k \cdot \omega \neq 0 \qquad \text{for } 0 < |k| \leq r+3 \ .$$

Thus we can write the new Hamiltonian as $H^{(r+1)} = \exp L_{\chi^{(r+1)}} H^{(r)}$.

Sketch of the procedure

H_0

$\mathcal{R}_1^{(0)}$

$\mathcal{R}_2^{(0)}$

$\mathcal{R}_3^{(0)}$

$\ddots$

$$\{\chi^{(1)}, \omega \cdot I\} + \mathcal{R}_1^{(0)}(x, y) = Z_1(I)$$

Sketch of the procedure

$$
\begin{array}{llll}
H_0 & & & \\
L_{\chi^{(1)}} H_0 & \mathcal{R}_1^{(0)} & & \\
\frac{1}{2!} L^2_{\chi^{(1)}} H_0 & L_{\chi^{(1)}} \mathcal{R}_1^{(0)} & \mathcal{R}_2^{(0)} & \\
\frac{1}{3!} L^3_{\chi^{(1)}} H_0 & \frac{1}{2!} L^2_{\chi^{(1)}} \mathcal{R}_1^{(0)} & L_{\chi^{(1)}} \mathcal{R}_2^{(0)} & \mathcal{R}_3^{(0)} \\
 & & & \ddots
\end{array}
$$

$$H^{(1)} = \exp L_{\chi^{(1)}} \, H^{(0)}$$

Sketch of the procedure

H_0

Z_1

$\mathcal{R}_2^{(1)}$

$\mathcal{R}_3^{(1)}$

$\ddots$

$$\{\chi^{(2)}, \omega \cdot I\} + \mathcal{R}_2^{(1)}(x, y) = Z_2(I)$$

Sketch of the procedure

H_0

Z_1

$L_{\chi^{(2)}} H_0$ $\quad$ $\mathcal{R}_2^{(1)}$

$L_{\chi^{(2)}} Z^{(1)}$ $\quad$ $\mathcal{R}_3^{(1)}$

$\ddots$

$$H^{(2)} = \exp L_{\chi^{(2)}} \, H^{(1)}$$

Sketch of the procedure

H_0

Z_1

Z_2

$\mathcal{R}_3^{(2)}$

$\ddots$

$$\{\chi^{(2)}, \omega \cdot I\} + \mathcal{R}_3^{(2)}(x, y) = Z_3(I)$$

Sketch of the procedure

H_0

Z_1

Z_2

$L_{\chi^{(3)}} H_0$

$\mathcal{R}_3^{(2)}$

$\ddots$

$$H^{(3)} = \exp L_{\chi^{(3)}} \, H^{(2)}$$

Sketch of the procedure

H_0

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$\ddots$

Effective stability time

The Hamiltonian $H^{(r)}$ admits approximated first integrals of the form

$$I_l = \frac{1}{2}(x_l^2 + y_l^2) \ ,$$

indeed

$$\dot{I}_l = \{I_l, H^{(r)}\} = \{I_l, \mathcal{R}^{(r)}\} \sim 2\{I_l, \mathcal{R}^{(r)}_{r+1}\} \ .$$

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Consider a polydisk

$$\Delta_{\rho R} = \left\{(x,\, y) :\, x_l^2 + y_l^2 \leq \rho^2 R_l^2\right\} \ .$$

Let $\rho_0 = \rho/2$ and $(x(0), y(0)) \in \Delta_{\rho_0 R}$, then

$$I(0) = \frac{x_j^2 + y_j^2}{2} \leq \frac{\rho_0^2 R_j^2}{2} \ .$$

Thus, there is $T(\rho_0) > 0$ such that for $|t| \leq T(\rho_0)$ we have

$$I(t) \leq \frac{\rho^2 R_j^2}{2} \quad \text{so that} \quad (x(t), y(t)) \in \Delta_{\rho R} \ .$$

Effective stability

Given a homogeneous polynomial $f(x, y)$ of degree s as

$$f(x,y) = \sum_{|j|+|k|=s} f_{j,k} x^j y^k \ ,$$

we define the quantity $|f|_R$ as

$$|f|_R = \sum_{|j|+|k|=s} |f_{j,k}| R^{j+k} \Theta_{j,k} \ , \quad \Theta_{j,k} = \sqrt{\frac{j^j k^k}{(j+k)^{j+k}}} \ .$$

Thus, for $\rho > 0$, we have

$$\sup_{(x,y)\in\Delta_{\rho R}} \big|f(x,y)\big| < \rho^s |f|_R \ .$$

Effective stability

We can now estimate

$$\sup_{(x,y)\in\Delta_{\rho R}} \left|\dot{I}_j(x,y)\right| \le 2\rho^{r+3}\left|\{I_j, \mathcal{R}^{(r)}_{r+1}\}\right|_R \ ,$$

and get a lower bound for the time stability $T(\rho_0)$ as

$$\tau(\rho_0, r) = \min_j \left(1 - \frac{1}{2^{r+1}}\right) \frac{R_j^2}{2(r+1)|\{I_j, \mathcal{R}^{(r)}_{r+1}\}|_R \, \rho_0^{r+1}} \ .$$

Finally we can set

$$T(\rho_0) = \max_r \tau(\rho_0, r) \ .$$

Stability of the secular problem for the planar Sun – Jupiter – Saturn – Uranus system

The Solar system. . .

Questions

Is the Solar System stable?
Can we apply the Kolmogorov and Nekhoroshev theorems to *realistic* models of planetary systems?

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Models considered

The complete Sun-Jupiter-Saturn system (SJS).
The planar Sun-Jupiter-Saturn-Uranus system (SJSU).

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Models considered

The complete Sun-Jupiter-Saturn system (SJS).
The planar Sun-Jupiter-Saturn-Uranus system (SJSU).

Answers

The KAM theorem was applied to the *realistic* SJS system (L.&G. 2007).
We applied the Nekhoroshev's like exponential estimates for the stability of the SJS system, in the neighborhood of a KAM torus (G.,L.&S. 2009).
We studied the secular problem for the SJSU system (S.L.&G. 2013).

Dynamics of the SJSU systems

In order to study the secular dynamics of the SJSU system we must take into account the triple $(3, -5, -7)$ mean-motion resonance. Indeed Saturn is close to the celebrated $5:2$ resonance with Jupiter, while Uranus is near to the $7:1$. Moreover, $2n_J - 5n_S \simeq 7n_U - n_J$.

Dynamics of the SJSU systems

In order to study the secular dynamics of the SJSU system we must take into account the triple $(3, -5, -7)$ mean-motion resonance. Indeed Saturn is close to the celebrated $5:2$ resonance with Jupiter, while Uranus is near to the $7:1$. Moreover, $2n_J - 5n_S \simeq 7n_U - n_J$.

