A Riemannian structure on the space of conformal metrics

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- Our metric will be defined for conformal classes on even-dimensional manifolds. Moreover, we will need to restrict to conformal metrics satisfying an 'admissibility' condition.
- Our main focus will be some results in four dimensions, where there are applications to a natural geometric variational problem. In the case of surfaces, it will correspond to the Mabuchi-Donaldson-Semmes metric for the natural Kähler class.

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- Note

$$\mathcal{C}^+ \neq \emptyset \;\Leftrightarrow\; \int_M K_g dA_g > 0,$$

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The inner product

- Note

$$T_u \mathcal{C}^{\pm} \cong C^{\infty}(M)$$

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Definition

For $\alpha, \beta \in T_u\mathcal{C}^{+} \cong C^{\infty}$, we define

$$\langle \alpha, \beta \rangle_u = \int \alpha \beta K_u \, dA_u$$

and for $\alpha, \beta \in T_u\mathcal{C}^{-} \cong C^{\infty}$ we define

$$\langle \alpha, \beta \rangle_u = \int \alpha \beta (-K_u) \, dA_u$$

The inner product, cont.

• If $g_u = e^{2u} g_0$, then the Gauss curvature and volume forms of g_u and g_0 are related by

$$K_u = e^{-2u}\left(K_0 - \Delta_0 u\right), \quad dA_u = e^{2u} dA_0,$$

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• To motivate our results in dimension four, we want to point out some geometric properties of $(\mathcal{C}^{\pm}, \langle \cdot, \cdot \rangle)$, beginning with geodesics.

The geodesic equation

• Let $u : [a, b] \to \mathcal{C}^+$ be a curve (i.e., $e^{2u}g \in \mathcal{C}^+$). We can use the inner product to define the length of a path:

$$L[u] = \int_a^b \|\frac{\partial u}{\partial t}\| dt = \int_a^b \left\{ \int_M \left(\frac{\partial u}{\partial t}\right)^2 K_u dA_u \right\}^{1/2} dt.$$

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• Chen-He ('08) proved partial regularity ($C^{1,1}$) of solutions.

- Let

$$F[u] = \int \left(|\nabla u|^2 + 2Ku\right) dA - \left(\int KdA\right) \log \fint e^{2u} dA.$$

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• Polyakov ('81) showed that if $g_u = e^{2u} g$ have the same area, then

$$\log \frac{\det(-\Delta_u)}{\det(-\Delta_g)} = -\frac{1}{12\pi} F[u].$$

Geodesic convexity

Claim

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Proof. Let $u = u(x,t)$ denote a geodesic in $\mathcal{C}^{+}$. Then

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$$\frac{d^2}{dt^2} F[u] = 2\overline{K}_u \Big\{ \int \frac{|\nabla u_t|^2}{K_u} dA_u - 2 \Big[\int u_t^2 dA_u - \frac{1}{A_u} \big(\int u_t dA_u \big)^2 \Big] \Big\}.$$

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To get a sign, we need a kind of "curvature-weighted Poincaré inequality".

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$$\frac{n}{n-1} \int \phi^2 \, dV \leq \int (Ric)^{-1}(\nabla\phi, \nabla\phi) \, dV,$$

with equality $\Leftrightarrow$ $\phi \equiv 0$ or $(M, g) = (S^n, g_0)$ and ϕ is a fist order spherical harmonic.

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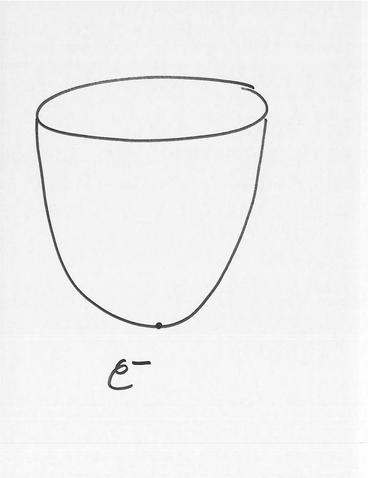
- When $n = 2$, $(Ric_u)^{-1} = \frac{1}{K_u} g_u$, and Andrews' inequality applied to $\phi = u_t - \bar{u}_t$ implies $\frac{d^2}{dt^2} F \geq 0$.

Convexity, cont.

• In $\mathcal{C}^-$, we have strict convexity of F and uniqueness of critical points (i.e., metrics of curvature -1):



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$$\psi_\alpha = \sigma^{-1} \circ \delta_\alpha \circ \sigma : S^2 \to S^2$$

defines a 1-parameter family of conformal maps:

$$\psi_\alpha^* g_0 = e^{2w_\alpha} g_0.$$

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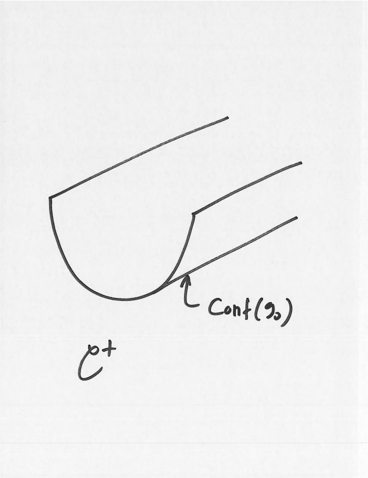
Then

$$u(x,t) = w_{e^t}(x)$$

is a geodesic in $\mathcal{C}^+$.

Convexity, cont.



Four dimensions

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$$2\pi\chi(M) = \int K dA,$$

$$n = 4 : \quad 8\pi^2\chi(M) = \int \Big(-\frac{1}{2}|Ric|^2 + \frac{1}{6}R^2 \Big) dV + \int |W|^2 dV,$$
$$\langle \alpha, \beta \rangle = \frac{1}{4} \int \alpha\beta \Big(-\frac{1}{2}|Ric|^2 + \frac{1}{6}R^2 \Big) dV.$$

Schouten tensor

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$$8\pi^2\chi(M) = \int |W|^2 + 4\int \sigma_2(A)dV,$$

and

$$\langle \alpha, \beta \rangle_g = \int \alpha\beta\sigma_2(A_g)dV_g.$$

The spaces

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(For reasons we'll touch on later, we will only consider $\mathcal{C}^{+}$.)

$\mathcal{C}^+$

Question: When is $\mathcal{C}^+ \neq \emptyset$?

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- In analogy with two dimensions, we have

Theorem.

(Chang-G-Yang, '96) If $\int \sigma_2(A) dV > 0$, $R > 0$, then $\mathcal{C}^+ \neq \emptyset$.

The σ_k-Yamabe problem:(J. Viaclovsky)

Given $[g]$, find $\tilde{g} \in [g]$ such that

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- When $k = 1$, this corresponds to the Yamabe problem.
- When $k \geq 2$ this equation fully nonlinear, and elliptic if

$$\mathcal{C}_k^+ = \{\tilde{g} \in [g] \ : \ \sigma_j(A_{\tilde{g}}) > 0, \ 1 \leq j \leq k\} \neq \emptyset$$

(or with A replaced by $-A$).

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• $\mathcal{C}^-$:

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• Non-uniqueness for solutions of the Yamabe problem is well known; Schoen constructed explicit examples on $S^n \times S^1$. Viaclovsky generalized Schoen's construction to the fully nonlinear Yamabe problem when $k < n/2$.

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- Define

$$F_{CY}[u] = \int \Big\{ 2\Delta u |\nabla u|^2 - |\nabla u|^4 - 2Ric(\nabla u, \nabla u) + R|\nabla u|^2 - 8u\sigma_2(A_g) \Big\} dV_g - 2\Big(\int \sigma_2(A_g) dV \Big) \log \Big(\fint e^{-4u} dV \Big).$$

This formula is due to Branson-Orsted/Chang-Yang.

The functional

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• When $(M, g) = (S^4, g_{S^4})$, Chang-Yang showed that $F\big|_{\mathcal{C}^+} \geq 0$, with equality given precisely by the image of the round metric under the conformal group.

Geodesics

• As in the case of surfaces, many of the geometric properties of the space $(\mathcal{C}^+, \langle \cdot, \cdot \rangle)$ depend on the existence/regularity of geodesics.

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There is a more convenient way to write this: let

$$E_u = u_{tt} A_u - \nabla u_t \otimes \nabla u_t,$$

then the geodesic equation can be written

$$\sigma_2(E_u) = 0.$$

• This is a degenerate elliptic, fully nonlinear equation. We regularize by letting

$$E_u^{\epsilon} = (1+\epsilon)u_{tt}A_u - \nabla u_t \otimes \nabla u_t,$$

and for $f > 0$, solve

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• Unlike the surface case, estimates for the regularized geodesic equation degenerate at the C^2-level: given endpoints $u_0, u_1 \in \mathcal{C}^+$ and $\epsilon > 0$, we can prove the existence of a solution of the regularized equation connecting u_0 and u_1 with

$$|u_\epsilon| + |(u_\epsilon)_t| + |\nabla u_\epsilon| + \epsilon\{|\nabla^2 u_\epsilon| + |(u_\epsilon)_{tt}| + |\nabla (u_\epsilon)_t|\} \leq C.$$

Main Results

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The functional F_{CY} is geodesically convex.

• The proof of Theorem 2 also depends on a version of Andrews' unequality, but in a very non-obvious way.

Main Results, cont.

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Theorem 3

If $\mathcal{C}^+ \neq \emptyset$ and (M, g) is not conformally equivalent to the sphere, then there is a **unique** metric $\tilde{g} \in \mathcal{C}^+$ which minimizes F_{CY}, and is therefore a solution of

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Also,

$$(i) \quad Vol(\tilde{g}) < Vol(S^4),$$

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$$(ii) \quad 0 < Ric(\tilde{g}) < \frac{1}{2} R_{\tilde{g}} \tilde{g}.$$

Some Remarks about Theorem 3

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The proof of Theorem 3, cont.

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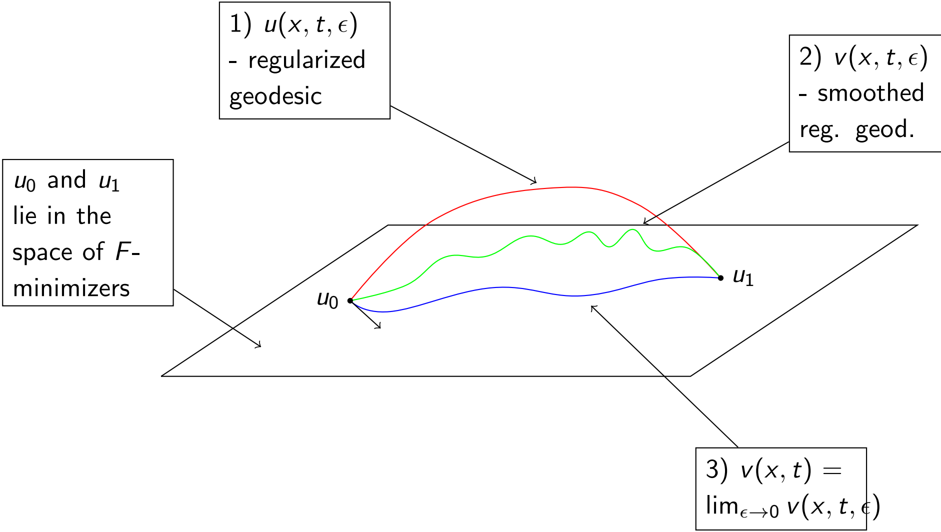
• The idea is to use the flow introduced by Guan-Wang:

$$\frac{\partial u}{\partial t} = \log \sigma_2(A_u).$$

The Proof, in a Picture:



Dimension $n \geq 6$

• In higher dimensions there are two ways to proceed. One way is to define

$$\langle \alpha, \beta \rangle_{g_u} = \int \alpha\beta \, \sigma_{n/2}(A_u) dV_u.$$

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• There is an alternate definition which has many nice formal properties, in which one replaces $\sigma_{n/2}(A_u)$ with the 'renormalized volume coefficient' (cf. Chang-Fang, Chang-Fang-Graham). The associated formulas become quite complicated, though.

Happy Birthday, et Bonne Anniversaire, Claude!