Weyl Orbit Functions

and

Conformal Field Theory

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IDEA

?

Discretized Weyl-orbit functions

$$S_\lambda(a) = \sum_{w\in W} (\det w) e^{2\pi i \langle w\lambda, a\rangle}$$

for example

$\Leftarrow$

$\Rightarrow$

Affine modular data, modular S-matrix of WZNW conformal field theory

$$S_{\lambda,\mu} \propto \sum_{w\in W} (\det w) e^{-2\pi i \langle w\lambda, \mu\rangle / M'}$$

... ?

Incidentally, the relation between affine modular data and Weyl-orbit functions, of both S and C type, was exploited in Gannon-Jakovljevic-Walton 1995.

Simple Lie algebra weight multiplicities were extracted using these objects and their symmetries.

Quella has done work similar in spirit (2002).

OUTLINE

- (Weyl-)Orbit Functions
- Affine (Wess-Zumino-Novikov-Witten) Modular Data
- Modified Multiplication, Galois Symmetry
- Conclusion

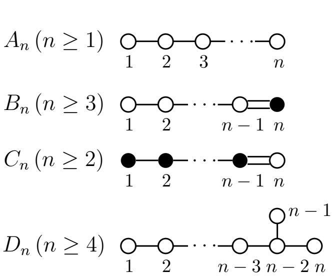
Notation

G compact, simple Lie group, rank n, Lie algebra g_n

Simple roots: $\Pi = \{\alpha_j \,|\, j \in \{1, \ldots, n\} =: I\}$, normalized $|\alpha_{\text{long}}|^2 = 2$.

Primitive reflections $r_{\alpha_j} = r_j$ generate Weyl group $W = \langle\, r_j \mid j \in I \,\rangle$

Coxeter-Dynkin diagram of g_n encodes W.



Notation

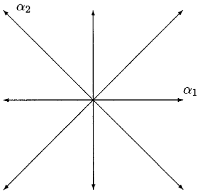
Fundamental weights $\Phi = \{\omega_j \mid j \in I\}$
weight space $P_{\mathbb{R}} := \mathbb{R}\,\Phi \cong \mathbb{R}^n$

Non-negative weights: $P_{+,\mathbb{R}} := \mathbb{R}_{\geq 0}\,\Phi$;
positive weights $P_{++,\mathbb{R}} := \mathbb{R}_{>0}\,\Phi$

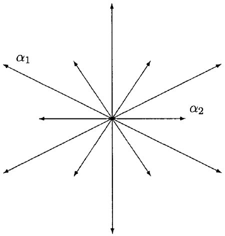
Weight lattice $P := \mathbb{Z}\,\Phi$;
dominant (regular) integral weights:
$P_{+} := \mathbb{N}_0\,\Phi$ ($P_{++} := \mathbb{N}\,\Phi$).

Roots Δ, long (short) roots Δ^{ℓ} (Δ^{s}),
positive roots: Δ_{+}, etc.

Root system of C_2:



Root system of G_2:



ORBIT FUNCTIONS

Weyl-orbit functions: program of study as "special functions"

initiated with Patera 2003; general, systematic study

launched with Klimyk-Patera 2006-08.

C-functions, or Weyl-orbit sums, are

$$C_\lambda(a) \ = \ \sum_{w \in W} e^{2\pi i \langle w\lambda, a \rangle} \ ,$$

where $\lambda, a \in P_{\mathbb{R}}$.

So-named since $g_n = A_1 \ \rightarrow \ C_\lambda(a) = 2\,\cos\left(2\pi i \langle \lambda, a \rangle\right)$

W-invariance: $C_{w\lambda}(a) \ = \ C_\lambda(wa) \ = \ C_\lambda(a) \ , \ \forall w \in W \ .$

Restrict to λ, $a \in P_{+,\mathbb{R}}$, the fundamental region of W.

S-functions are the antisymmetric Weyl-orbit sums

$$S_\lambda(a) \;=\; \sum_{w\in W} (\det w)\, e^{2\pi i\langle w\lambda, a\rangle}\;.$$

$g_n = A_1 \;\to\; S_\lambda(a) = 2i\,\sin\left(2\pi i\langle\lambda, a\rangle\right)$

W-antisymmetry: $S_{w\lambda}(a) = S_\lambda(wa) = (\det w)\, S_\lambda(a)\,,\;\forall w\in W\,.$

S-functions vanish on $\partial P_{+,\mathbb{R}}$, $\mathrm{int}(P_{+,\mathbb{R}}) =: \tilde{P}_{+,\mathbb{R}}$.

Other "trig" possibility:

$$E_\lambda(a) \;=\; \sum_{w\in W_e} e^{2\pi i\langle w\lambda, a\rangle}$$

$W_e \subset W$ subgroup of even elements.