Remark:

From the study of the Sun-Jupiter-Saturn system we know that a careful handling of *the secular part of the Hamiltonian is crucial*. Before starting to manipulate the secular part of the Hamiltonian, we need to reduce the main part of the perturbation depending on the fast angles.

The Hamiltonian of the planetary system

The Hamiltonian is

$$F(\mathbf{r}, \tilde{\mathbf{r}}) = T^{(0)}(\tilde{\mathbf{r}}) + U^{(0)}(\mathbf{r}) + T^{(1)}(\tilde{\mathbf{r}}) + U^{(1)}(\mathbf{r})\,,$$

where $\mathbf{r}$ are the heliocentric coordinates and $\tilde{\mathbf{r}}$ the conjugated momenta.

$$T^{(0)}(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_{j=1}^{3} \|\tilde{\mathbf{r}_j}\|^2 \left(\frac{1}{m_0} + \frac{1}{m_j}\right)\,,$$

$$U^{(0)}(\mathbf{r}) = -\mathcal{G}\sum_{j=1}^{3} \frac{m_0 m_j}{\|\mathbf{r}_j\|}\,,$$

$$T^{(1)}(\tilde{\mathbf{r}}) = \frac{\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_2}{m_0} + \frac{\tilde{\mathbf{r}}_1 \cdot \tilde{\mathbf{r}}_3}{m_0} + \frac{\tilde{\mathbf{r}}_2 \cdot \tilde{\mathbf{r}}_3}{m_0}\,,$$

$$U^{(1)}(\mathbf{r}) = -\mathcal{G}\left(\frac{m_1 m_2}{\|\mathbf{r}_1 - \mathbf{r}_2\|} + \frac{m_1 m_3}{\|\mathbf{r}_1 - \mathbf{r}_3\|} + \frac{m_2 m_3}{\|\mathbf{r}_2 - \mathbf{r}_3\|}\right)\,.$$

The Poincaré variables in the plane

$$\underbrace{\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{\mathcal{G}(m_0 + m_j)a_j} \qquad \lambda_j = M_j + \omega_j}_{\text{fast variables}}$$

$$\underbrace{\xi_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\cos(\omega_j) \quad \eta_j = -\sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\sin(\omega_j)}_{\text{secular variables}}$$

where a_j, e_j, M_j and ω_j are the semi-major axis, the eccentricity, the mean anomaly and perihelion argument of the j-th planet, respectively.

How to expand the Hamiltonian

1. The development of the Hamiltonian is a quite standard matter.
2. Choose a Λ^* such that
$$\left.\frac{\partial\langle F\rangle_\lambda}{\partial\Lambda_j}\right|_{\substack{\Lambda=\Lambda^*\\ \xi=\eta=0}} = n_j^* \,, \qquad j=1,2,3\,.$$
   - $\langle . \rangle_\lambda$ means the average over the fast angles ,
   - n_j^* are the fundamental frequencies of the mean motion .
3. Introduce new actions $L_j = \Lambda_j - \Lambda_j^*$.
4. Perform the canonical transformation $\mathcal{T}_F$ translating the fast actions.
5. Expand the Hamiltonian in power series of $\mathbf{L}$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and in Fourier series of $\boldsymbol{\lambda}$.

The expansion of the Hamiltonian

The transformed Hamiltonian reads

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where $h_{j_1,0}^{(Kep)}$ is an homogeneous polynomial of degree j_1 in $\mathbf{L}$ and

$$h_{j_1,j_2}^{(\mathcal{T}_F)} \text{is a } \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } \mathbf{L}\,, \\ \text{hom. pol. of degree } j_2 \text{ in } \boldsymbol{\xi}, \boldsymbol{\eta}\,, \\ \text{with coeff. that are trig. pol. in } \boldsymbol{\lambda}\,. \end{cases}$$

Truncation limits of the expansion

This is the Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

Truncation limits of the expansion

This is the computed Hamiltonian,

$$H^{(\mathcal{T}_F)} = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{2} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{1} \sum_{j_2=0}^{12} h_{j_1,j_2}^{(\mathcal{T}_F)}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})$$

where we also truncate all the coefficients with harmonics of degree greater than 16.

These are the lowest limits to include the fundamental features of the system.

The scheme of the preliminary perturbation reduction

We now aim to kill the terms

$$\left\lceil \mu\, h_{0,0}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}),\ \left\lceil \mu\, h_{0,1}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}),\ \ldots,\ \left\lceil \mu\, h_{0,6}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}),$$

and

$$\left\lceil \mu\, h_{1,0}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}),\ \left\lceil \mu\, h_{1,1}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}),\ \ldots,\ \left\lceil \mu\, h_{1,6}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}).$$

where $\lceil \cdot \rceil_{\boldsymbol{\lambda}:K}$ means the truncation of the harmonics of degree greater than K.

The details of the transformation

This procedure is essentially a "Kolmogorov's like" step of normalization.

In order to kill the term $\left\lceil h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K}$, one has to solve the equation

$$\{\chi,\, \mathbf{n}^* \cdot \mathbf{L}\} + \left\lceil \mu\, h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K} = 0 \ .$$

and find the generating function χ .

The generating function χ has the same structure of $h_{j_1,j_2}^{(\mathcal{T}_F)}$, is of order $\mathcal{O}(\mu)$ and must depends on the fast angles $\boldsymbol{\lambda}$.

The details of the transformation

We now perform a canonical transformation of the Hamiltonian

$$\exp \mathcal{L}_{\chi} H = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}_{\chi}^{j} H \ .$$

This transformation, by construction, kill the terms $\left\lceil \mu \, h_{j_1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:K}$, but the transformed Hamiltonian still has a term of the same type, but at least of order $\mathcal{O}(\mu^2)$.

This effect is due to Lie series algorithm, for example take $j_1 = 0$, $j_2 = 0$ and consider the Poisson bracket

$$\left\{ \chi \, , \mu \, h_{1,0}^{(\mathcal{T}_F)} \right\} \to \mu^2 \, \widetilde{h}_{0,0}^{(\mathcal{T}_F)} \, .$$

Partial preliminary reduction of the perturbation

First step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_1^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil h_{0,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ \widetilde{H} = \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}^j_{\chi_1^{(\mathcal{O}2)}} H\,. \end{cases}$$

Second step

$$\begin{cases} \mathbf{n}^* \cdot \dfrac{\partial \chi_2^{(\mathcal{O}2)}}{\partial \boldsymbol{\lambda}} + \mu \displaystyle\sum_{j_2=0}^{6} \left\lceil \tilde{h}_{1,j_2}^{(\mathcal{T}_F)} \right\rceil_{\boldsymbol{\lambda}:8} (\boldsymbol{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = 0 \\ H^{(\mathcal{O}2)} = \exp \mathcal{L}_{\chi_2^{(\mathcal{O}2)}} \circ \exp \mathcal{L}_{\chi_1^{(\mathcal{O}2)}} H\,. \end{cases}$$

Why these limits?

