Quantum ergodicity on regular graphs

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M Riemannian manifold
Let $(\psi_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(M)$, with

$$-\Delta\psi_k = \lambda_k\psi_k, \qquad \lambda_k \leq \lambda_{k+1}.$$

QE theorem (simplified) :

Theorem (Shnirelman 74, Zelditch 85, Colin de Verdière 85)

Assume that the action of the geodesic flow is **ergodic** *for the Liouville measure. Let $a \in C^0(M)$. Then*

$$\frac{1}{N(\lambda)}\sum_{\lambda_k\leq\lambda}\left|\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x)\right|^2 \underset{\lambda\longrightarrow\infty}{\longrightarrow} 0.$$

Equivalently, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density 1, such that

$$\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k \longrightarrow +\infty]{k \in \mathcal{S}} \int_M a(x) d\mathrm{Vol}(x).$$

Equivalently,

$$|\psi_k(x)|^2 d\mathrm{Vol}(x) \xrightarrow[k \longrightarrow +\infty]{k \in \mathcal{S}} d\mathrm{Vol}(x)$$

in the weak topology.

The Quantum Unique Ergodicity conjecture

QUE conjecture :

Conjecture (Rudnick, Sarnak 94)

On a negatively curved manifold, we have convergence of the whole sequence : $\int_M a(x)|\psi_k(x)|^2 d\mathrm{Vol}(x) \longrightarrow \int_M a(x) d\mathrm{Vol}(x)$ *(for all a).*

and more generally,
$\langle \psi_k, a(x, D_x)\psi_k\rangle_{L^2(M)} \longrightarrow \int_{(x,\xi)\in SM} a(x,\xi)dxd\xi.$

Other questions or conjectures

- Random wave Ansatz. In the case of a chaotic geodesic flow, M. Berry' Ansatz is that eigenfunctions of Δ should locally resemble a "monochromatic gaussian random field"

$$\psi_\lambda(x) = \sum_{\alpha=1}^{N} a_\alpha e^{i\sqrt{\lambda} u_\alpha \cdot x}$$

  with the u_α equidistributed over the unit sphere, a_α iid gaussian random variables, $N \longrightarrow +\infty$.

- Spectral statistics : eigenvalues of Δ, $\lambda_k \in (E - E^{1/2}, E + E^{1/2})$ $(E \longrightarrow +\infty)$ should after suitable rescaling resemble the eigenvalues of large $N \times N$ gaussian symmetric matrices.

QE on discrete graphs

Since the 90s there has been the idea of using graphs as a testing ground / toy model for quantum chaos.

Smilansky, Kottos, Elon,...

Bogomolny, Keating, Berkolaiko, Winn, Piotet, Marklof, Gnutzmann...

studied mostly "metric graphs" with Kirchhoff matching conditions for the derivatives at the vertices.

For a fixed metric graph, it is known that QE does NOT hold in the limit $\lambda_j \longrightarrow +\infty$ (at least for rationally lengths of the edges : Colin de Verdière 2014, Tanner 2001).

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For the limit of large graphs, spectral statistics have been studied (star graphs), "scarring", as well as as the validity of random wave ansatz.

Here we focus on the case of large regular (discrete) graphs (cf Smilansky).
Let $G = (V, E)$ be a $(q+1)$-regular graph.

Discrete laplacian : $f : V \longrightarrow \mathbb{C}$,

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)) = \sum_{y \sim x} f(y) - (q+1) f(x).$$

$$\Delta = \mathcal{A} - (q+1) I$$

$$Sp(\mathcal{A}) \subset [-(q+1), q+1]$$

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This implies convergence of the spectral measure (Kesten-McKay)

$$\frac{1}{N}\sharp\{i, \lambda_i \in I\} \underset{N \longrightarrow +\infty}{\longrightarrow} \int_I m(\lambda) d\lambda$$

for any interval I. m is a completely explicit density, supported in $(-2\sqrt{q}, 2\sqrt{q})$

Theorem

(Brooks-Lindenstrauss 2011) Assume that G_N has "few" loops of length $\leq c \log N$.
For $\epsilon > 0$, there exists $\delta > 0$ s.t. for every eigenfunction ϕ,

$$B \subset V_N, \sum_{x \in B} |\phi(x)|^2 \geq \epsilon \Longrightarrow |B| \geq N^{\delta}.$$

Proof also yields that $\|\phi\|_\infty \leq |\log N|^{-1/4}$.

Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family $=$ *uniform spectral gap for $\mathcal{A}$.*
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $a = a_N : V_N \longrightarrow \mathbb{C}$ be such that $|a(x)| \leq 1$ for all $x \in V_N$.
Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \sum_{x \in V_N} a(x) |\phi_i^{(N)}(x)|^2 - \langle a \rangle \right|^2 = 0.$$

$$\langle a \rangle = \frac{1}{N} \sum_{x \in V_N} a(x)$$

- Also works on shrinking spectral intervals

Examples

Deterministic examples :

- the Ramanujan graphs of Lubotzky-Phillips-Sarnak 1988 (arithmetic quotients of the q-adic symmetric space $PGL(2, \mathbb{Q}_q)/PGL(2, \mathbb{Z}_q)$).

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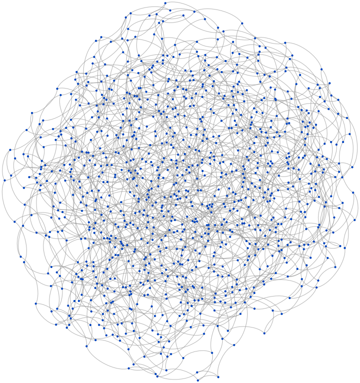
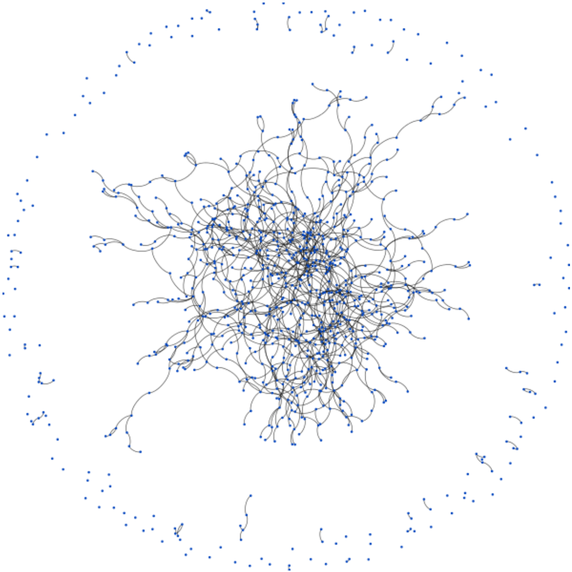
- Cayley graphs of $SL_2(\mathbb{Z}/p\mathbb{Z})$, p ranges over the primes, (Bourgain-Gamburd, based on Helfgott 2005)
- Extended to all simple Chevalley groups (Breuillard-Green-Tao), perfect groups and square-free p (Salehi-Golsefidy–Varju)

Random regular graphs

Our result applies in particular to random regular graphs. In that case there also exists a probabilistic proof (Geisinger 2013) in the case where $a(x)$ is chosen independently of G_N.



Just as a comparison... the Erdös-Renyi model



More general version

Theorem

(A-Le Masson, 2013) Assume that G_N has "few" short loops and that it forms an expander family.
Let $(\phi_i^{(N)})_{i=1}^N$ be an ONB of eigenfunctions of the laplacian on G_N.
Let $K = K_N : V_N \times V_N \longrightarrow \mathbb{C}$ be a matrix such that $d(x,y) > D \Longrightarrow K(x,y) = 0$. Assume $|K(x,y)| \leq 1$. Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \sum_{i=1}^{N} \left| \langle \phi_i^{(N)}, K\phi_i^{(N)} \rangle - \langle K \rangle_{\lambda_i} \right|^2 = 0.$$

$$\langle K \rangle_\lambda = \frac{1}{N} \sum_{x,y} K(x,y) \Phi_{sph,\lambda}(d(x,y)).$$

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$$\Phi_\lambda(d) = q^{-d/2} \left(\frac{2}{q+1} \cos(ds \ln q) + \frac{q-1}{q+1} \frac{\sin((d+1)s \ln q)}{\sin(s \ln q)} \right)$$

if $\lambda = q^{1/2+is} + q^{1/2-is} = 2\sqrt{q} \cos(s \ln q)$.