$g_n = A_1 \;\to\; E_\lambda(a) = e^{2\pi i\langle w\lambda, a\rangle}$.

Non-simply-laced algebras, generalize to S^ℓ- and S^s-functions, via

$$\varphi_\lambda^\sigma(a) \;=\; \sum_{w\in W} \sigma(w)\, e^{2\pi i\langle w\lambda,a\rangle} \;.$$

$\sigma(w)$ is a sign homomorphism on W.

$\sigma = 1$, det $\Rightarrow$ C-, S-functions, respectively.

$\sigma^s(r_i) = -1$ $(+1)$, if α_i is a short (long) simple root, respectively.

Opposite definition for σ^ℓ.

$\sigma = \sigma^s$, σ^ℓ $\;\Rightarrow\;$ $\varphi = S^s$-, S^ℓ-functions, respectively.

Ratios $\Rightarrow$ characters, and "hybrid characters":

$$\chi_\lambda(a) = \frac{S_{\lambda+\rho}(a)}{S_\rho(a)}, \quad \chi^\ell_\lambda(a) = \frac{S^\ell_{\lambda+\rho^\ell}(a)}{S^\ell_{\rho^\ell}(a)}, \quad \chi^s_\lambda(a) = \frac{S^s_{\lambda+\rho^\ell}(a)}{S^s_{\rho^\ell}(a)};$$

where

$$\rho^\ell = \frac{1}{2}\sum_{\alpha\in\Delta^\ell_+}\alpha = \sum_{\alpha_i\in\Pi\cap\Delta^\ell_+}\omega_i \quad ;$$

ρ^s similarly; and ρ is the usual Weyl vector.

Focus here on the C- and S-functions, and characters χ.

Properties of orbit functions

Restricting weight-labels $\lambda \in P = \mathbb{Z}\Phi \;\Rightarrow\;$ affine Weyl symmetry:

$$C_\lambda(wa) \;=\; C_\lambda(a)\;, \quad S_\lambda(wa) \;=\; (\det w)\, S_\lambda(a)\;,$$

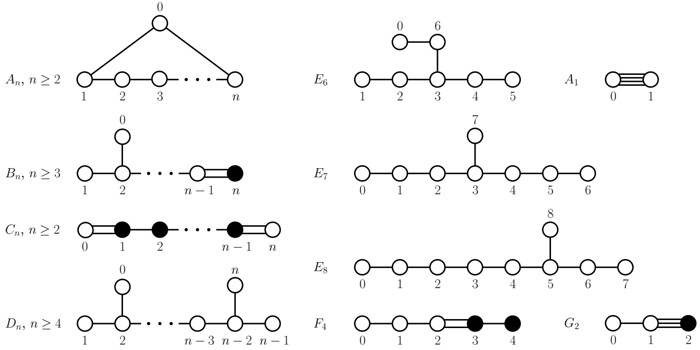
$$\text{or} \qquad \varphi^\sigma_\lambda(wa) \;=\; \sigma(w)\, \varphi^\sigma_\lambda(a)\;, \qquad \forall w \in W^{\mathrm{aff}}\;.$$

Affine Weyl group $W^{\mathrm{aff}} \;=\; Q^\vee \rtimes W \;=\; \langle r_j \mid j \in \hat{I} \rangle\;,$

where $\hat{I} := \{0, 1, \ldots, n\}$.

0-th simple reflection: $r_0\, a \;=\; r_\xi a + \frac{2\xi}{\langle \xi, \xi \rangle}\,,\;$ ξ highest root of g_n.

Extended Coxeter-Dynkin diagram encodes the affine Weyl group W^{aff}:



Fundamental domain of W^{aff},

$$F = \mathrm{Conv}\left\{0, \frac{\omega_1^\vee}{m_1}, \dots, \frac{\omega_n^\vee}{m_n}\right\}.$$

$m_j \in \mathbb{N}$ known as *marks*, and $\xi = m_1\alpha_1 + \cdots + m_n\alpha_n$. Put $m_0 = 1$.

$\Phi^\vee := \{\omega_1^\vee, \dots, \omega_n^\vee\}$ denote the dual fundamental weights, satisfying $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{i,j}$.

They are the fundamental weights of the dual Lie algebra $g_n^\vee$.

Short $\leftrightarrow$ long roots: Coxeter-Dynkin diagram of $g_n \Rightarrow$ diagram of $g_n^\vee$.

Dual roots $\alpha_j^\vee$ satisfy $\langle \alpha_i^\vee, \omega_j \rangle = \delta_{i,j}$ $(i, j \in I)$.

