Holomorphic discs, null geodesic and gravitational scattering

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Based on Math.DG/0806.3761, Math Res. Lett., 16, p291 (2009), Math.DG/1002.2993, Comm. Anal. & Geom., (2011) joint with Claude Le Brun.
&
With David Skinner arxiv:1311.2564 and Yvonne Geyer & Arthur Lipstein 1404.6219. [Cf. also Berkovits, Witten, Roiban Spradlin & Volovich 2003/4, Skinner 2012, Cachazo, He, Yuan 2013]

Slides at: http://people.maths.ox.ac.uk/lmason/lebrunfest.pdf

Null geodesics and scattering

Let (M^d, g) be manifold with Lorentzian metric g such that

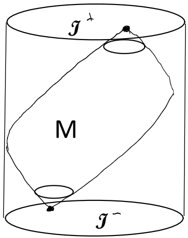
- it is Globally hyperbolic, asymptotically flat/de Sitter
- has conformal compactification $\widetilde{M} = M \cup \mathscr{I}^+ \cup \mathscr{I}^-$,
- null geodesics end on $\mathscr{I}^-$ in past and $\mathscr{I}^+$ in future.

Scattering thru M gives symplectic maps for

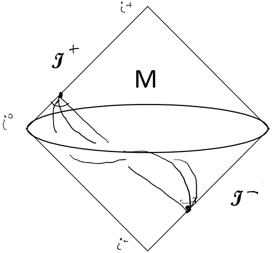
- null geodesics $T^*\mathscr{I}^- \to T^*\mathscr{I}^+$
- Gravitational field data on $\mathscr{I}^-$ to data at $\mathscr{I}^+$.

Can we relate null geodesic scattering to gravitational field?

- With integrability can obtain precise fully nonlinear results.
- Strings suggest geometric connection without integrability.



Asymptotically de Sitter



Asymptotically Flat

Einstein-Weyl geometry

Definition (An Einstein-Weyl space)

is a smooth 3-manifold M with

- *conformal metric* $[g]$
- *torsion-free connection,* ∇ *compatible with* $[g]$, $\nabla[g] \in [g]$.
- *with* $Sym_0 \, \mathrm{Ricci}(\nabla) = 0$.

Cartan (1943):

- The Einstein-Weyl equations determine evolution from initial data (4 free functions of 2 variables).
- If $\exists \, g \in [g]$ with $\nabla g = 0$, metric is flat or (anti-) de Sitter.

Theorem

Einstein-Weyl equations $\Leftrightarrow \exists$ *totally geodesic null two-surfaces,* Minitwistors*, orthogonal to each null covector at each point.*

This is a 'Lax pair' $\rightsquigarrow$ equations are 'integrable'.

Main theorem

Theorem (LeBrun & M.)

There is a natural 1:1 correspondence between

- *Lorentzian Einstein-Weyl spaces* $(\mathcal{M}, [g], \nabla)$ *that are*
  1. *globally hyperbolic,* C^∞*,*
  2. *space and time oriented*
  3. *conformally compact (asymptotically de Sitter),*

  and
- *orientation reversing diffeomorphisms* $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ *up to Mobius tranformations.*

Forwards: $\psi : \mathscr{I}^- \to \mathscr{I}^+$ from null geodesic scattering.
For de Sitter, ψ is the antipodal map

$$\zeta \to -1/\bar{\zeta}.$$

Conformal compactification & infinity

Definition

$(\mathcal{M}, [g], \nabla)$ is conformally compact (asymptotically de Sitter) if

- *$\exists$ manifold with boundary $\widetilde{M}$ with C^∞ Lorentzian metric $\hat{g}$*
- *$\hat{g}$ is nondegenerate at $\partial\widetilde{M}$,*
- *$\exists$ diffeomorphism $\Phi : \mathcal{M} \simeq \widetilde{M} - \partial\widetilde{M}$ and $[\Phi^*\hat{g}] = [g]$*
- *For u a nondegenerate defining function for $\partial\widetilde{M}$ and $\nabla\hat{g} = \alpha \otimes \hat{g}$, then $\alpha - 2u^{-1}\mathrm{d}u$ is C^∞ on $\widetilde{M}$.*

Globally hyperbolic and space & time oriented gives

$$\widetilde{M} = \Sigma \times [-1, 1]$$

with $t \in [-1, 1]$ time and Σ space, but with $t = \pm 1$ being future/past infinity $\mathscr{I}^\pm$ resp., so

$$\partial\widetilde{M} = \mathscr{I}^+ \cup \mathscr{I}^- , \qquad \mathscr{I}^\pm \simeq \Sigma \, .$$

From Einstein-Weyl to Scattering map

Let $\mathbb{T}_{\mathbb{R}}$ be the 2-dim space of Cartan's null 2-surfaces Σ_Z.

Lemma

The Σ_Z, $Z \in \mathbb{T}_{\mathbb{R}}$, are lightcones of points of $\in \mathscr{I}^{\pm}$.

As $\mathscr{I}^-$ is approached, each 2-surface, Σ_Z, focuses on some point $p \in \mathscr{I}^-$ and refocuses at a unique point $q \in \mathscr{I}^+$.

