Quantum ergodicity of Wigner induced spherical harmonics

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Setup

- (M, g) compact Riemannian manifold
- $\Delta = \Delta_g$ the Laplace-Beltrami operator
- Eigenvalue problem

$$(\Delta + \lambda_k)\varphi_k = 0, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$$

- Delocalization/diffuseness of eigenfunctions in phase space

Quantum ergodicity

- $A \in \Psi^0(M)$ a zeroth order pseudo-differential operator
- $\sigma_A(x,\xi)$ is the principal symbol of A
- $d\mu_L$ normalized Liouville measure on S^*M

Definition

The Laplacian eigenfunctions φ_k are quantum ergodic if

$$\lim_{\lambda\to\infty} \frac{1}{\#\{\lambda_k \le \lambda\}} \sum_{\lambda_k \le \lambda} \left| \langle A\varphi_k, \varphi_k\rangle - \int_{S^*M} \sigma_A(x,\xi)\, d\mu_L \right|^2 = 0$$

for every $A \in \Psi^0(M)$. For a density one subsequence we have

$$\langle A\varphi_{k_j}, \varphi_{k_j}\rangle \to \int_{S^*M} \sigma_A(x,\xi)\, d\mu_L$$

Spherical harmonics

- Specialize to $(M, g) = (S^2, \text{round metric})$
- Let $\frac{\partial}{\partial\theta}$ generate z-axis rotation
- The standard spherical harmonics (YLMs) are joint eigenfunctions of the Laplacian $\Delta = \Delta_{S^2}$ and the z-component of the angular momentum operator:

$$\begin{cases} \Delta Y_N^k = -N(N+1) Y_N^k, & \\ \dfrac{1}{i}\dfrac{\partial}{\partial\theta} Y_N^k = k Y_N^k, & -N \leq k \leq N \end{cases}$$

- The standard spherical harmonics are not QE
- But random spherical harmonics are QE almost surely

Random spherical harmonics

- Eigenspaces $\mathcal{H}_N := \text{span}\{Y_N^k\colon\ -N \leq k \leq N\}$
- $\dim \mathcal{H}_N = 2N + 1 =: d_N$
- A random change-of-basis matrix

$$\big(u_{N,k}(\alpha)\big)_{-N\leq k,\alpha\leq N} \in (\mathrm{U}(d_N), \mathrm{Haar}_N)$$

defines a random basis for $\mathcal{H}_N$:

$$\psi_{N,k} = \sum_{-N\leq\alpha\leq N} u_{N,k}(\alpha) Y_N^\alpha, \qquad -N \leq k \leq N$$

- A random basis for $L^2(S^2) = \bigoplus_{N\geq 0} \mathcal{H}_N$ is thus an element of the product probability space

$$\prod_{N\geq 0} (\mathrm{U}(d_N), \mathrm{Haar}_N)$$

QE of random bases

Theorem (Zelditch '92)

A random orthonormal basis of spherical harmonics (as described in the previously slide) for $L^2(S^2)$ is almost surely QE.

- VanderKam '97: QUE of random bases for S^2
- General M with additional hypotheses on the subspaces $\mathcal{H}_N$:
  - Zelditch '12: QE of random bases for sequences of eigenspaces with dimensions $d_N \to \infty$
  - Maples '13: QUE of random bases for sequences of eigenspaces with dimensions $d_N > CN^\varepsilon$ for $\varepsilon > 0$
- Chatterjee and Galkowski '16: QE for random small perturbations of the Laplacian

Goal of talk: QE of a more general class of "random" bases

Generalized Wigner matrices

Definition

Generalized Wigner $H = (h_{jk})_{-N \le j,k \le N} \in \mathrm{Herm}(d_N)$ *satisfies*