Reflections $r_j^\vee = r_j$ $(j \in I)$ associated with the $\alpha_i^\vee$ generate the same Weyl group W, but a different affine Weyl group arises:

$\widehat{W}^{\rm aff} \;=\; Q \rtimes W$.

$$r_0^\vee\, a \;=\; r_\eta\, a + \frac{2\eta}{\langle \eta, \eta \rangle} \;.$$

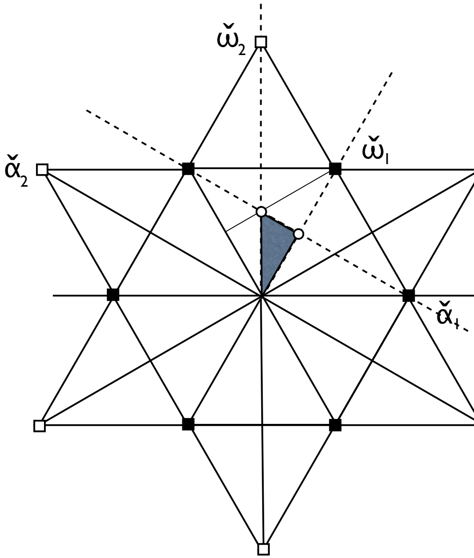
Highest dual root $\eta =: -\alpha_0^\vee = m_1^\vee \alpha_1^\vee + \ldots + m_n^\vee \alpha_n^\vee$

defines the *dual marks* $m_j^\vee,\ j \in I$; put $m_0^\vee = 1$.

$\lambda \in P \;\Rightarrow$ consider $\varphi_\lambda^\sigma(a)$ a function on the fundamental domain F of $W^{\rm aff}$.

Orbit function φ_λ^σ is an eigenfunction of the Laplace operator on F with Neumann (Dirichlet) boundary conditions on $\sigma(r) = +1$ $(\sigma(r) = -1)$ hyperplanes.

$g_n = G_2$ example of a fundamental region F, shaded here:



Dual affine Weyl symmetry:

Restricting weight-arguments $a \in P^\vee = \mathbb{Z}\Phi^\vee \;\Rightarrow\;$ affine dual Weyl symmetry:

$$C_{w\lambda}(a) \;=\; C_\lambda(a)\;, \quad S_{w\lambda}(a) \;=\; (\det w)\, S_\lambda(a)\;,$$

$$\text{or} \qquad \varphi^\sigma_{w\lambda}(a) \;=\; \sigma(w)\,\varphi^\sigma_\lambda(a)\;, \qquad \forall w \in \widehat{W}^{\mathrm{aff}}\;.$$

The fundamental region of $\widehat{W}^{\mathrm{aff}}$, or dual fundamental domain, is $F^\vee = \mathrm{Conv}\{0, \frac{\omega_1}{m_1^\vee}, \dots, \frac{\omega_n}{m_n^\vee}\}$.

Generalizations of Chebyshev polynomials

(Nesterenko-Patera-Tereszkiewicz 2011):

Polynomials constructed 3 ways:

- Most familiar: $X_j := e^{2\pi i x_j}$

- When C_λ can be written as a sum of cosines, then it can be written in terms of basic ones using trig identities. Then these basic cosines can become the new variables.

- $X_j := C_{\omega_j}(x)$ and $S := S_\rho(x)$.

Decompose $X_j\, C_\lambda \;\rightarrow$ recursion relations defining the polynomials.

(Also $X_j := S_{\omega_i}(x)$ and $X_j := \chi_{\omega_i}(x)$, of course.)

Continuous orthogonality:

$$\langle C_\lambda, C_{\lambda'} \rangle \;:=\; |F|^{-1} \int_F C_\lambda(x) \overline{C_{\lambda'}(x)}\, dx \;=\; |W\lambda|\, \delta_{\lambda,\lambda'}$$

$$\text{for } \lambda, \lambda' \in P_+ \ .$$

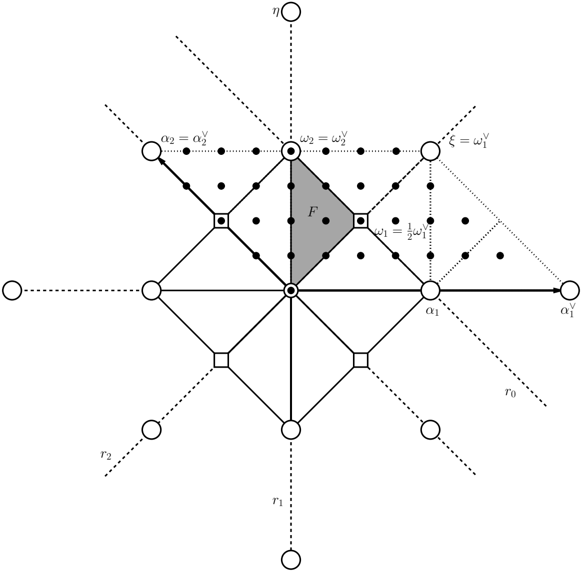
Discrete orthogonality:

$$\langle f, g \rangle_M \;:=\; |W| \sum_{x \in F_M} f(x)\, \overline{g(x)}$$

where $F_M \;:=\; \frac{1}{M} P^\vee / Q^\vee \cap F$ is the discretized fundamental domain, or "grid"; M controls the fineness, or resolution of the discretization.

$g_n = C_2$ example: grid $F_4 = \frac{1}{4}P^\vee/Q^\vee \cap F$.

Dots in $\frac{1}{4}P^\vee/Q^\vee$, and $|F_4| = 9$ (fundamental region F shaded).



Discrete orthogonality (Moody-Patera 2006):

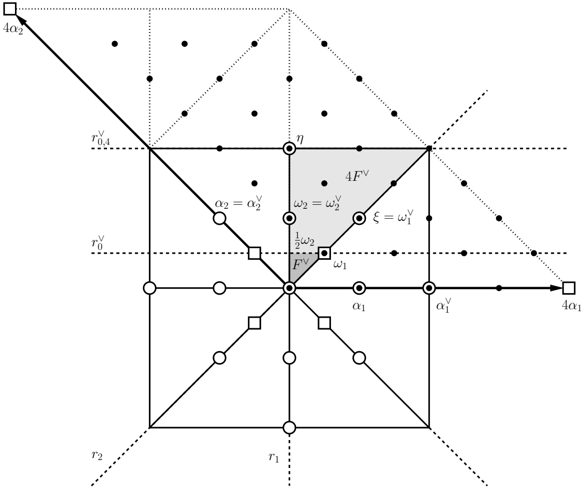
$$\langle C_\lambda, C_{\lambda'} \rangle_M \ = \ c_G \, M^n \, |W\lambda| \, \delta_{\lambda,\lambda'} \ ,$$

$$\text{for} \ \ \lambda, \lambda' \in \Lambda_M \ := \ MF^\vee \cap P/MQ \ .$$

$c_G = \det(\text{C}\mathit{artan\ matrix}) =$ order of the centre of G.

$g_n = C_2$ example: "dual grid" $\Lambda_4 = 4F^\vee \cap \frac{P}{4Q}$.

Dots $\in\ 4F^\vee \cup \frac{P}{4Q}$, $|\Lambda_4| = |F_4| = 9$.



(Finite) orbit function transforms:

Consider data with support F, or $\tilde{F}$. The Weyl-orbit functions provide useful expansion bases for the analysis of functions on F, or $\tilde{F}$.

Digitized data, the values on the grid F_M in F, or $\tilde{F}_M$ in $\tilde{F}$. $M \in \mathbb{N}$ will determine the resolution $\sim 1/M$ of the digital data of interest.

Interpolate, by requiring

$$f(x) \ = \ \sum_{\lambda \in \Lambda_M} \mathcal{F}_\lambda \, C_\lambda(x) \ , \quad \forall \, x \in F_M \ .$$

Discrete orthogonality $\Rightarrow$ Discrete C-Transform

$$\mathcal{F}_\lambda \ = \ \frac{1}{c_G \, M^n |W|} \sum_{x \in F_M} f(x) \, \overline{C_\lambda(x)} \ .$$

Cubature = quadrature in higher dimensions

Cubature formula: a weighted sum of function evaluations used to approximate a multivariate integral.

$$\int_{\Omega} f(x)\, w(x)\, dx \;\approx\; \sum_{j=1}^{N} w_j\, L_j[f]\;,$$

$$L_j[f] \,=\, f(x_j)\,,\ \text{e.}g.$$

Exact ($\approx\, \to\, =$) cubature formulas are possible if one restricts to a certain class of functions.

Such exact cubature formulas are found using the orbit functions (H. Li & Y. Xu 2010, Moody-Patera 2011).

Define domain $\Omega = \{ (X_1(x), \ldots, X_n(x)) \ : \ x \in \tilde{F} \} \ \subset \ \mathbb{C}^n$.

Put m-degree of X_i as $m_i^\vee$, then m-deg(S_ρ)$= h - 1$.

For any function f on Ω, define

$$\tilde{f}(x) \ := \ f\,(\,\chi_{\omega_1}(x), \ldots, \chi_{\omega_n}(x)\,) \ .$$

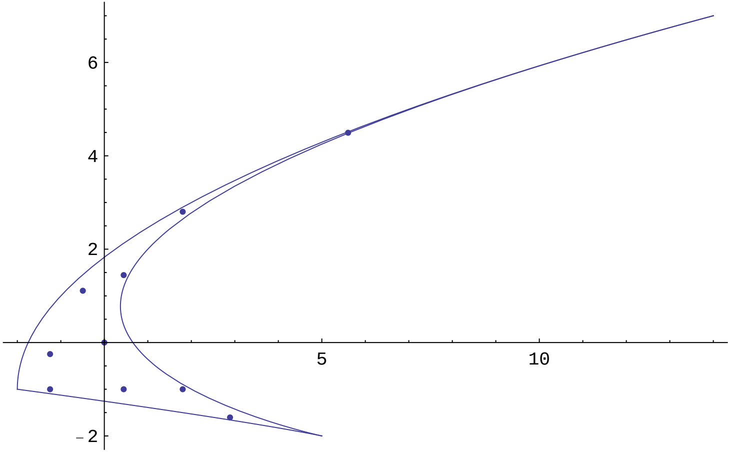
Cubature formula:

Polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ with m-degree $\leq 2M+1$, have

$$\int_\Omega f\, K^{1/2}\, dX_1 \cdots dX_n \ = \ \frac{1}{c_G} \left(\frac{2\pi}{M+h} \right)^n \sum_{x \in \tilde{F}_{M+h}} \tilde{f}(x)\, \tilde{K}(x) \ .$$

Here $K(x) := |S_\rho(x)|^2$ and the Coxeter number $h = \sum_{j=0}^n m_j$.

Example of domain Ω for cubature formula ($M = 8$ for G_2):



Wess-Zumino-Novikov-Witten conformal field theories: AFFINE MODULAR DATA

Objects related to the modular data of Wess-Zumino-Novikov-Witten (WZNW) models bear a striking resemblance to the discretized Weyl-orbit functions. This modular data is associated with the affine Kac-Moody algebras of untwisted type, at a fixed level, and so is also called affine modular data.

As for any RCFT, the WZNW 1-loop partition function is invariant under the modular group $SL(2;\mathbb{Z})$, with generators S and T.

genus-1 conformal blocks $\cong$ characters of the untwisted affine Kac-Moody algebra at fixed *level*, sometimes denoted $g_{n,k}$.

The affine characters form a finite-dimensional representation of the modular group (Kac-Peterson 1984).

Put $\quad P_+^M = \{\sum_{i\in\hat{I}} \lambda_i\omega_i \,|\, \lambda_i \in \mathbb{N}_0,\, \sum_{i\in\hat{I}} \lambda_i m_i^\vee = M\}$,

and $\quad P_{++}^M = \{\sum_{i\in\hat{I}} \lambda_i\omega_i \,|\, \lambda_i \in \mathbb{N},\, \sum_{i\in\hat{I}} \lambda_i m_i^\vee = M\}$.

Similarly, $\quad P_+^{\vee M} = \{\sum_{i\in\hat{I}} \lambda_i\omega_i^\vee \,|\, \lambda_i \in \mathbb{N}_0,\, \sum_{i\in\hat{I}} \lambda_i m_i = M\}$,

and $\quad P_{++}^{\vee M} = \{\sum_{i\in\hat{I}} \lambda_i\omega_i^\vee \,|\, \lambda_i \in \mathbb{N},\, \sum_{i\in\hat{I}} \lambda_i m_i = M\}$.

Grid $F_M \;:=\; P_+^{\vee\, M}/M$. Grid interior $\tilde{F}_M :=\; P_{++}^{\vee\, M}/M$.

Dual grid $\Lambda_M = P_+^M$; interior $\tilde{\Lambda}_M = P_{++}^M$.

Note that $m_0 + m_1 + \ldots + m_n \;=\; m_0^\vee + m_1^\vee + \ldots + m_n^\vee \;=\; h$, the Coxeter number.

Highest root of g_n: $\xi \ = \ \sum_{j\in I} m'_j \, \alpha_j^\vee$, m'_j are known as co-marks.

Dual Coxeter number $h^\vee := 1 + \sum_{j\in I} m'_j$.

Let Δ_+ denote the set of positive roots of g_n and put $M' := k + h^\vee$.

The affine characters can be labeled by weights in $P_{++}^{M'}$, and the

Kac-Peterson modular S matrix has the form

$$S_{\lambda',\mu'} \ = \ R_{M'} \sum_{w\in W} (\det w) \, e^{-2\pi i \langle w\lambda', \mu' \rangle / M'} \quad ,$$

for $\lambda', \mu' \in P_{++}^{M'}$.

But $P_{++}^{M'} = P_+^k + \rho$; rewrite as

$$S_{\lambda,\mu} \;=\; R_{k+h^\vee} \sum_{w\in W} (\det w)\, \exp\left\{ \frac{-2\pi i \langle w(\lambda+\rho), \mu+\rho\rangle}{k+h^\vee} \right\}$$
$$= \; R_{k+h^\vee}\, S_{\lambda+\rho}\left(\frac{\mu}{k+h^\vee} \right) \;,$$

with $\lambda, \mu \in P_+^k$, and $R_{k+h^\vee} = i^{\|\Delta_+\|}\, |P/Q^\vee|^{-\frac{1}{2}}\, (k+h^\vee)^{-\frac{r}{2}}$.