Corollary

$\mathscr{I}^{\pm} \simeq S^2 = \mathbb{T}_{\mathbb{R}}$. *By assumption, conformal structure on* $\mathscr{I}^{\pm}$ *is non-degenerate, so* $\mathscr{I}^{\pm} \simeq \mathbb{CP}^1$.

Definition

Thus, identification along lightrays $\psi : \mathscr{I}^- \to \mathscr{I}^+$ *is well defined and defines the* Scattering map

$$\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$$

a diffeo defined up to Mobius transformations.

For de Sitter, ψ is the antipodal map.

Einstein-Weyl space from scattering map

Definition

The twistor space $\mathbb{T} := \mathscr{I}^+ \times \mathscr{I}^- = \mathbb{CP}^1 \times \mathbb{CP}^1$.

We have:

- The graph of the scattering map in $\mathbb{CP}^1 \times \mathbb{CP}^1$ defines $\mathbb{T}_\mathbb{R}$ as a totally real submanifold of $\mathbb{T}$.
- Away from $\mathbb{T}_\mathbb{R}$, points of $\mathbb{T} \leftrightarrow$ to timelike geodesics.
- Each $x \in \mathcal{M}$ corresponds to a holomorphic disc $D_x \subset \mathbb{T}$ with boundary $\partial D_x \subset \mathbb{T}_\mathbb{R}$ via the timelike geodesics thru x.

Example For de Sitter, discs are images of unit disc $|\zeta| \leq 1$ under

$$\zeta \to \left(\frac{a\zeta + b}{c\zeta + d}, \frac{-\bar{d}\zeta - \bar{c}}{\bar{b}\zeta + \bar{a}} \right) .$$

This realizes de Sitter as the quotient $SL(2, \mathbb{C})/SL(2, \mathbb{R})$.
Here, the boundaries of the discs are the round circles in $\mathbb{T}_\mathbb{R}$.

Recovering $\widetilde{M}$ from embedded $\mathbb{T}_{\mathbb{R}} \subset \mathbb{T}$

Let $\widetilde{M} =$ moduli space of hol. discs $D \subset \mathbb{T}$ with $\partial D \subset \mathbb{T}_{\mathbb{R}}$.

- Finding such holomorphic discs is a Fredholm problem of index 3.
- The discs are stable under deformations.
- Energy estimates $\rightsquigarrow$ existence & Gromov compactness.
- Gives compact 3d moduli space $\widetilde{M}$ with boundary.
- $x \in M := \widetilde{M} - \partial\widetilde{M} \leftrightarrow$ a holomorphic disc D_x, whereas $\partial\widetilde{M} \leftrightarrow$ generators of $\mathbb{CP}^1 \times \mathbb{CP}^1$.
- $x \in \mathcal{M} \leftrightarrow \partial \mathrm{D}_x \subset \mathbb{T}_{\mathbb{R}}$ topological circles that shrink down to a point as $x \to \partial\widetilde{M}$.
- That the above define bona fide $[g], \nabla$ follows by standard twistor methods (uses Liouville's theorem).

Other twistor constructions using holomorphic discs

LeBrun & M, math.DG/0211021, J. Diff. Geom. **61**, 2002:

$$\left\{\begin{array}{l}\text{Zoll projective structures}\\ \text{on } S^2\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Deformations of embedding}\\ \mathbb{RP}^2 \subset \mathbb{CP}^2\end{array}\right\}$$

LeBrun & M, Zoll Metrics, Branched Covers, and Holomorphic Disks, arxiv:1002.2993, Comm. Anal. Geom..

LeBrun & M, math.DG/0504582, Duke:

$$\left\{\begin{array}{l}\text{Self-dual conformal}\\ \text{structures on } S^2 \times S^2\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Deformations of embedding}\\ \mathbb{RP}^3 \subset \mathbb{CP}^3\end{array}\right\}$$

M, math-ph/0505039, Crelle:

$$\left\{\begin{array}{l}\text{global self-dual } U(n)\\ \text{Yang-Mills fields in split}\\ \text{signature on } S^2 \times S^2,\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Hol. Vector bundle}\\ E \to \mathbb{CP}^3 \text{ \& hermitian metric}\\ H \text{ on } E|_{\mathbb{RP}^3}\end{array}\right\}.$$

LeBrun, Twistors, Holomorphic Disks, and Riemann Surfaces with Boundary, arxiv:math/0508038, Proc CRM.

Conclusions and generalization

- Asymptotically de Sitter Einstein-Weyl spaces are 1:1 with orientation reversing diffeos $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$.
- Relies on integrability:
  null geodesic scattering $PT^*\mathscr{I}^- \to PT^*\mathscr{I}^+$ is function of 3-variables, but lifted from $\psi : \mathscr{I}^- \to \mathscr{I}^+$ a function of 2.

But:

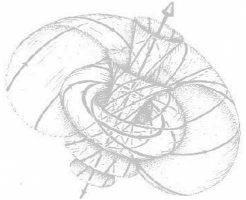
- