# Symmetric polynomials, generalized Jacobi-Trudy identities and $\tau$-functions$^*$

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## Outline

1. **Schur functions, Grassmannians and $\tau$-functions**
   - Jacobi-Trudy Identity
   - Grassmannian $\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$, partitions, Plücker coordinates
   - Generalized Jacobi-Trudy Identity for symmetric polynomials
2. **Integrable systems and $\tau$-functions**
   - The polynomials $S^{\phi}_{\lambda}$ as KP $\tau$-functions
   - Hilbert space Grassmannian $Gr_{\mathcal{H}_+}(\mathcal{H})$ and linear group actions
   - The $\tau$-function as a determinant
3. **Fermionic operator approach**
   - Fermionic Fock space
   - Fermionic VEV representations
   - Schur function expansions
4. **Examples and Applications**

# Jacobi-Trudy Identity

## Jacobi-Trudy Identity for Schur polynomials

$$S_\lambda = \det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j})$$

Where $h_j$, $e_j$ are the complete and elementary symmetric functions:

$$h_j := S_{(j)}, \quad e_j := S_{((1)^j)}.$$

The **Generating function** representations in terms of $(x_1, \dots, x_n)$:

$$\prod_{a=1}^{n} (1 - z x_a)^{-1} = \sum_{j=0}^{\infty} h_j z^j, \qquad \prod_{a=1}^{n} (1 + z x_a) = \sum_{j=0}^{n} e_j z^j$$

Here $\lambda$ denotes an integer **partition**

$$\lambda = (\lambda_1 \geq \lambda_2, \dots \geq \lambda_{\ell(\lambda)} > 0)$$

and $\lambda'$ the conjugate partition (**transposed Young diagram**).

## Grassmanian $\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ :

Let $\mathbf{F} := \mathbf{C}(x_1, \dots, x_n)$ denote the extension of the field of complex numbers by the indeterminates $\{x_1, x_2, \dots, x_n\}$,

$\mathcal{H}_+ :=$ space of **square integrable complex functions on the unit circle $|z| = 1$ admitting a holomorphic extension to the interior disc.**

$\mathcal{H}_+(\mathbf{F}) := \mathbf{F} \otimes_{\mathbf{C}} \mathcal{H}_+$: Its extension to an **F-module**,

$\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ : : **Grassmannian** of $n$-dimensional subspaces of $\mathcal{H}_+(\mathbf{F})$ :

**Monomial basis**:

$$\{\mathbf{b}_i := z^{i-1}\}_{i \in \mathbf{N}^+}$$

These may be viewed as elementary column vectors, labelled increasingly from the bottom element upward:

$$\mathbf{b}_1 \sim (\dots, 0, 0, 1)^t, \mathbf{b}_2 \sim (\dots, 0, 1, 0)^t \cdots$$

## Elements $w \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ and Plücker coordinates

Elements $w \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ may be represented by equivalence classes $[W]$ of semi-infinite $(\infty \times n)$ rank $n$ matrices $W \equiv W$ , $g \in GL(\mathcal{H}_+)$ whose columns $(W^{(1)}, \dots, W^{(n)})$ span $w$.

$$W^{(i)} = \sum_{j=1}^{\infty} W_{ij} \mathbf{b}_j$$

Define Particle coordinates $\{l_i \in \mathbf{Z}\}_{i \in \mathbf{N}^+}$ on the integer lattice

$$l_i := \lambda_i - i + n, \quad i \in \mathbf{N}^+,$$

The Plücker coordinates of $w$ are, up to projective equivalence

$$\pi_\lambda(w) := \det(W_\lambda)$$

where $W_\lambda$ denotes the $n \times n$ minor of $W$ whose $i$th row (counting from the top down) is $W_{l_i+1}$.