Modular S-matrix is unitary and symmetric, with symmetric affine Weyl symmetry:

$$S_{w.\lambda,\mu} \;=\; S_{\lambda,w.\mu} \;=\; (\det w)\, S_{\lambda,\mu} \;, \quad \forall\, w \in W^{\mathrm{aff}} \;;$$

shifted action $w.\lambda := w(\lambda+\rho) - \rho$.

Modified multiplication

The affine Weyl symmetry $\to$ modified (truncated) multiplication of characters.

Verlinde formula:

$$\left(\frac{S_{\lambda,\sigma}}{S_{\rho,\sigma}}\right)\left(\frac{S_{\mu,\sigma}}{S_{\rho,\sigma}}\right) = \sum_{\nu\in P_+^k} {}^{(k)}N_{\lambda,\mu}^{\nu}\left(\frac{S_{\nu,\sigma}}{S_{\rho,\sigma}}\right) \quad .$$

The ratios are (discretized) Weyl characters of integrable, highest-weight representations of g_n:

$$\left(\frac{S_{\lambda,\sigma}}{S_{\rho,\sigma}}\right) = \chi_\lambda(\sigma) \; .$$

Products of characters decompose as tensor products of representations do:

$$\chi_\lambda(\sigma)\,\chi_\mu(\sigma) \;=\; \sum_{\phi\in P_+} T^{\phi}_{\lambda,\mu}\,\chi_\phi(\sigma)\;,$$

where $T^{\phi}_{\lambda,\mu}$ is the tensor product coefficient. Using the affine Weyl symmetry to compare this with the Verlinde formula $\rightarrow$ (Kac-W)

$${}^{(k)}N^{\nu}_{\lambda,\mu} \;=\; \sum_{w\in W^{\mathrm{aff}}} (\det w)\; T^{w.\nu}_{\lambda,\mu}\;.$$

The affine Weyl symmetry valid when the character is discretized results in a modified multiplication, so that the tensor product coefficients are modified, truncated to the fusion coefficients.

The modified multiplications of discretized Weyl-orbit functions will work essentially the same way.

Galois symmetry

Galois symmetry is an important property of all RCFTs (Coste-Gannon 1994). The Galois symmetry of WZNW models is the motivation for a Galois symmetry of discretized orbit functions.

Kac-Peterson $S_{\lambda,\mu}$ is a linear combination of roots of unity $\exp\{-2\pi i\langle w(\lambda+\rho), \mu+\rho\rangle/(k+h^{\vee})\} =: \varphi$ with rational coefficients. Let $N \in \mathbb{N}$ denote the minimum such that $\varphi^N = 1$ for all such φ.

Suppose $\gcd(\ell, N) = 1$ for $\ell \in \mathbb{N}$. Define the Galois transformation t_ℓ by

$$t_\ell\,(\varphi) := \ \varphi^{\ell} \ \ ,$$

and extend it linearly to sums of such terms. This transformation swaps one primitive root of unity for another.

Equivalently, it maps $\lambda + \rho$ to $\ell(\lambda + \rho)$, possibly outside the dominant sector. Transforming back is possible, however, using the affine Weyl group:

$$\ell(\lambda + \rho) \ = \ w_\ell[\lambda]\ (t_\ell[\lambda] + \rho) \ \ , \ \ \ w_\ell[\lambda] \in W^{\mathrm{aff}} \ .$$

The Galois symmetry of the modular S-matrix then follows:

$$t_\ell\,(\,S_{\lambda,\mu}\,) \ = \ \epsilon_\ell[\lambda]\,S_{t_\ell[\lambda],\mu} \ = \ \epsilon_\ell[\mu]\,S_{\lambda,t_\ell[\mu]}$$

ORBIT FUNCTIONS

The affine Weyl symmetry of the orbit functions can be summarized as follows. With $a \in P$, we have

$$\begin{aligned} C_\lambda(wa) &= C_\lambda(a) , \quad w \in W^{\mathrm{aff}} ; \\ C_{\hat{w}\lambda}(a) &= C_\lambda(a) , \quad \hat{w} \in \widehat{W}^{\mathrm{aff}} ; \\ S_{\tilde{\lambda}}(wa) &= (\det w)\, S_\lambda(a) , \quad w \in W^{\mathrm{aff}} ; \\ S_{\hat{w}\lambda}(a) &= (\det \hat{w})\, S_\lambda(a) , \quad \hat{w} \in \widehat{W}^{\mathrm{aff}} ; \end{aligned} \tag{1}$$

provided $\lambda, \tilde{\lambda} \in P^\vee/M$.