Our result says that

$$\langle \phi_i^{(N)}, K\phi_i^{(N)} \rangle = \sum_{x,y} \overline{\phi_i^{(N)}(x)} K(x,y) \phi_i^{(N)}(y)$$

$$\sim \frac{1}{N} \sum_{x,y} K(x,y) \Phi_{sph,\lambda_i}(d(x,y))$$

for most i.

Sketch of proof : phase space analysis on regular graphs

- Fourier-Helgason transform on the $(q+1)$-regular tree ;
- "phase space analysis" on the tree and on a finite regular graph ;
- use of the geodesic dynamics to study eigenfunctions of the laplacian.

Fourier-Helgason transform on the $(q+1)$-regular tree

$f : \mathfrak{X} \longrightarrow \mathbb{C}$. Its Fourier transform is

$$\hat{f}(\omega, s) = \sum_{x \in \mathfrak{X}} f(x) \overline{e_{s,\omega}(x)}$$

where $s \in \mathbb{T}_q = \mathbb{R}/(2\pi/\log q)$, $\omega \in \partial\mathfrak{X}$, and

$$e_{s,\omega}(x) = q^{(1/2+is)h_\omega(x)}$$

satisfies

$$\mathcal{A} e_{s,\omega} = 2\sqrt{q}\cos(s \log q) e_{s,\omega}.$$

Inversion formula, Plancherel theorem

$$f(x) = \int_{s\in\mathbb{T}_q,\omega\in\partial\mathfrak{X}} \hat{f}(\omega, s)e_{s,\omega}(x)dm(s)d\nu_o(\omega)$$

$$\sum_{x\in\mathfrak{X}} |f(x)|^2 = \int_{s\in\mathbb{T}_q,\omega\in\partial\mathfrak{X}} |\hat{f}(\omega, s)|^2 dm(s)d\nu_o(\omega)$$

Paley-Wiener type theorem

(Cowling-Setti)

Decay of $f(x)$ when $d(x, o) \longrightarrow \infty \leftrightsquigarrow$ "smoothness" of $\hat{f}(\omega, s)$.

Phase space

We define it as

$$\mathfrak{X} \times \partial\mathfrak{X} \times \mathbb{T}_q$$

"Pseudodifferential" calculus on a regular tree

For a function $a(x,\omega,s)$ on $\mathfrak{X}\times\partial\mathfrak{X}\times\mathbb{T}_q$, we define an operator $\mathrm{Op}(a)$ on $\ell^2(\mathfrak{X})$ by

$$\mathrm{Op}(a)e_{s,\omega}(x) = a(x,\omega,s)e_{s,\omega}$$

in other words

$$\mathrm{Op}(a)f(x) = \int_{s\in\mathbb{T}_q,\omega\in\partial\mathfrak{X}} a(x,\omega,s)\hat{f}(\omega,s)e_{s,\omega}(x)dm(s)d\nu_o(\omega).$$

The adjacency operator $\mathcal{A}$ corresponds to $a(x,\omega,s) = 2\sqrt{q}\cos(s\log q)$.

According to the "Paley-Wiener" theorem,
Decay of $K_a(x, y)$ when $d(x, y) \longrightarrow \infty \leftrightsquigarrow$ "smoothness" of $a(x, \omega, s)$ wrt (ω, s).

Le Masson (2012) studied the operators $\mathrm{Op}(a)$ (behaviour under composition, boundedness on $\ell^2(\mathfrak{X})$ etc).

Definition of Op(a) on a finite regular graph

$G_N = (V_N, E_N) = \Gamma_N \backslash \mathfrak{X}$ where Γ is a group of automorphisms acting without fixed point on the regular tree $\mathfrak{X}$.

Op(a) preserves the Γ-invariant functions on $\mathfrak{X}$ (and thus act on $\ell^2(V)$)

iff

$a(x, \omega, s)$ is Γ-invariant ($a(\gamma \cdot x, \gamma \cdot \omega, s) = a(x, \omega, s)$ for $\gamma \in \Gamma$)

Quantum variance

$$G_N = (V_N, E_N) = \Gamma_N \backslash \mathfrak{X},\ N = |V_N|.$$

$$\mathcal{A}\phi_i^{(N)} = \lambda_i^{(N)}\phi_i^{(N)}$$

$$Var(K) = \frac{1}{N}\sum_{i=1}^{N} |\langle \phi_i^{(N)}, K\phi_i^{(N)}\rangle|^2.$$

• Trivially, $Var([\mathcal{A}, K]) = 0$ for any operator K.