## $S_\lambda$ as Plücker coordinate of the canonical element $\overset{0}{\Phi} \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$

Define the **canonical element** $\overset{0}{\Phi} \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ having matrix elements (homogeneous coordinates) that are monomials in $(x_1, \dots, x_n)$.

$$\overset{0}{\Phi}_{ij} := x_j{}^i$$

with column vectors:

$$\overset{0}{\Phi}{}^{(j)} = \frac{1}{1 - x_j z} = \sum_{i=0}^{\infty} x_j{}^i \mathbf{b}_{i+1}$$

Bi-alternant formula for $S_\lambda$ (Weyl character formula):

$$S_\lambda = \frac{\det(\overset{0}{\Phi}_\lambda)}{\det(\overset{0}{\Phi}_0)} = \frac{\pi_\lambda(\overset{0}{\Phi})}{\pi_{(0)}(\overset{0}{\Phi})}$$

# Jacobi-Trudy formula as change of basis

Define a new basis for the canonical element $\overset{0}{\Phi} \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$

$$\mathbf{h}_i := z^{i-1}\left(\prod_{a=1}^{n}(1 - x_a z)^{-1}\right) = \sum_{j=0}^{\infty} h_j \mathbf{b}_{i+j}$$

$$\overset{0}{H} := \left(\overset{0}{H}{}^{(1)}\ \overset{0}{H}{}^{(2)} \dots \overset{0}{H}{}^{(n)}\right) = \overset{0}{\Phi} T_0$$

$$\overset{0}{H}_{ij} := h_{i+j-n-1}$$

Then the **Jacobi-Trudy identity** is just the equality of Plücker coordinates:

$$S_\lambda = \frac{\pi_\lambda(\overset{0}{\Phi})}{\pi_{(0)}(\overset{0}{\Phi})} = \frac{\pi_\lambda(\overset{0}{H})}{\pi_{(0)}(\overset{0}{H})} = \pi_\lambda(\overset{0}{H}) = \det(\overset{0}{H}_\lambda)$$

## Monic polynomial bases

For any monic polynomial basis for $\mathbf{C}[x]$:

$$\{\phi_i(x),\ \deg \phi_i = i\}_{i\in\mathbf{N}}, \quad \phi_i(x) = \sum_{j=0}^{i} \phi_{i,j} x^j, \quad \Phi_{i,i} = 1,$$

we associate an $\infty \times n$ matrix $\Phi$ with components

$$\Phi_{ij} := \phi_{i-1}(x_j)$$

that determines an element $[\Phi] \in \mathrm{Gr}_n(\mathcal{H}_+^n, \mathbf{F})$. Let $\Lambda :=$ semi-infinite upper triangular shift matrix representing multiplication by $z$:

$$\Lambda : \mathbf{b}_i \to \mathbf{b}_{i+1}.$$

Define

$$X := \mathrm{diag}(x_1, \dots x_n).$$

$\exists!$ semi-infinite upper triangular recursion matrix $J^+$ such that

$$\Phi X = J\Phi, \quad J := \Lambda^t + J^+.$$

# Recursion matrices and dressing transfornations

Define the infinite triangular matrix $A^\phi$, with 1's along the diagonal, whose rows are the coefficients of the polynomials $\{\phi_i\}$,

$$A^\phi_{ij} := \phi_{i-1,j-1} \quad \text{if} \quad i \geq j, \quad A^\phi_{ij} = 0 \quad \text{if} \quad i < j, \quad i, j \in \mathbf{N}^+,$$

We then have

$$\Phi = A^\phi \overset{0}{\Phi}.$$

Since $\overset{0}{\Phi}$ satisfies the recursion relations

$$\overset{0}{\Phi} X = \Lambda^t \overset{0}{\Phi},$$

the intertwining relation may be solved for $J$:

$$A^\phi \Lambda^t = J A^\phi$$
$$J = A^\phi \Lambda^t \left(A^\phi\right)^{-1}.$$

## Generalized Schur functions

Now define [ cf. Sergeev-Veselov (2009)], the following generalizations of the usual Schur functions $S_\lambda(x_1, \dots, x_n)$,

$$S^\phi_\lambda(x_1, \dots, x_n) := \frac{\det(\Phi_\lambda)}{\det(\Phi(0))} = \frac{\pi_\lambda(\Phi)}{\pi_0(\Phi)}.$$

The analogs of the *complete symmetric functions* are denoted

$$h^{(0)}_i := S^\phi_{(i)} \quad \text{for} \quad i \geq 0,$$
$$h^{(0)}_{-i} := 0 \quad 1 \leq i \leq n-1.$$