Modified multiplication of discretized Weyl-orbit functions

For any $a \in P_{\mathbb{R}}$, write

$$C_\lambda(a)\, C_\mu(a) \;=\; \sum_{\nu \in P_+} \langle C|CC\rangle^{\nu}_{\lambda,\mu}\; C_\nu(a)\;,$$

$$C_\lambda(a)\, S_{\tilde\mu}(a) \;=\; \sum_{\tilde\nu \in P_{++}} \langle S|CS\rangle^{\tilde\nu}_{\lambda,\tilde\mu}\; S_{\tilde\nu}(a)\;,$$

$$S_{\tilde\lambda}(a)\, S_{\tilde\mu}(a) \;=\; \sum_{\nu \in P_+} \langle C|SS\rangle^{\nu}_{\tilde\lambda,\tilde\mu}\; C_\nu(a)\;,$$

for all $\lambda, \mu, \nu \in P_+$, and all $\tilde\lambda, \tilde\mu, \tilde\nu \in P_{++}$.

Similarly, if $\lambda, \mu, \nu \in P_+^M$, and all $\tilde{\lambda}, \tilde{\mu}, \tilde{\nu} \in P_{++}^M$, then

$$C_\lambda(a)\, C_\mu(a) \;=\; \sum_{\nu \in P_+^M} {}_M\langle C|CC\rangle_{\lambda,\mu}^{\nu}\; C_\nu(a) \;,$$

$$C_\lambda(a)\, S_{\tilde{\mu}}(a) \;=\; \sum_{\tilde{\nu} \in P_{++}^M} {}_M\langle S|CS\rangle_{\lambda,\tilde{\mu}}^{\tilde{\nu}}\; S_{\tilde{\nu}}(a)$$

$$S_{\tilde{\lambda}}(a)\, S_{\tilde{\mu}}(a) \;=\; \sum_{\nu \in P_+^M} {}_M\langle C|SS\rangle_{\tilde{\lambda},\tilde{\mu}}^{\nu}\; C_\nu(a) \;,$$

for any $a \in F_M \supset \tilde{F}_M$.

We find

$$
{}_M\langle C|CC\rangle^{\nu}_{\lambda,\mu} \;=\; \sum_{\hat{w}\in\widehat{W}^{\mathrm{aff}}} \langle C|CC\rangle^{\hat{w}\nu}_{\lambda,\mu}\ ,
$$

$$
{}_M\langle S|CS\rangle^{\tilde{\nu}}_{\lambda,\tilde{\mu}} \;=\; \sum_{\hat{w}\in\widehat{W}^{\mathrm{aff}}} (\det w)\,\langle S|CS\rangle^{\hat{w}\tilde{\nu}}_{\lambda,\tilde{\mu}}\ ,
$$

$$
{}_M\langle C|SS\rangle^{\nu}_{\tilde{\lambda},\tilde{\mu}} \;=\; \sum_{\hat{w}\in\widehat{W}^{\mathrm{aff}}} \langle C|SS\rangle^{\hat{w}\nu}_{\tilde{\lambda},\tilde{\mu}}\ .
$$

Galois symmetry of Weyl-orbit functions

Let N denote the minimum positive integer such that

$$\left(e^{2\pi i\langle\lambda,a\rangle}\right)^N = e^{2\pi i\langle N\lambda,a\rangle} = 1 ,$$

for all $\lambda \in \Lambda_M,\ a \in F_M$.

Suppose $\gcd(\ell, N) = 1$ for $\ell \in \mathbb{N}$. Define the Galois transformation t_ℓ by

$$t_\ell\left(e^{2\pi i\langle\lambda,a\rangle}\right) := e^{2\pi i\ell\langle\lambda,a\rangle} \quad ,$$

and extend it linearly to sums of such terms. This transformation swaps one primitive root of unity for another.

Applying the Galois transformation to the orbit functions, one gets

$$t_\ell\Big(C_\lambda(a)\Big) \;=\; C_{\ell\lambda}(a) \;=\; C_\lambda(\ell a)\;,$$
$$t_\ell\Big(S_{\tilde\lambda}(\tilde a)\Big) \;=\; S_{\ell\tilde\lambda}(\tilde a) \;=\; S_{\tilde\lambda}(\ell\tilde a)\;. \qquad (2)$$

Now suppose that $\lambda \in P_+^M$, $\tilde\lambda \in P_{++}^M$, $a \in F_M$, and $\tilde a \in \tilde F_M$. Multiples of these weights by a factor of ℓ will not also, in general, be part of the same sets.