The secular variables:

$$\widetilde{H} \qquad \chi_2^{(\mathcal{O}2)} \qquad \longrightarrow \qquad H^{(\mathcal{O}2)}$$

The fast angles:

$(3, -5, -7)$ harmonics of order 15

Why these limits?

The secular variables:

$$\begin{array}{ccccc} \widetilde{H} & \chi_2^{(\mathcal{O}2)} & \longrightarrow & H^{(\mathcal{O}2)} \\ \downarrow & \downarrow & & \downarrow \\ \cos(1\lambda_1 - 7\lambda_3) & \sin(1\lambda_1 - 7\lambda_3) & & \text{secular terms} \end{array}$$

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\downarrow{\scriptstyle (\xi,\eta)} & \downarrow{\scriptstyle (\xi,\eta)} & & \downarrow{\scriptstyle (\xi,\eta)} \\
6 & 6 & & \boxed{12}
\end{array}
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The fast angles:

$(3, -5, -7)$ harmonics of order $15 < \boxed{16}$.

The Hamiltonian *up to order two in the masses*

- $H^{(\mathcal{O}2)}$ is the Hamiltonian up to order two in the masses.
- **No** terms corresponding to any *triple resonances* **before** the "Kolmogorov's like" step.
- The "Kolmogorov's like" step introduce the *triple resonances*, in particular the $(3, -5, -7)$ resonance.
- Small limits **don't** mean small expansion!
- After the "Kolmogorov's like" step, we have $94\,109\,751$ coefficients.

The secular part *up to order two in the masses*

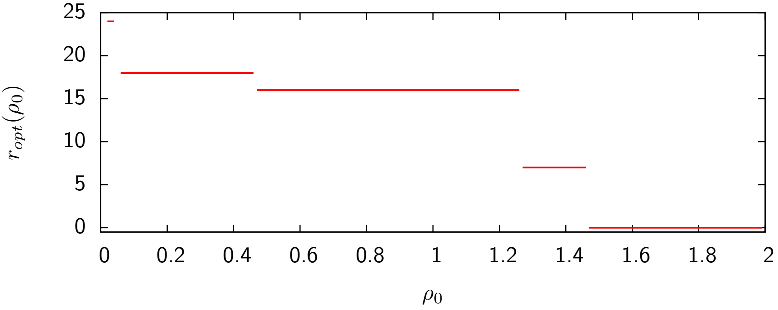
- Reduction to the secular system:
  - average over the fast angles $\boldsymbol{\lambda}$, and put $\mathbf{L} = 0$;
  - hereafter, we are considering a system with *three degrees of freedom*.
- From the D'Alembert rules, it follows that

$$H^{(sec)} = H_0 + H_2 + H_4 + \ldots \,,$$

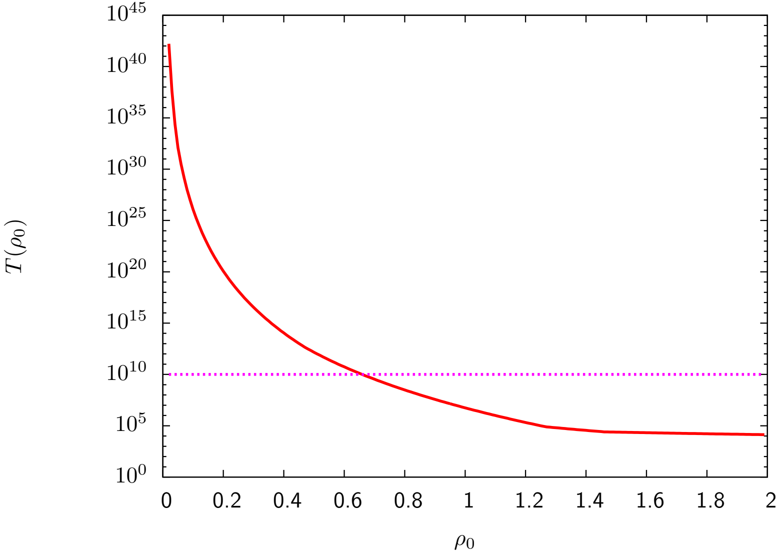
  where H_{2j} is a hom. pol. of degree $(2j+2)$ in $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, $\forall\ j \in \mathbf{N}$.
- $\boldsymbol{\xi} = \boldsymbol{\eta} = 0$ is an elliptic equilibrium point.
- We diagonalize the quadratic term by a linear canonical transformation $\mathcal{D}$:

$$H_2^{(\mathcal{D})} = \sum_{j=1}^{3} \frac{\nu_j}{2} \left(\xi_j^2 + \eta_j^2 \right) \,.$$

The optimal normalization order



The estimated "stability time" of the secular Hamiltonian



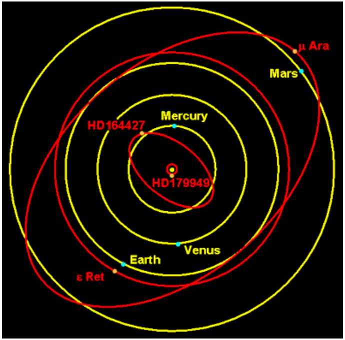
Comments about our results

- We considered a secular Hamiltonian model of the **planar Sun-Jupiter-Saturn-Uranus system**, providing an approximation of the motions of the secular variables *up to order two in the masses*. Our results ensure that such a system is *stable for a time comparable to the age of the universe* just in a domain with a radius that is about **a half of the real distance of the initial secular variables from the origin**.

- This exponential stability estimate around the equilibrium point is a "too lazy option". Indeed, we show that a preliminary construction of a KAM torus for the planar SJSU system allows much better estimates.

Secular evolution of extrasolar systems

Extrasolar systems vs. Solar System



The main difference between the extrasolar systems and the Solar System regards the **shape of the orbits**.

In the extrasolar systems, the majority of the orbits describe **true ellipses** (high eccentricities) and no more almost circles like in the Solar System.

The classical approach uses the circular approximation as a reference. Dealing with systems with high eccentricities we need to **compute the expansion at high order** to study the long-term evolution of the extrasolar planetary systems.

Aim of the work

1. Can we predict the **long-term evolution** of extrasolar systems?

2. How can we evaluate the influence of a **mean-motion resonance**?

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   - Numerical integrations are really accurate, but have high computational cost. One has to compute a numerical integration for each initial condition.
   - Normal forms provide non-linear approximations of the dynamics in a neighborhood of an invariant object. In addition, an accurate analytic approximation is the starting point for the study of the effective long-time stability.
2. How can we evaluate the influence of a **mean-motion resonance**?