We may view these as the components of the column $H^{(1)}$ of an $\infty \times \infty$ matrix **H**, whose columns are recursively defined:

$$H^{(j)}_i = JH^{(j-1)}, \quad H^{(j)}_i := H_{ij} := h^{(j-1)}_{i-n},$$

# The generalized Jacobi-Trudi formula

## Theorem

**Generalized Jacobi-Trudi identity**: *The semi-infinite matrices $H$ and $\Phi$ represent the same element $[H] = [\Phi] \in \mathrm{Gr}_n(\mathcal{H}_+, \mathbf{F})$ of the Grassmannian. It follows that their Plucker coordinates coincide:*

$$\det(H_\lambda) = \frac{\det(\Phi_\lambda)}{\det(\Phi_0)} = S^\phi_\lambda,$$

*which is equivalent to the generalized Jacobi-Trudi identity*

$$S^\phi_\lambda = \det\left(h^{(j-1)}_{\lambda_i - i+1}\right)|_{1\le i,j\le \ell(\lambda)}.$$

## Proof.

(**Sketch**)Two methods of proof: 1) Using the recursion relations.

2) Using the dressing transformation $H = A^\phi \overset{0}{H}$. □

# $S^\phi_\lambda$ as KP $\tau$-functions

**Monomial sums**:

$$t_i := \frac{1}{i}\sum_{a=1}^{n} x_a^i,$$
$$[x] = \mathbf{t} = (t_1, t_2, \dots)$$

Define coefficients $\phi^{(n)}_{\lambda\mu}$ as the **Plücker coordinates** $\pi_\mu(C^{(\lambda,n)})$ of the element $C^{(\lambda,n)}$ of the Grassmannian $\mathrm{Gr}_n(\mathcal{H}_+, \mathbf{C})$ spanned by the polynomials $\{\phi_{\lambda_i - i+n}(z)\}_{i=1,\dots n}$

$$\phi^{(n)}_{\lambda\mu} := \pi_\mu(C^{(\lambda,n)}) = \det\left(\phi_{l_i, m_j}\right)|_{1\leq i,j\leq n},$$

where

$$l_i := \lambda_i - i + n, \quad m_j := \mu_j - j + n, \quad i, j \in \mathbf{N}^+,$$

are the **particle coordinates** associated to partitions $\lambda$ and $\mu$.

## Schur function expansions

The following shows that the $\phi^{(n)}_{\lambda\mu}$'s are the coefficients in the expression for $S^{\phi}_{\lambda,n}([x])$ as a linear combination of Schur functions $S_\lambda([x])$.

**Lemma**

$$S^{\phi}_{\lambda,n}([x]) = \sum_{\substack{\mu \\ \ell(\mu)\leq n \\ |\mu|\leq|\lambda|}} \phi^{(n)}_{\lambda\mu} S_\mu([x]).$$

**Corollary**

*$S^{\phi}_{\lambda,n}(\mathbf{t})$ is a KP $\tau$-function.*

**Proof.**

Use the Cauchy-Binet identity. □

# Hilbert Space Grassmannians (Sato-Segal-Wilson )

## Model for Hilbert space

$$\mathcal{H} := L^2(S^1) = \mathcal{H}_+ + \mathcal{H}_-,$$
$$\mathcal{H}_+ = \mathrm{span}\{z^i\}_{i\in\mathbf{N}}, \quad \mathcal{H}_- = \mathrm{span}\{z^{-i}\}_{i\in\mathbf{N}^+},$$

The **infinite dimensional Grassmannian** $Gr_{\mathcal{H}_+}(\mathcal{H})$ is defined as

$$Gr_{\mathcal{H}_+}(\mathcal{H}) = \{\text{closed subspaces } w \subset \mathcal{H} \text{ ``commensurable'' with } \mathcal{H}_+\}$$

i.e., such that **orthogonal projection** to $\mathcal{H}_+$ along $\mathcal{H}_-$

$$\pi^\perp : w \to \mathcal{H}_+$$

is a **Fredholm map** and orthogonal projection to $\mathcal{H}_-$

$$\pi^\perp : w \to \mathcal{H}_-$$

is “small” (e.g. compact). ($\mathcal{H}_+ \in Gr_{\mathcal{H}_+}(\mathcal{H})$ is the “origin".)