- h_{jk} *independent for* $j \le k$.
- **Normalization**: *Mean zero, variances satisfy* $\sum_{j=-N}^{N} \sigma_{jk}^2 = 1$
- **Non-degeneracy**: $\exists c_1, c_2 > 0$ *independent of* N *such that*

  $$c_1^{-1} \le d_N \sigma_{jk}^2 \le c_1 \quad \textit{and} \quad \mathbb{E}(\mathbf{h}_{jk}^* \mathbf{h}_{jk}) \ge c_2 d_N^{-1}$$

  in the sense of inequality between 2×2 *positive matrices. Here* $\mathbf{h}_{jk} = (\mathrm{Re}\, h_{jk}, \mathrm{Im}\, h_{jk})$.
- **Bounded moments**: $\forall p \in \mathbb{N}\ \exists C_p > 0$ *such that*

  $$\mathbb{E}\left|\sqrt{d_N} h_{jk}\right|^p < C_p \quad \textit{for all } j, k, N.$$

Induced measure

- Let $\tilde{H} \in \mathrm{Herm}(d_N)$ be a GUE matrix, i.e., upper triangular entries are iid $N(0,1)_{\mathbb{C}}$, and diagonal entries are iid $N(0,1)_{\mathbb{R}}$
- Consider the decomposition $\tilde{H} = U \Lambda U^*$, where $U \in \mathrm{U}(d_N)$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{d_N})$
- Decomposition is unique up to ordering of the λ_j's and a diagonal unitary matrix
- The distribution

$$\nu_{\tilde{H}} = c_N \left(\prod_{i<j} e^{-|\tilde{h}_{ij}|^2} \right) \left(\prod_{i} e^{-|\tilde{h}_{ii}|^2/2} \right) d\tilde{H}$$

of the GUE matrix $\tilde{H}$ defines a continuous probability measure on $\mathrm{Herm}(d_N)$ (here $d\tilde{H}$ = Lebesgue measure).

Induced measure

- Can rewrite the distribution function as

$$\nu_{\tilde{H}} = c_N e^{-\operatorname{tr}(\tilde{H}^2)/2}\, d\tilde{H}$$

- From the unitary invariance of the above line, see that $\nu_{\tilde{H}}$ induces the invariant Haar measure on $\mathrm{U}(d_N)$ under the decomposition $\tilde{H} = U\Lambda U^*$
- Now let $H \in \mathrm{Herm}(d_N)$ be a generalized Wigner matrix, ν_H its distribution
- Denote by μ_N the induced measure on $\mathrm{U}(d_N)$, that is, if $\pi\colon H \mapsto U$, then

$$\mu_N := \pi_* \nu_H.$$

Asymptotic normality of Wigner eigenvectors

- Let $\left(u_{N,k}(\alpha)\right)_{-N\leq\alpha\leq N}$ be eigenvectors of $\{H_N\}$
- Then $u_{N,k}(\alpha)$ are asymptotically Gaussian random variables:

$$\sqrt{d_N}u_{N,k}(\alpha) \to \mathcal{N}^{(1)} + i\mathcal{N}^{(2)}$$

THEOREM (Bourgade-Yau '13)

Given any polynomial Q in $2m$ variables, there exists $\varepsilon = \varepsilon(Q)$ such that for all N sufficiently large

$$\sup_{\substack{J\subset[-N,\dots,N]\\ |J|=m\\ k\in[-N,N]}} \Big|\mathbb{E}Q\big(\sqrt{d_N}(e^{i\omega}u_{N,k}(\alpha), e^{-i\omega}\overline{u_{N,k}(\alpha)})_{\alpha\in J}\big) - \mathbb{E}Q\big((\mathcal{N}_j^{(1)} + i\mathcal{N}_j^{(2)}, \mathcal{N}_j^{(1)} - i\mathcal{N}_j^{(2)})_{j=1}^m\big)\Big| \leq d_N^{-\varepsilon}.$$

Here ω is independent of H_N and uniform on $(0, 2\pi)$.