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1. Can we predict the **long-term evolution** of extrasolar systems?
   - Numerical integrations are really accurate, but have high computational cost. One has to compute a numerical integration for each initial condition.
   - Normal forms provide non-linear approximations of the dynamics in a neighborhood of an invariant object. In addition, an accurate analytic approximation is the starting point for the study of the effective long-time stability.
2. How can we evaluate the influence of a **mean-motion resonance**?
   - The semi-major axis ratio gives a rough indication of the proximity to the main mean-motion resonance.
   - However, the impact of the proximity to a mean-motion resonance on the secular evolution of a planetary system depends on many parameters. This is due to the non-linear character of the system.

Expansion of the Hamiltonian

In order to compute the Taylor expansion of the Hamiltonian around the fixed value $\mathbf{\Lambda}^*$, we introduce the *translated fast actions*,

$$\mathbf{L} = \mathbf{\Lambda} - \mathbf{\Lambda}^* \ .$$

The Hamiltonian reads,

$$H = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1=2}^{\infty} h_{j_1,0}^{(Kep)}(\mathbf{L}) + \mu \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} h_{j_1,j_2}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) \ .$$

where $h_{j_1,0}^{(Kep)}$ is a hom. pol. of degree j_1 in $\mathbf{L}$ and

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$$h_{j_1,j_2} \text{is a } \begin{cases} \text{hom. pol. of degree } j_1 \text{ in } \mathbf{L} \ , \\ \text{hom. pol. of degree } j_2 \text{ in } \boldsymbol{\xi}, \boldsymbol{\eta} \ , \\ \text{with coeff. that are trig. pol. in } \boldsymbol{\lambda} \ . \end{cases}$$

We will choose the **lowest possible limits** in order to include the **fundamental features** of the system.

First order averaging

We consider the **averaged Hamiltonian**,

$$H(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) = \langle H(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) \rangle_{\lambda} \ .$$

Namely, we **get rid of the fast motion** removing from the expansion of the Hamiltonian all the terms that depend on the fast angles $\boldsymbol{\lambda}$.

This is the so called **first order averaging**.

We end up with the Hamiltonian,

$$H^{(sec)} = \mu \sum_{j_2=0}^{12} h_{0,j_2}(\boldsymbol{\xi}, \boldsymbol{\eta}) \ .$$

Secular dynamics

- Doing the averaging over the fast angles (as we are interested in the **secular motions** of the planets), the system pass from 4 to 2 degrees of freedom,

$$H^{(sec)} = H_0(\boldsymbol{\xi}, \boldsymbol{\eta}) + H_2(\boldsymbol{\xi}, \boldsymbol{\eta}) + H_4(\boldsymbol{\xi}, \boldsymbol{\eta}) + \ldots \ ,$$

  where H_{2j} is a hom. pol. of degree $(2j+2)$ in $(\boldsymbol{\xi}, \boldsymbol{\eta})$, for all $j \in \mathbb{N}$.

- $\boldsymbol{\xi} = \boldsymbol{\eta} = \mathbf{0}$ is an **elliptic equilibrium** point, thus we can introduce **action-angle** variables via **Birkhoff normal form**.

- Having the Hamiltonian in Birkhoff normal form, we can **easily solve the equations of motion** and finally obtain the motion of the orbital parameters.

Secular dynamics

If the remainder, $\mathcal{R}_r$, is *small enough*, we can neglect it!

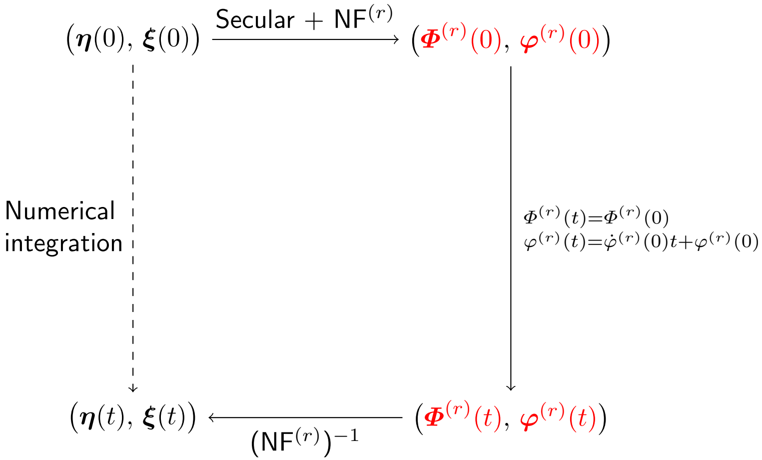
- The **equations of motion** are

$$\dot{\Phi}_j(0) = 0 \ , \qquad \dot{\varphi}_j(0) = \left. \frac{\partial H^{(r)}}{\partial \Phi_j} \right|_{(\Phi(0),\varphi(0))} \ .$$

- The **solutions** are

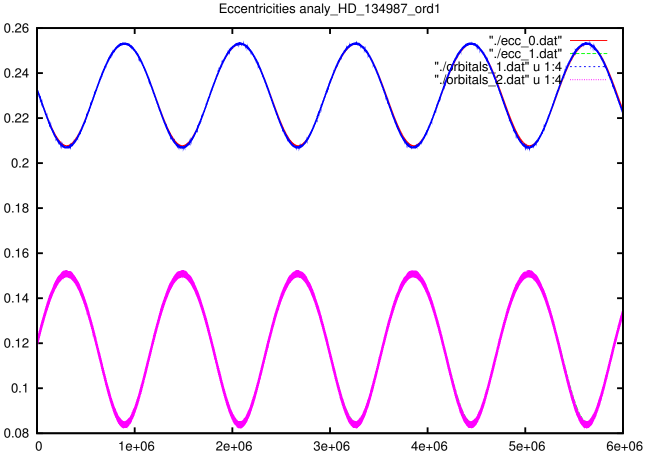
$$\Phi_j(t) = \Phi_j(0) \ , \qquad \varphi_j(t) = \dot{\varphi}_j(0) \ t + \varphi_j(0) \ .$$

Analytical integration



First order approximation (HD 134987)

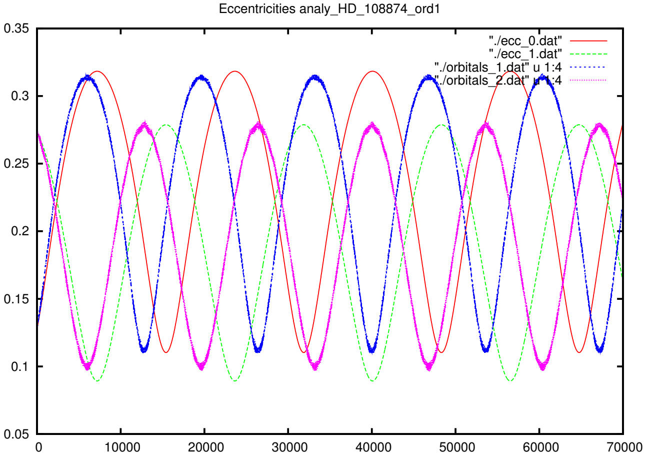
HD 134987: the system is secular.