# Basis labelling and frames

**Orthonormal basis for $\mathcal{H}$:**

$$\{e_i := z^{-i-1}\}_{i \in \mathbf{Z}},$$

In terms of **frames**, let

$$w = \mathrm{span}\{w_1, w_2, \dots\},$$

and expand the basis vectors $w_i$ in the **orthonormal basis** $\{e_j\}$

$$w_i := \sum_{j \in \mathbf{Z}} W_{ji} e_j.$$

Define **doubly $\infty$ column vectors** $\{\mathbf{W}_i\}_{i=1,2\dots}$ with components

$$(\mathbf{W}_i)_j := W_{ji}$$

and the **rectangular** $2\infty \times \infty$ **matrix** $W$ with columns $\{\mathbf{W}_i\}_{i=1,2\dots}$

$$W := (\mathbf{W}_1, \mathbf{W}_2, \cdots)$$

# Linear and abelian group actions $\Gamma_\pm \subset GL(\mathcal{H})$ on $\mathcal{H}$ and $Gr_{\mathcal{H}_+}(\mathcal{H})$

## $GL(\mathcal{H})$ action on $Gr_{\mathcal{H}_+}(\mathcal{H})$

More generally, we have the **general linear group action**:

$$GL(\mathcal{H}) \times Gr_{\mathcal{H}_+}(\mathcal{H}) \rightarrow Gr_{\mathcal{H}_+}(\mathcal{H})$$
$$(g \in GL(\mathcal{H}), W) \rightarrow gW$$

represented by **doubly infinite, invertible matrices**

$$g = e^A, \quad A \in \mathfrak{gl}(\infty). \quad A = (A_{ij})|_{i,j,\in \mathbf{Z}}$$

## Sato-Segal-Wilson definition of KP $\tau$-functions

For $w =\in Gr_{\mathcal{H}_+}(\mathcal{H})$, the KP $\tau$-function $\tau_w(\mathbf{t})$ is obtained as the Fredholm determinant of the orthogonal projection of $W(\mathbf{t})$ to $\mathcal{H}_+$:

### KP $\tau$-function

$$\tau_w(\mathbf{t}) = \det(\pi^\perp : w(\mathbf{t}) \to \mathcal{H}_+), \quad \mathbf{t} = (t_1, t_2, \dots)$$

or, equivalently if

$$W(\mathbf{t}) = e^{\sum_{i=1}^\infty t_i \Lambda^i} W := \begin{pmatrix} W_+(\mathbf{t}) \\ W_-(\mathbf{t}) \end{pmatrix}, \tag{2.1}$$

$$\tau_w(\mathbf{t}) = \det W_+(\mathbf{t}).$$

The Hirota bilinear relations are equivalent to the Plücker relations defining the image of $Gr_{\mathcal{H}_+}(\mathcal{H})$ under the Plücker map embedding.

$$\mathcal{P} : Gr_{\mathcal{H}_+}(\mathcal{H}) \to \mathcal{F} := \Lambda\mathcal{H}, \quad W = \mathrm{span}(w_1, w_2, \dots) \mapsto w_1 \wedge w_2 \wedge \dots$$

## Fermionic Fock space $\mathcal{F}$

For every partition $\lambda = (\lambda_1, \lambda_2, \dots)$ and integer $N \in \mathbf{Z}$ extended to the extended semi-infinite sequence

$$\lambda = (\lambda_1, \dots \lambda_{\ell(\lambda)}), 0, 0, \dots)$$

define the **"particle positions"**

$$l_j := \lambda_j - j + N$$

The **fermionic Fock space $\mathcal{F}$** is the **exterior space** (orthogonal direct sum of charge $N$ subspaces )

$$\mathcal{F} := \Lambda\mathcal{H} = \bigoplus_{N\in\mathbf{Z}} \mathcal{F}_N.$$

spanned by semi-infinite wedge products (orthonormal basis for $\mathcal{F}_N$)

$$|\lambda, N\rangle := e_{l_1} \wedge e_{l_2} \wedge \cdots$$

Each charge $N$ sector $\mathcal{F}_N$ has a charged **vacuum vector**

$$|0, N\rangle = e_{N-1} \wedge e_{N-2} \wedge \dots,$$

# Fermionic creation and annihilation operators

In terms of the **Orthonormal basis for** $\mathcal{H}$, and **dual basis for** $\mathcal{H}^*$

$$\{e_i := z^{-i-1}\}_{i\in\mathbf{Z}}, \qquad \{\tilde{e}_i\}_{i\in\mathbf{Z}}, \qquad \tilde{e}_i(e_j) = \delta_{ij}$$

define the Fermi **creation and annihilation operators** (exterior and interior muliplication):

$$\psi_i v := e_i \wedge v, \quad \psi_i^\dagger v := i_{\tilde{e}^i} v, \quad v \in \mathcal{H}.$$

These satisfy the usual anti-commutation relations

$$[\psi_i, \psi_j]_+ = [\psi_i^\dagger, \psi_j^\dagger]_+ = 0, \quad [\psi_i, \psi_j^\dagger]_+ = \delta_{ij}.$$

determining the $\infty$ dimensional Clifford algebra of fermionic operators.

# Plücker map and Plücker coordinates

The **Plücker map** $\mathcal{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbf{P}(\mathcal{F})$ into the projectivization of $\mathcal{F}$,

$$\mathcal{P} : \mathrm{span}(w_1, w_2, \dots) \mapsto [w_1 \wedge w_2 \wedge \cdots],$$

embeds $\mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$ in $\mathbf{P}(\mathcal{F})$ as the intersection of an infinite number of quadrics. If orthogonal projection to $\mathcal{H}_+$

$$\pi^{\perp} : w \to \mathcal{H}_+$$

has Fredholm index $N$, $\mathcal{P}(w)$ is in the charge $N$ sector: $\mathcal{P}(w) \subset \mathcal{F}_N$. Expanding in the standard orthonormal basis,

$$\mathcal{P}(w) = w_1 \wedge w_2 \wedge \cdots = \sum_{\lambda} \pi_\lambda(w, N) |\lambda, N>,$$

the coefficients $\pi_\lambda(w, N)$ are the **Plücker cordinates** of $w$ (which satisfy the infinite set of bilinear **Plücker equations**.)

# Fermionic representation of group actions and flows

The **Plücker map**

$$\mathcal{P} : \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathbf{P}(\mathcal{F})$$

interlaces the action of the abelian groups

$$\Gamma_\pm \times \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) \to \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H})$$

with the following representations on $\mathcal{F}$ (and its projectivization)

$$\gamma_\pm(\mathbf{t}) : v \mapsto \hat{\gamma}_\pm(\mathbf{t}) v, \quad \hat{\gamma}_\pm(\mathbf{t}) := e^{\sum_{i=1}^\infty t_i J_{\pm i}}, \quad v \in \mathcal{F}$$

where

$$J_i := \sum_{n\in\mathbb{Z}} \psi_n \psi^\dagger_{n+i}, \quad i \in \mathbb{Z}$$

More generally, if $g = e^A \in GL(\mathcal{H})$, $A \in \mathfrak{gl}(\mathcal{H})$ has the fermionic representation

$$\hat{g} := e^{\sum_{i,j\in\mathbb{Z}} A_{ij} :\psi_i \psi_j^\dagger:},$$

# VEV (vacuum expectations value) representations of $\tau$-functions

## Fermionic representation of KP-chain and 2-Toda $\tau$-function

For $w \in \mathrm{Gr}_{\mathcal{H}_+}(\mathcal{H}) = g(\mathcal{H}_+)$, $g \in GL(\mathcal{H})$, with $\mathcal{P}(w) \subset \mathcal{F}_N$ in the charge-$N$ sector, the KP chain $\tau$-function has the **fermionic representation**:

$$\tau_w(\mathbf{t}, N) = \langle N|\hat{\gamma}_+(\mathbf{t})\hat{g}|N\rangle =: \tau_g(\mathbf{t}, N)$$

Similarly, for the **2-Toda $\tau$-function**:

$$\tau_w^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N) = \langle N|\hat{\gamma}_+(\mathbf{t})\hat{g}\hat{\gamma}_-(\tilde{\mathbf{t}})|N\rangle := \tau_g^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N)$$

## Schur function expansions

It follows that we have the **Schur function expansions**

$$\tau_g(\mathbf{t}, N) = \sum_\lambda \pi_\lambda(g(\mathcal{H}_+), N) s_\lambda(\mathbf{t}),$$
$$\tau_g^{(2)}(\mathbf{t}, \tilde{\mathbf{t}}, N) = \sum_\lambda \sum_\mu B_{\lambda,\mu}(g, N) s_\lambda(\mathbf{t}) s_\mu(\tilde{\mathbf{t}}).$$

where

$$\pi_\lambda(g(\mathcal{H}_+), N) = \langle \lambda, N | \hat{g} | N \rangle$$
$$B_{\lambda,\mu}(g, N) = \langle \lambda, N | \hat{g} | \mu, N \rangle$$

are the Plücker coordinates along the basis direction $|\lambda, N\rangle$ .