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where

$$b(x, \omega, s) = i \sin(s \ln q)(a \circ \sigma - La) + \cos(s \ln q)(a \circ \sigma + La - 2a)$$

Although we do not know if $\langle \phi_i^{(N)}, \mathrm{Op}(a \circ \sigma - a)\phi_i^{(N)} \rangle$ is small for every i, we can prove that

$$Var(\mathrm{Op}(a \circ \sigma - a)) \underset{N \longrightarrow +\infty}{\longrightarrow} 0.$$

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We deduce (density argument) that $Var(\mathrm{Op}(a)) \longrightarrow 0$ (as $N \longrightarrow \infty$) if

$$\frac{1}{N}\sum_{x\in\mathcal{D}_N}\int_{\partial\mathfrak{X}\times\mathbb{T}_q} a(x,\omega,s)d\nu_x(\omega) = 0$$

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$$Var(\mathrm{Op}(a)) \leq \frac{1}{N} \sum_{x \in \mathcal{D}_N} \int_{\partial \mathfrak{X} \times \mathbb{T}_q} |a(x, \omega, s)|^2 d\nu_x(\omega) dm(s)$$

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for all s.

Do graphs teach us something about manifolds ?

By approximation of a manifold by graphs (triangulation) ? A priori NO.

Continuous analogue of our theorem...

Let (S_N) be a sequence of hyperbolic surfaces, whose genus goes to ∞.
Assume the first eigenvalue $\lambda_1(N)$ of $-\Delta$ is bounded away from 0 as $N \longrightarrow \infty$. Assume there are few short geodesics.

Fix an interval $I \subset (1/4, +\infty)$.

Let $(\phi_i^{(N)})$ be an ONB of eigenfunctions of the laplacian on S_N.
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Then

$$\lim_{N \longrightarrow +\infty} \frac{1}{Vol(S_N)} \sum_{\lambda_i(N) \in I} \left| \int_{S_N} a(x) |\phi_i^{(N)}(x)|^2 dx - \langle a \rangle \right|^2 = 0.$$

(Le Masson–Sahlsten 2016)

Application to Arithmetic Quantum Unique Ergodicity

M arithmetic surface.
Assume

$$\langle \psi_k, a(x, D_x)\psi_k \rangle_{L^2(M)} \underset{\lambda_k \longrightarrow \infty}{\longrightarrow} \int_{(x,\xi)\in SM} a(x,\xi) d\mu(x,\xi).$$

Lindenstrauss 2001 : if the ψ_k are eigenfunctions of Δ and ALL the Hecke operators, then μ is the Liouville (uniform) measure.

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One step is to prove that all the ergodic components of μ have positive Kolmogorov-Sinai entropy

Application to Arithmetic Quantum Unique Ergodicity

(on arithmetic surfaces)

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Brooks-Lindenstrauss 2011 : if the ψ_k are eigenfunctions of Δ and ONE Hecke operator, then μ is the Liouville (uniform) measure.

Idea : approximate Hecke trees by finite regular graphs, use a modification of the previously quoted result of BL.

QE on the sphere, revisited

(Brooks-Le Masson-Lindenstrauss 2015)
For $g_1, \ldots, g_k$ a finite set of rotations in $SO(3)$,

$$T_k f(x) = \sum_{j=1}^{k} (f(g_j x) + f(g_j^{-1} x))$$

commutes with $\Delta_{\mathbb{S}^2}$.

Theorem

Assume that $g_1, \ldots, g_k$ generate a free subgroup of $SO(3)$. For each ℓ, let $(\psi_j^{(\ell)})_{j=1}^{2\ell+1}$ be an o-n family of eigenfunctions of $-\Delta_{\mathbb{S}^2}$ of eigenvalue $\ell(\ell+1)$, that are also eigenfunctions of T_k. Then for any continuous function a on $\mathbb{S}^2$, we have

$$\frac{1}{2\ell+1} \sum_{j=1}^{2\ell+1} \left| \int_M a(x) |\psi_j^{(\ell)}(x)|^2 d\mathrm{Vol}(x) - \int_M a(x) d\mathrm{Vol}(x) \right|^2 \underset{\ell \longrightarrow \infty}{\longrightarrow} 0.$$

Rk :