But. . . near mean-motion resonance (HD 108874)

HD 108874: the system is "close" to the $4{:}1$ mean-motion resonance.

First order averaged Hamiltonian failed.



Second order averaging

Coming back to the original Hamiltonian,

$$H = \mathbf{n}^* \cdot \mathbf{L} + \sum_{j_1 \geq 2} h_{j_1,0}(\mathbf{L}) + \mu \sum_{j_1 \geq 0} \sum_{j_2 \geq 0} h_{j_1,j_2}(\mathbf{L}, \boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta}) \ .$$

If we consider the point $\boldsymbol{L}(0) = \mathbf{0}$, we have

$$\dot{L}_j = -\mu \sum_{j_2 \geq 0} \frac{\partial h_{0,j_2}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})}{\partial \lambda_j} \ .$$

In order to get rid of the fast motion, instead of simply erasing the terms depending on fast angles $\boldsymbol{\lambda}$, we perform a **canonical transformation via Lie Series** to kill the terms

$$\frac{\partial h_{0,0}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})}{\partial \boldsymbol{\lambda}} \ , \ \frac{\partial h_{0,1}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})}{\partial \boldsymbol{\lambda}} \ , \ \frac{\partial h_{0,2}(\boldsymbol{\lambda}, \boldsymbol{\xi}, \boldsymbol{\eta})}{\partial \boldsymbol{\lambda}} \ , \ \ldots \ .$$

The scheme of the preliminary perturbation reduction

We perform a "Kolmogorov-like" step of normalization.

We determine the generating function, χ, by solving the equation

$$\mathbf{n}^* \frac{\partial \chi}{\partial \boldsymbol{\lambda}} + \sum_{j_2=0}^{K_S} \left\lceil h_{0,j_2} \right\rceil_{\boldsymbol{\lambda}:K_F} = 0 \ .$$

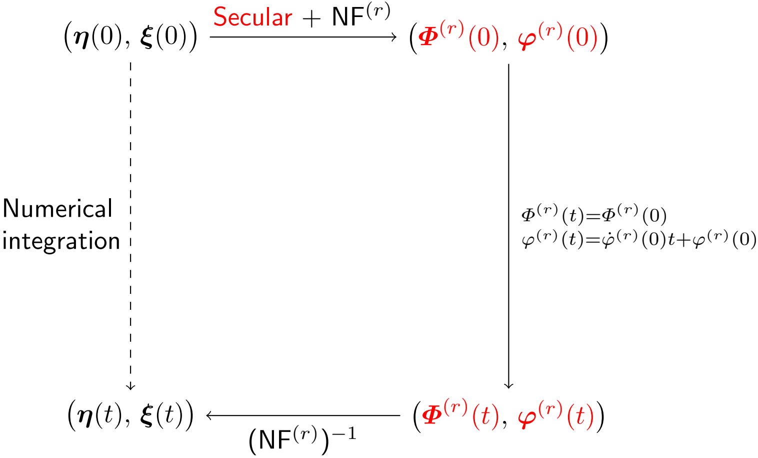
where $\lceil \cdot \rceil_{\boldsymbol{\lambda}:K_F}$ means that we keep only the terms depending on $\boldsymbol{\lambda}$ and at most of degree K_F . The parameters K_S and K_F are chosen so as to **include in the secular model** the main effects due to the possible **proximity to a mean-motion resonance**.

The transformed Hamiltonian reads

$$H^{(\mathcal{O}2)} = \exp \mathcal{L}_{\mu\chi} H = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}_{\mu\chi}^{j} H \ .$$

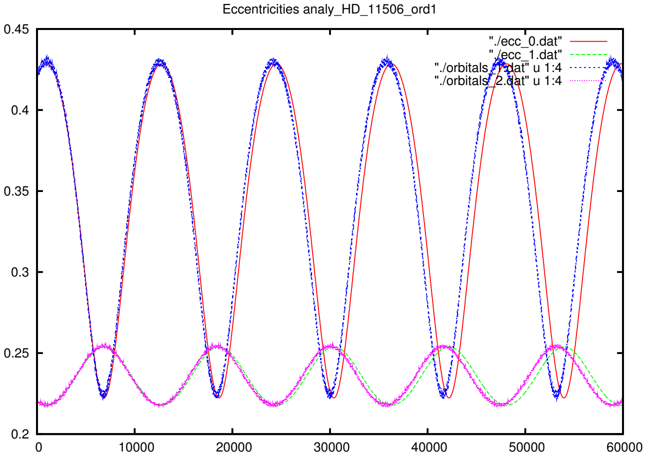
This is our Hamiltonian **at order two in the masses**.

Analytical integration



First order approximation (HD 11506)

HD 11506: the system is “close” to the $7{:}1$ MMR (weak MMR).



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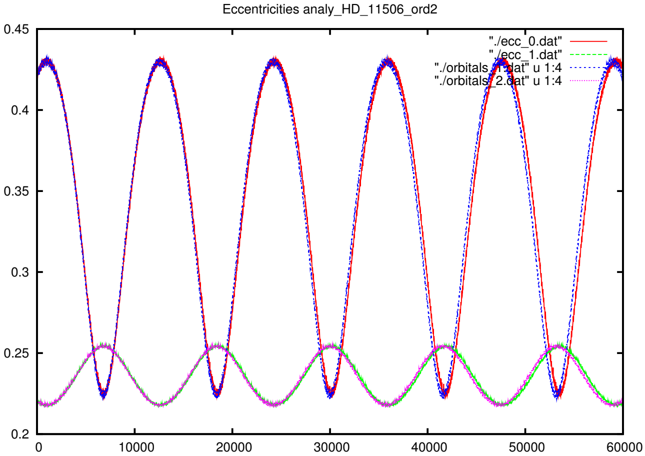
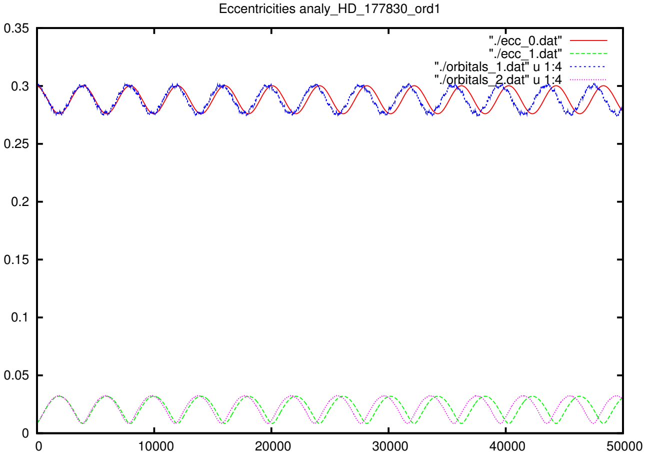