# Fermionic representation of the matrices $H$ and $E$

## Fermionic representation of the matrices $H$ and $E$

We may express the infinite lower triangular matrix with coefficients $(\phi_{ij})_{i,j\in\mathbf{N}}$ that are all equal to 1 on the diagonal as the exponential of a strictly lower triangular matrix $\alpha$

$$\phi = e^{\alpha}$$

with matrix elements $(\alpha_{ij})_{i,j\in\mathbf{N}}$ satisfying

$$\alpha_{ij} = 0 \quad \text{if} \quad j \geq i.$$

The fermionic representation of this group element is then

$$g_\phi := \exp \sum_{i>j\geq 0} \alpha_{ij}\psi_i\psi_j^\dagger.$$

## Theorem

*The matrix elements $H_i^{(j)}$ and $E_{(i)}^j$ are given as fermionic matrix elements as follows*

$$H_i^{(j)} = \langle (i+j-n-1); n-j+1 | g_\phi \gamma_-([x]) | n-j+1 \rangle = h_{i-n}^{(j-1)}$$
$$E_{(i)}^j = (-1)^{n-i-j+1} \langle (1)^{n-i-j+1}; n-i | g_\phi \gamma_-([x]) | n-i \rangle = (-1)^{n-i-j+1} e_{(-i)}^{n-j+1}$$

## Theorem

$$\phi_{\lambda\mu}^{(n)} = \langle \lambda; n | g_\phi | \mu; n \rangle = \langle \mu; n | g_{\phi^t} | \lambda; n \rangle .$$

## Corollary

$$S_{\lambda,n}^{\phi}([x]) = \langle \lambda; n | g_\phi \gamma_-([x]) | n \rangle = \langle n | \gamma_+([x]) g_{\phi^t} | \lambda; n \rangle .$$

## 2D-Toda chain of $\tau$-functions

The polynomials $S^{\phi}_{\lambda}([x])$ can themselves be used as coefficients in a Schur function expansion to define a family of KP $\tau$-functions, in which the indeterminates $(x_1, \dots x_n)$ are interpreted as complex parameters

$$\tau_{\phi}(n, \mathbf{t}, [x]) := \sum_{\lambda} S^{\phi}_{\lambda,n}([x]) S_{\lambda}(\mathbf{t}).$$

Here, the KP flow parameters $\mathbf{t} = (t_1, t_2, \dots)$ are independent and

$$\mathbf{s} := (s_1, s_2, \dots) := [x]$$

may be viewed as a second set of flow parameters.

**Theorem**

*The functions $\tau_{\phi}(n, \mathbf{t}, \mathbf{s})$ form a 2D-Toda chain of $\tau$-functions which may be expressed fermionically as*

$$\tau_{\phi}(n, \mathbf{t}, \mathbf{s}) = \langle n | \gamma_+(\mathbf{t}) g_{\phi} \gamma_-(\mathbf{s}) | n \rangle.$$

# Nontriangular 2-D Toda $\tau$-functions

We may furthermore choose a pair of polynomial systems $\{\phi_i\}_{i\in\mathbf{N}}$ and $\{\theta_i\}_{i\in\mathbf{N}}$ and associate to them the generalized Schur functions $S^{\phi}_{\lambda,n}$ and $S^{\theta}_{\lambda,n}$. Forming the sum of their products

$$\tau_{\phi,\theta}(n,\mathbf{t},\mathbf{s}) := \sum_{\lambda} S^{\phi}_{\lambda,n}(\mathbf{s}) S^{\theta}_{\lambda,n}(\mathbf{t}),$$

we obtain a more general, nontriangular form of 2D Toda $\tau$-functions.