They can all, however, be moved there by appropriate elements of the relevant affine Weyl group:

$$
\begin{aligned}
\hat{w}_\ell[\lambda]\,(\,\ell\,\lambda\,) \;&=:\; t_\ell[\lambda] \;\in\; P_+^M \quad , \qquad \hat{w}_\ell[\lambda] \in \widehat{W}^{\rm aff}\;; \\
\hat{w}_\ell[\tilde{\lambda}]\,\big(\,\ell\,\tilde{\lambda}\,\big) \;&=:\; t_\ell[\tilde{\lambda}] \;\in\; P_{++}^M \quad , \qquad \hat{w}_\ell[\tilde{\lambda}] \in \widehat{W}^{\rm aff}\;; \\
w_\ell[a]\,(\,\ell\,a\,) \;&=:\; t_\ell[a] \;\in\; F_M \quad , \qquad w_\ell[a] \in W^{\rm aff}\;; \\
w_\ell[\tilde{a}]\,(\,\ell\,\tilde{a}\,) \;&=:\; t_\ell[\tilde{a}] \;\in\; \tilde{F}_M \quad , \qquad w_\ell[\tilde{a}] \in W^{\rm aff}\;.
\end{aligned}
\tag{3}
$$

Using the affine Weyl symmetries $\Rightarrow$ the Galois symmetry of the orbit functions:

$$t_\ell\Big(C_\lambda(a)\Big) \ = \ C_{t_\ell[\lambda]}(a) \ = \ C_\lambda(t_\ell[a]) \ ,$$
$$t_\ell\Big(S_{\tilde\lambda}(\tilde a)\Big) \ = \ \hat\epsilon_\ell[\lambda]\, S_{t_\ell[\tilde\lambda]}(\tilde a) \ = \ \epsilon_\ell[\tilde a]\, S_{\tilde\lambda}(t_\ell[\tilde a]) \ . \tag{4}$$

Here we have defined the signs

$$\hat\epsilon_\ell[\tilde\lambda] \ := \ \det\big(\,\hat w_\ell[\tilde\lambda]\,\big) \ , \quad \hat w_\ell[\tilde\lambda] \ \in \ \widehat{W}^{\rm aff} \ ;$$
$$\epsilon_\ell[\tilde a] \ := \ \det\big(\,w_\ell[\tilde a]\,\big) \ , \quad w_\ell[\tilde a] \ \in \ W^{\rm aff} \ . \tag{5}$$

Galois symmetry also produces relations involving the decomposition coefficients discussed above. For example, because the Galois transformation exchanges one root of unity for another, and because the coefficients ${}_M\langle C|\ SS\rangle^{\nu}_{\tilde\lambda,\tilde\mu}$ are rational, we have

$$t_\ell\Big(\ S_{\tilde\lambda}(a)\Big)\, t_\ell\Big(\ S_{\tilde\mu}(a)\ \Big)\ =\ \sum_{\nu\in P_+^M}\ {}_M\langle C|SS\rangle^{\nu}_{\tilde\lambda,\tilde\mu}\ t_\ell\Big(\ C_\nu(a)\ \Big)\ .$$

$\Rightarrow$

$$\hat\epsilon_\ell[\tilde\lambda]\, S_{t_\ell[\tilde\lambda]}(a)\, \hat\epsilon_\ell[\tilde\mu]\, S_{t_\ell[\tilde\mu]}(a)\ =\ \sum_{\nu\in P_+^M}\ {}_M\langle C|SS\rangle^{\nu}_{\tilde\lambda,\tilde\mu}\ C_{t_\ell[\nu]}(a)\ .$$

so that

$$\hat{\epsilon}_\ell[\tilde{\lambda}]\,\hat{\epsilon}_\ell[\tilde{\mu}] \sum_{\nu\in P_+^M} {}_M\langle C|SS\rangle^{\nu}_{t_\ell[\tilde{\lambda}],t_\ell[\tilde{\mu}]}\ C_\nu(a)$$

$$= \sum_{\nu\in P_+^M} {}_M\langle C|SS\rangle^{\nu}_{\tilde{\lambda},\tilde{\mu}}\ C_{t_\ell[\nu]}(a)\ .$$

Orthogonality relations for $C_\nu(a) \Rightarrow$

$$\hat{\epsilon}_\ell[\tilde{\lambda}]\,\hat{\epsilon}_\ell[\tilde{\mu}]\,{}_M\langle C|SS\rangle^{\nu}_{t_\ell[\tilde{\lambda}],t_\ell[\tilde{\mu}]} \;=\; {}_M\langle C|SS\rangle^{t_\ell[\nu]}_{\tilde{\lambda},\tilde{\mu}}\quad .$$

Similar relations for the other decomposition coefficients.

CONCLUSION (Future Work?)

(Generalized?) Orbit Functions $\Leftrightarrow$ Conformal Field Theory

$\checkmark$ $\Leftarrow$ Modified multiplication, Galois symmetries (Hrivnák-W)

? $\Leftarrow$ Fusion generators and bases (Gannon-Walton 1999)

? $\Rightarrow$ Pasquier algebras, NIM-reps, boundary conditions, . . . ?

Chevalley groups $\Rightarrow$?