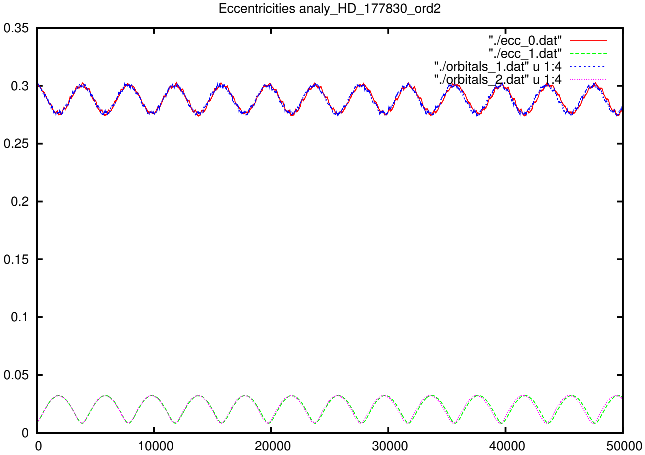
First order approximation (HD 177830)

HD 177830: the system is "close" to the $3{:}1$ and $4{:}1$ MMR.



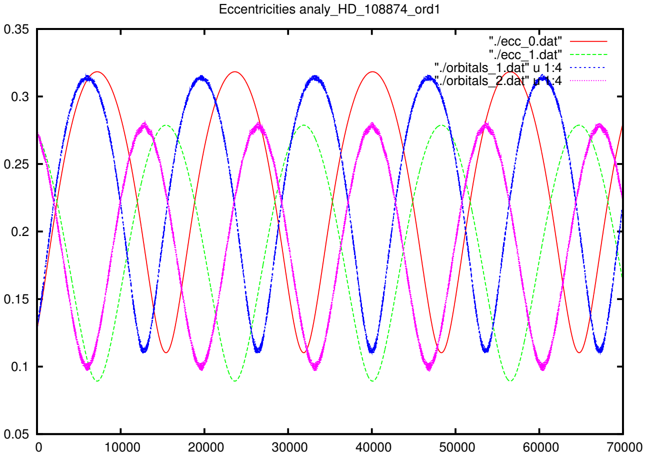
Second order approximation (HD 177830)

HD 177830: the system is "close" to the $3{:}1$ and $4{:}1$ MMR.



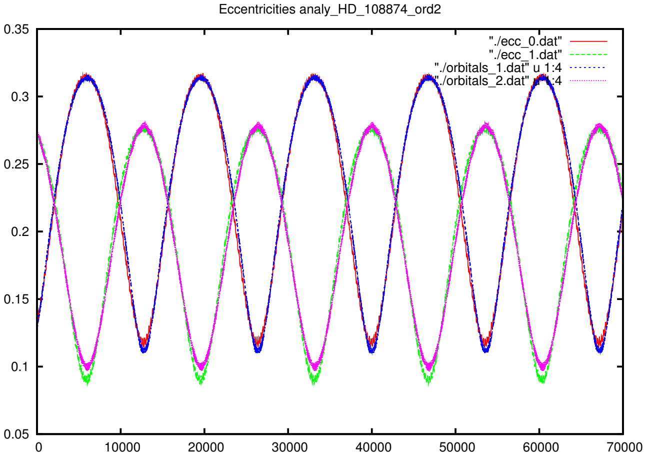
First order approximation (HD 108874)

HD 108874: the system is “close” to the $4{:}1$ MMR (strong MMR).



Second order approximation (HD 108874)

HD 108874: the system is "close" to the $4{:}1$ MMR (strong MMR).



Proximity to a mean-motion resonance

We now want to evaluate the **proximity to a mean-motion resonance**.

The idea is to rate the proximity by looking at the **canonical change of coordinates** induced by the approximation at order two in the masses.

$$\xi_j' = \xi_j - \mu \frac{\partial \chi}{\partial \eta_j} = \xi_j \left(1 - \frac{\mu}{\xi_j} \frac{\partial \chi}{\partial \eta_j}\right) ,$$
$$\eta_j' = \eta_j - \mu \frac{\partial \chi}{\partial \xi_j} = \eta_j \left(1 - \frac{\mu}{\eta_j} \frac{\partial \chi}{\partial \xi_j}\right) .$$

In particular, we focus on the **coefficients** of the terms

$$\delta\xi_j = \frac{\mu}{\xi_j} \frac{\partial \chi}{\partial \eta_j} \qquad \text{and} \qquad \delta\eta_j = \frac{\mu}{\eta_j} \frac{\partial \chi}{\partial \xi_j} .$$

δ-criterion

	System	a_1/a_2	δ
Secular	HD 11964	0.072	$9.897 \times 10^{-4} \sin(-\lambda_1 + 2\lambda_2)$
	HD 74156	0.075	$9.681 \times 10^{-4} \cos(4\lambda_1 - \lambda_2)$
	HD 134987	0.140	$9.897 \times 10^{-4} \sin(-\lambda_1 + 2\lambda_2)$
	HD 163607	0.149	$1.376 \times 10^{-3} \cos(3\lambda_1 - \lambda_2)$
	HD 12661	0.287	$1.760 \times 10^{-3} \sin(-\lambda_1 + 2\lambda_2)$
	HD 147018	0.124	$2.455 \times 10^{-3} \sin(-2\lambda_1 + \lambda_2)$
near a MMR	HD 11506	0.263	$2.943 \times 10^{-3} \cos(\lambda_1 - 7\lambda_2)$
	HD 177830	0.420	$2.551 \times 10^{-3} \cos(\lambda_1 - 4\lambda_2)$
	HD 9446	0.289	$2.328 \times 10^{-3} \sin(-2\lambda_1 + \lambda_2)$
	HD 169830	0.225	$2.316 \times 10^{-2} \cos(\lambda_1 - 9\lambda_2)$
	υ Andromedae	0.329	$1.009 \times 10^{-2} \cos(\lambda_1 - 5\lambda_2)$
	Sun-Jup-Sat	0.546	$2.534 \times 10^{-2} \cos(2\lambda_1 - 5\lambda_2)$
MMR	HD 108874	0.380	$4.314 \times 10^{-2} \sin(-\lambda_1 + 4\lambda_2)$
	HD 128311	0.622	$6.421 \times 10^{-1} \sin(-\lambda_1 + 2\lambda_2)$
	HD 183263	0.347	$5.253 \times 10^{-2} \cos(\lambda_1 - 5\lambda_2)$

Results

1. Can we predict the long-term evolution of extrasolar systems?
   - If the system is not *too close* to a mean-motion resonance, providing an approximation of the motions of the secular variables *up to order two in the masses*, **the secular evolution is well approximated via Birkhoff normal form**.

2. How can we evaluate the influence of a mean-motion resonance?
   - The secular Hamiltonian at order two in the masses is **explicitly constructed** via Lie Series, so the generating function contains the information about the **proximity to a mean-motion**.
   - We introduce an heuristic and quite rough **criterion** that we think is useful to *discriminate* between the different behaviors:

(i) $\delta \leq 2.6 \times 10^{-3}$: secular;
(ii) $2.6 \times 10^{-3} < \delta \leq 2.6 \times 10^{-2}$: near mean-motion resonance;
(iii) $\delta > 2.6 \times 10^{-2}$: in mean-motion resonance.

Effective stability around the Cassini state

in the spin-orbit problem

Hamiltonian formulation

We consider a rotating body (e.g., Titan) with mass m and equatorial radius R_{e}, orbiting around a point body (e.g., Saturn) with mass M.

The rotating body is considered as a triaxial rigid body whose principal moments of inertia are A, B and C, with $A \leq B < C$.

We closely follow the Hamiltonian formulation that has already been used in previous works, see, e.g., Henrard & Schwanen (2004) for a general treatment of synchronous satellites.

Reference planes

In order to describe the spin-orbit motion we need four reference frames, centered at the center of mass of the rotating body,

(i) the *inertial frame*, (X_0, Y_0, Z_0), with X_0 and Y_0 in the ecliptic plane;

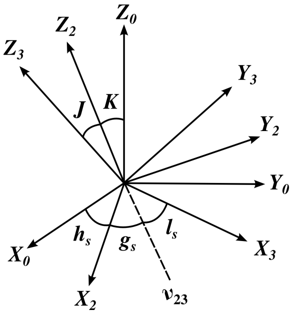
(ii) the *orbital frame*, (X_1, Y_1, Z_1), with Z_1 perpendicular to the orbit plane;

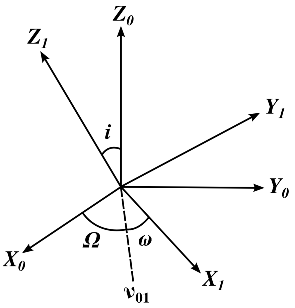
(iii) the *spin frame*, (X_2, Y_2, Z_2), with Z_2 pointing to the spin axis direction and X_2 to the ascending node of the equatorial plane on the ecliptic plane;

(iv) the *body frame*, (X_3, Y_3, Z_3), with Z_3 in the direction of the axis of greatest inertia and X_3 of the axis of smallest inertia.

Reference planes

The four reference frames and the relevant angles related to the Andoyer (left) and Delaunay (right) canonical variables.





Andoyer variables

For the rotational motion we adopt the Andoyer variables,

$$
\begin{aligned}
L_s &= G_s \cos J \ , \quad & l_s \ , \\
G_s & \ , & g_s \ , \\
H_s &= G_s \cos K \ , & h_s \ ,
\end{aligned}
$$

where G_s is the norm of the angular momentum.

In order to remove the two virtual singularity ($J = 0$ and $K = 0$), we introduce the modified Andoyer variables,

$$
\begin{aligned}
L_1 &= \frac{G_s}{n_o C} \ , & l_1 &= l_s + g_s + h_s \ , \\
L_2 &= \frac{G_s - L_s}{n_o C} \ , & l_2 &= -l_s \ , \\
L_3 &= \frac{G_s - H_s}{n_o C} \ , & l_3 &= -h_s \ ,
\end{aligned}
$$

where n_o is the orbital mean-motion of the rotating body.

Delaunay variables

For the orbital motion, we introduce the classical Delaunay variables,

$$\begin{aligned} L_o &= m\sqrt{\mu a}\ , & l_o\ , \\ G_o &= L_o\sqrt{1-e^2}\ , & g_o = \omega\ , \\ H_o &= G_o \cos i\ , & h_o = \Omega\ , \end{aligned}$$

Again, to remove the singularity ($e = 0$ and $i = 0$), we introduce the modified Delaunay variables,

$$\begin{aligned} L_4 &= L_o\ , & l_4 = l_o + g_o + h_o\ , \\ L_5 &= L_o - G_o\ , & l_5 = -g_o - h_o\ , \\ L_6 &= G_o - H_o\ , & l_6 = -h_o\ . \end{aligned}$$

Free rotation

The rotational kinetic energy, (Deprit, 1967), reads

$$T = \frac{L_s^2}{2C} + \frac{1}{2}(G_s^2 - L_s^2)\left(\frac{\sin^2 l_s}{A} + \frac{\cos^2 l_s}{B}\right) \ .$$

Thus, in our set of variables, we get

$$\frac{T}{n_o C} = \frac{n_o L_1^2}{2} + \frac{n_0 L_3(2L_1 - L_3)}{2}\left(\frac{\gamma_1 + \gamma_2}{1 - \gamma_1 - \gamma_2}\sin^2(l_3) + \frac{\gamma_1 - \gamma_2}{1 - \gamma_1 + \gamma_2}\cos^2(l_3)\right) \ ,$$

where

$$\gamma_1 = \frac{2C - B - A}{2C} \quad \text{and} \quad \gamma_2 = \frac{B - A}{2C} \ .$$

Perturbation by another body

The perturbation induced by the point body mass on the rotation of the rigid body, can be expressed via a gravitational potential, V, in the form

$$V = \frac{3}{2}\frac{\mathcal{G}M}{a^3}\left(\frac{a}{r}\right)^3 \left(C_2^0\big(x_3^2 + y_3^2\big) - 2C_2^2\big(x_3^2 - y_3^2\big)\right) \ ,$$

where (x_3, y_3, z_3) are the components (in the body frame) of the unit vector pointing to the perturbing body. The coefficients C_2^0 and C_2^2, can be written in terms of the moments of inertia and of the dimensionless parameters J_2 and C_{22}, as

$$C_2^0 = \frac{A + B - 2C}{2} = -mR_\mathrm{e}^2 J_2 \ ,$$
$$C_2^2 = \frac{B - A}{4} = mR_\mathrm{e}^2 C_{22} \ .$$

Simplified model

We now consider a simplified spin-orbit model, making some strong assumptions on the system.

(i) We assume that the wobble, J, is equal to zero. This means that the spin axis is aligned with figure one.

(ii) We introduce the resonant variables

$$\Sigma_1 = L_1 \ , \quad \sigma_1 = l_1 - l_4 \ ,$$
$$\Sigma_3 = L_3 \ , \quad \sigma_3 = l_3 - l_6 \ ,$$

and make an over the fast angle, l_4, namely

$$\langle V \rangle_{l_4} = \frac{1}{2\pi} \int_0^{2\pi} V \, \mathrm{d}l_4 \ .$$

(iii) We neglect the influence of the rotation on the orbit of the body and we model the time dependence of the Hamiltonian via the two angular variables,

$$l_4(t) = n\,t + l_4(0) \qquad \text{and} \qquad l_6(t) = \dot{\Omega}\,t + l_6(0) \ .$$

Simplified model

Finally, we end up with a Hamiltonian that reads

$$H = \frac{n_o \Sigma_1^2}{2} - n_o \Sigma_1 + \dot{\Omega}\Sigma_3 + \langle V \rangle_{l_4} \ .$$

This Hamiltonian possesses an equilibrium, the Cassini state, defined by

$$\sigma_1 = 0 \ , \quad \frac{\partial H}{\partial \Sigma_1} = 0 \ ,$$
$$\sigma_3 = 0 \ , \quad \frac{\partial H}{\partial \Sigma_3} = 0 \ .$$

We denote by Σ_1^* and Σ_3^* the values at the equilibrium.