**Theorem**

$$\tau_{\phi,\theta}(n,\mathbf{t},\mathbf{s}) = \langle n|\gamma_{+}(\mathbf{t}) g^{t}_{\theta} g_{\phi} \gamma_{-}(\mathbf{s})|n\rangle.$$

# Examples and applications

## Example

### 1. Character expansions of classical groups.

The subgroup reductions:
$Sp(n) \supset U(n), \quad O(2n) \supset U(n), \quad O(2n+1) \supset U(n)$ give rise to the Littlewood character expansions

$$\chi_\lambda^{Sp(2n)}(x_1, \ldots, x_n) = \sum_{\substack{\ell(\mu) \leq n \\ \mu \leq |\lambda|}} \sum_{\alpha} (-1)^{|\alpha|} C^\lambda_{D'(\alpha),\mu} S_\mu(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$$

$$\chi_\lambda^{SO(2n)}(x_1, \ldots, x_n) = \sum_{\substack{\ell(\mu) \leq n \\ \mu \leq |\lambda|}} \sum_{\alpha} (-1)^{|\alpha|} C^\lambda_{D(\alpha),\mu} S_\mu(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$$

$$\chi_\lambda^{SO(2n+1)}(x_1, \ldots, x_n) = \sum_{\substack{\ell(\mu) \leq n \\ \mu \leq |\lambda|}} \sum_{\alpha} (-1)^{|\alpha|} C^\lambda_{D(\alpha),\mu} S_\mu(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}$$

where the sums in $\alpha$ are taken over strict partitions.

## Example

**Maximal tori** of $O(2n), Sp(2n) \subset U(2n)$ or $O(2n+1) \subset U(2n+1)$ consist of elements of the form $\mathrm{diag}(x_1, \dots, x_n, x_1^{-1}, \dots x_n^{-1})$ or $\mathrm{diag}(x_1, \dots, x_n, x_1^{-1}, \dots x_n^{-1}, 1)$.

Systems of orthogonal polynomials $\{\phi_i^{Sp(2n)}(z)\}$, $\{\phi_i^{SO(2n)}(z)\}$ and $\{\phi_i^{SO(2n+1)}(z)\}$ are expressed in terms of the variables $z := x + x^{-1}$

$$\phi_i^{Sp(2n)}(z) = \sum_{j=0}^{i} x^{i-2j} = \sum_{j=0}^{i} \phi_{ij}^{Sp(2n)} z^j,$$

$$\phi_i^{SO(2n)}(z) = x^i + x^{-i} = \sum_{j=0}^{i} \phi_{ij}^{SO(2n)} z^j,$$

$$\phi_i^{SO(2n+1)}(z) = \sum_{j=0}^{2i} x^{i-j} = \sum_{j=0}^{i} \phi_{ij}^{SO(2n+1)} z^j,$$

As noted by **Sergeev-Veselov**, the corresponding generalized Schur functions coincide with the irreducible characters

$$S_\lambda^{Sp(2n)}(z_1, \dots, z_n) = \chi_\lambda^{Sp(2n)}(x_1, \dots, x_n)$$
$$S_\lambda^{SO(2n)}(z_1, \dots, z_n) = \chi_\lambda^{SO(2n)}(x_1, \dots, x_n)$$
$$S_\lambda^{SO(2n+1)}(z_1, \dots, z_n) = \chi_\lambda^{SO(2n+1)}(x_1, \dots, x_n)$$

where

$$z_i := x_i + x_i^{-1}. \quad i = 1, \dots, n$$

and

$$S_\lambda^{G}(z_1, \dots, z_n) := S_\lambda^{\phi^G}(z_1, \dots, z_n)$$

for $G = Sp(2n),\ SO(2n),\ \text{or}\ SO(2n+1)$.

For any pair of partitions $(\lambda, \mu)$, , define

$$\phi^G_{\lambda\mu} := \det\left(\phi^G_{\lambda_i - i + n, \mu_j - j + n}\right)_{1 \le i, j, \le n}.$$

for $G = Sp(2n),\ SO(2n),\ \text{or}\ SO(2n+1)$. By the above Schur expansionsm we have

$$S^{Sp(2n)}_\lambda(z_1, \dots, z_n) = \sum_{\substack{\ell(\mu) \le n \\ \mu \le |\lambda|}} \phi^{Sp(2n)}_{\lambda\mu} S_\mu(z_1, \dots, z_n),$$

$$S^{SO(2n)}_\lambda(z_1, \dots, z_n) = \sum_{\substack{\ell(\mu) \le n \\ \mu \le |\lambda|}} \phi^{SO(2n)}_{\lambda\mu} S_\mu(z_1, \dots, z_n),$$

$$S^{SO(2n+1)}_\lambda(z_1, \dots, z_n) = \sum_{\substack{\ell(\mu) \le n \\ \mu \le |\lambda|}} \phi^{SO(2n+1)}_{\lambda\mu} S_\mu(z_1, \dots, z_n).$$

## Example

**2. Moment matrices and 1-matrix models.**

Factorization of the **Hankel matrix of moments**:

$$M_{ij} := \int_\Gamma d\mu(z) z^{i+j}$$

of some measure $d\mu$, supported on a curve $\Gamma$ in the complex plane:

$$\sum_{k=\max(i,j)}^{\infty} \phi_{ki}\phi_{kj} = M_{ij}, \quad i, j \in \mathbf{N}.$$

$$\tau_{\phi,\phi}(n, \mathbf{t}, \mathbf{0}) = \frac{1}{n!} \left( \prod_{a=1}^{n} \int_\Gamma d\mu(z_a) \right) \Delta^2(\mathbf{z}) \exp\left( \sum_{j=1}^{\infty} \sum_{a=1}^{n} t_j z_a^j \right)$$

$$= \sum_{\lambda,\, \ell(\lambda) \le n} B_{\lambda,n}(d\mu) S_\lambda(\mathbf{t}) = \sum_{\lambda,\, \ell(\lambda) \le n} S^\phi_\lambda(\mathbf{0}) S^\phi_\lambda(\mathbf{t}),$$

where

$$\Delta(\mathbf{z}) = \prod_{i<j}^{n}(z_i - z_j)$$

is Vandermonde determinant and

$$B_{\lambda,n}(d\mu) := \det(M_{\lambda_i - i + j})_{1\le i,j\le n}$$

$$= \sum_{\nu,\, \ell(\nu)\le n} \phi^{(n)}_{\nu\lambda}\phi^{(n)}_{\nu(0)}$$

## Example

### 3. Bimoments of two variable measures and 2-matrix models.

We may choose the product of matrices $\theta^t\phi$ entering in the 2D-Toda chain of $\tau$-functions $\tau_{\theta,\phi}(n,\mathbf{t},\mathbf{s})$ to be the upper/lower factorization of the **matrix of bimoments**

$$M_{ij} := \sum_{p,q} c_{pq} \int_{\Gamma_p} \int_{\tilde{\Gamma_q}} d\mu(z,w) z^i w^j = (\theta^t\phi)_{ij}$$

of a two-variable measure. The 2D-Toda $\tau$-function $\tau_{\theta,\phi}(n,\mathbf{t},\mathbf{s})$ is then the **partition function of a coupled two-matrix model**:

$$\begin{aligned}\tau_{\theta,\phi}(n,\mathbf{t},\mathbf{s}) &= \frac{1}{n!}\prod_{a=1}^{n}\left(\int\int d\mu(z_a,w_a) e^{\sum_{i=1}^{\infty} t_i z_a^l} e^{\sum_{j=1}^{\infty} s_j w_a^j}\right)\Delta(\mathbf{z})\Delta(\mathbf{w}) \\ &= \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} \sum_{\substack{\nu \\ \ell(\nu)\leq n}} B_{\lambda,\nu,n}(d\nu) S_\lambda(\mathbf{t}) S_\nu(\mathbf{s}) = \sum_{\substack{\lambda \\ \ell(\lambda)\leq n}} S_\lambda^\psi(\mathbf{t}) S_\lambda^\phi(\mathbf{s}),\end{aligned}$$

$$B_{\lambda,\nu,n}(d\mu) := \det(M_{\lambda_i-i+n,\,\nu_j-j+n})_{1\leq i.j\leq n} := \sum_{\rho,\,\ell(\rho)\leq n} \psi_{\rho\lambda}^{(n)} \phi_{\rho\nu}^{(n)}.$$

# Background and related work

## Fermionic approach to $\tau$-functions

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