Probabilistic local QUE

- Let $a_N\colon \{-N,\dots,N\} \to [-1,1]$ be functions
- Assume $\sum_{\alpha=-N}^{N} a_N(\alpha) = 0$
- Let $|a_N| := \#\{\alpha\colon a_N(\alpha) \neq 0\}$ denote the size of the support

Theorem (Bourgade-Yau '13)

Let $\{H_N\}$ be a sequence of generalized Wigner matrices. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there exists $C > 0$ so

$$\mathbb{P}\left(\left|\frac{d_N}{|a_N|}\sum_{\alpha} a_N(\alpha)|u_{N,k}(\alpha)|^2\right| > \delta\right) \leq C(d_N^{-\varepsilon} + |a_N|^{-1})$$

for all sequences of functions a_N as above.

QE of Wigner induced random bases

- Define a (Wigner induced) random basis for $\mathcal{H}_N$ by

$$\psi_{N,k} = \sum_{-N \le \alpha \le N} u_{N,k}(\alpha) Y_N^\alpha, \qquad -N \le k \le N$$

- Here, $\left(u_{N,k}(\alpha)\right)_{\alpha=-N}^{N}$ are eigenvectors of some Wigner matrix;
- Or equivalently, the unitary matrix $\left(u_{N,k}(\alpha)\right)_{-N \le k,\alpha \le N}$ is a random element of $(\mathrm{U}(d_N), \mu_N)$

Theorem (C '16)

A Wigner induced random orthonormal basis of spherical harmonics is almost surely QE.

QE of Wigner induced random bases

- Define random variables

$$X_N := \frac{1}{d_N} \sum_{k=-N}^{N} \left| \langle A\psi_{N,k}, \psi_{N,k} \rangle - \int_{S^*M} \sigma_A \, d\mu_L \right|^2$$

- The following implies QE:

Theorem

$\mathbb{E}X_N = O(d_N^{-\varepsilon})$ and $\mathbb{E}X_N^2 = O(d_N^{-\varepsilon'})$.

- Kolmogorov's convergence criterion allows us to apply SLLN to the X_N's
- SLLN implies $\frac{1}{M} \sum_{N=0}^{M} X_N \to 0$ a.s.

Weingarten formula

Theorem (Weingarten '78)

- *Let* $\left(u_{N,k}(\alpha)\right)_{k,\alpha} \in \mathrm{U}(d_N)$ *be a unitary matrix;*
- *Let* $k_j, k_j', \alpha_j, \alpha_j' \in [-N, N]$ *be indices for* $1 \leq j \leq m$*;*

Then the integral

$$I_N(m) := \int_{\mathrm{U}(d_N)} u_{N,k_1}(\alpha_1) \cdots u_{N,k_m}(\alpha_m) \times \overline{u_{N,k_1'}(\alpha_1')} \cdots \overline{u_{N,k_m'}(\alpha')}\, d\mathrm{Haar_N}$$

is asymptotically

$$d_N^{-m} \sum \delta_{k_1,k_{j_1}'} \delta_{\alpha_1,\alpha_{j_1}'} \cdots \delta_{k_m,k_{j_m}'} \delta_{\alpha_m,\alpha_{j_m}'} + O(d_N^{-m-1}),$$

where the sum is over all choices $j_1, \cdots, j_m$ *as a permutation of* $1, \ldots, m$.

Proposition

- *Let* $\left(u_{N,k}(\alpha)\right)_{k,\alpha} \in \mathrm{U}(d_N)$ *be a unitary matrix;*
- *Let* $k_j, k_j', \alpha_j, \alpha_j' \in [-N, N]$ *be indices for* $1 \leq j \leq m$*;*

Then the integral

$$\mathbb{E}\left[u_{N,k_1}(\alpha_1)\cdots u_{N,k_m}(\alpha_m)\overline{u_{N,k_1'}(\alpha_1')}\cdots\overline{u_{N,k_m'}(\alpha_m')}\right]$$

is asymptotically

$$d_N^{-m}\sum \delta_{k_1,k_{j_1}'}\delta_{\alpha_1,\alpha_{j_1}'}\cdots\delta_{k_m,k_{j_m}'}\delta_{\alpha_m,\alpha_{j_m}'} + O(d_N^{-m-\varepsilon}),$$

where the sum is over all choices $j_1, \cdots, j_m$ *as a permutation of* $1, \ldots, m$.