Stability around the Cassini state

We now aim to study the dynamics in the neighborhood of the Cassini state defined here above. We introduce the translated canonical variables

$$\Delta\Sigma_1 = \Sigma_1 - \Sigma_1^* \, , \qquad \sigma_1 \, ,$$
$$\Delta\Sigma_3 = \Sigma_3 - \Sigma_3^* \, , \qquad \sigma_3 \, ,$$

and, with a little abuse of notation, in the following we will denote again $\Delta\Sigma_i$ by Σ_i, with $i = 1, 3$.

In these new coordinates, the equilibrium is set at the origin, thus we can expand the Hamiltonian in power series of (Σ, σ). Let us remark that the linear terms disappear, as the origin is an equilibrium, thus the lower order terms in the expansion are quadratic in (Σ, σ).

Stability around the Cassini state

Precisely, we can write the Hamiltonian as

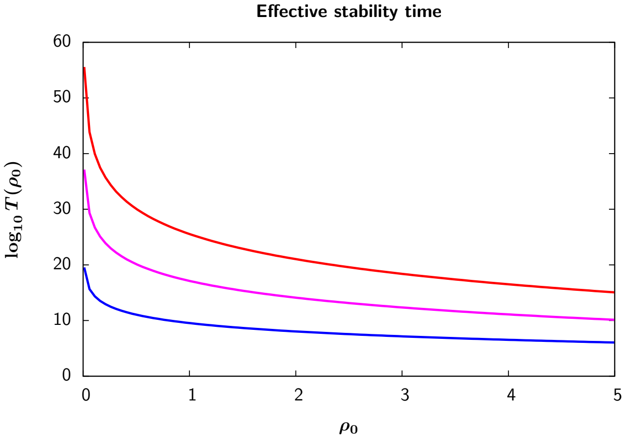
$$H(\Sigma,\sigma) = H_0(\Sigma,\sigma) + \sum_{j>0} H_j(\Sigma,\sigma)\ , \qquad (1)$$

where H_j is an homogeneous polynomial of degree $j+2$ in $(\Sigma,\,\sigma)$.
The Hamitonian is almost in the "right" form, we just need to diagonalize the quadratic part, H_0, via the so-called 'untangling transformation" (Henrard & Lemaître, 2005), perform a rescaling and introduce the action-angle coordinates.
Finally, the transformed Hamiltonian can be expanded in Taylor-Fourier series and reads

$$H^{(0)}(U,\,u) = \omega_u \cdot U + \sum_{j>0} H_j^{(0)}(U,\,u)\ ,$$

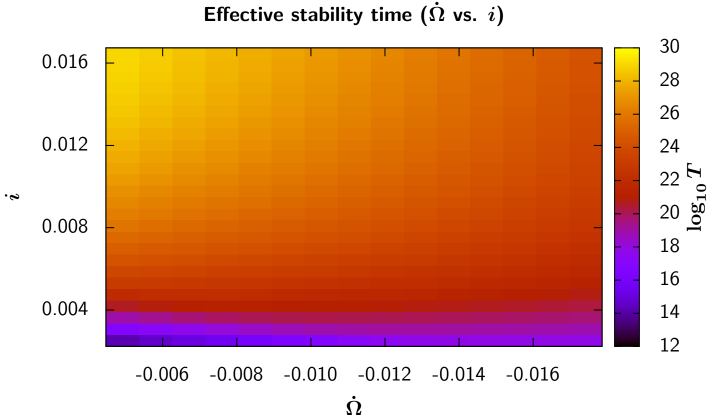
where the terms H_j are homogeneous polynomials of degree $j/2+1$ in U, whose coefficients are trigonometric polynomials in the angles u.

Estimated effective stability time



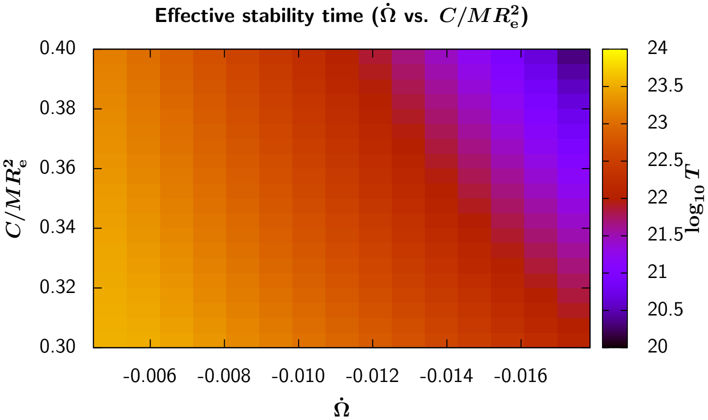
Estimated effective stability time. The time unit is the year. The three lines correspond (from down to top) to three different normalization orders: $r = 10$ (blue), $r = 20$ (pink) and $r = 30$ (red).

Estimated effective stability time



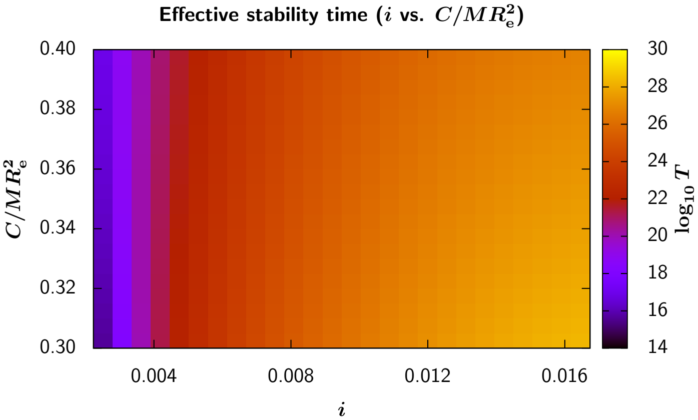
Effective stability time as a function of the mean inclination, i, and the mean precession of the ascending node of Titan orbit, $\dot{\Omega}$.

Estimated effective stability time



Effective sability time as a function of the normalized greatest moment of inertia, C/MR_e^2, and the mean precession of the ascending node of Titan orbit, $\dot{\Omega}$.

Estimated effective stability time



Effective stability time as a function of the mean inclination, i, and the normalized greatest moment of inertia, C/MR_e^2.

Thanks for your attention!

Questions?

Comments?