Rotationally invariant case

- Assume $\int_{S^*M} \sigma_A d\mu_L = 0$ and assume $A \in \Psi^0(S^2)$ is invariant with respect to z-axis rotation, then

$$\langle AY_N^\alpha, Y_N^\beta \rangle = \begin{cases} \langle AY_N^\alpha, Y_N^\alpha \rangle & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

- Using the definition of random bases $\psi_{N,k} = \sum_\alpha u_{N,k}(\alpha) Y_N^\alpha$,

$$X_N = \frac{1}{d_N} \sum_{k=-N}^{N} |\langle A\psi_{N,k}, \psi_{N,k} \rangle|^2$$
$$= \frac{1}{d_N} \sum_{k=-N}^{N} \left| \sum_{\alpha,\beta=-N}^{N} \langle AY_N^\alpha, Y_N^\beta \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\beta)} \right|^2$$

Rotationally invariant case

- Using rotational invariance and expanding the square gives

$$X_N = \frac{1}{d_N} \sum_k \sum_{\alpha,\beta} \langle A Y_N^\alpha, Y_N^\alpha \rangle \langle A Y_N^\beta, Y_N^\beta \rangle |u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2$$

- Compute the expected value using Weingarten:

$$\mathbb{E}\left(|u_{N,k}(\alpha)|^2 |u_{N,k}(\beta)|^2\right) = d_N^{-2}(1 + \delta_{\alpha\beta}) + O(d_N^{-2-\varepsilon})$$

- Hence

$$\mathbb{E} X_N = \underbrace{\left(\frac{1}{d_N} \sum_\alpha \langle A Y_N^\alpha, Y_N^\alpha \rangle \right)^2}_{=O(d_N^{-1})} + \underbrace{\frac{1}{d_N^2} \sum_\alpha \langle A Y_N^\alpha, Y_N^\alpha \rangle^2}_{=O(d_N^{-1})} + O(d_N^{-\varepsilon})$$

General case

Proposition

Fix $A \in \Psi^0(S^2)$. Then for $n \in \mathbb{Z}$ there exist $\hat{A}(n) \in \Psi^0(S^2)$ with

- $\left\| A - \sum_{|n| \leq K} \hat{A}(n) \right\|_{L^2 \to L^2} \to 0$
- *For $n \neq 0$, have $\|\hat{A}(n)\|_{L^2 \to L^2} = O(n^{-\ell})$ for every $\ell \geq 1$*
- $\displaystyle\int_{S^*M} \sigma_{\hat{A}(n)} \, d\mu_L = \begin{cases} \displaystyle\int_{S^*M} \sigma_A \, d\mu_L & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$
- *The matrix elements of A and $\hat{A}(n)$ are related by*

$$\langle \hat{A}(n) Y_N^\alpha, Y_N^\beta \rangle = \begin{cases} \langle A Y_N^\alpha, Y_N^{\alpha - n} \rangle & \text{if } \alpha = \beta + n \\ 0 & \text{if } \alpha \neq \beta + n \end{cases}$$

simultaneously for all $N \geq 0$

General case

- Approximate operator A by a finite sum of $\hat{A}(n)$
- Equivalently, approximate

$$X_N = \frac{1}{d_N} \sum_{k=-N}^{N} \left| \sum_{\alpha,\beta=-N}^{N} \langle AY_N^\alpha, Y_N^\beta \rangle | u_{N,k}(\alpha) \overline{u_{N,k}(\beta)} \right|^2$$

by a finite sum of

$$Y_{n,m,N} := \frac{1}{d_N} \sum_{k=-N}^{N} \sum_{\alpha,\beta} \langle AY_N^\alpha, Y_N^{\alpha-n} \rangle u_{N,k}(\alpha) \overline{u_{N,k}(\alpha-n)} \times \langle AY_N^\beta, Y_N^{\beta-m} \rangle u_{N,k}(\beta) \overline{u_{N,k}(\beta-m)}$$

- Compute (as before) expectation/variance for each of $Y_{n,m,N}$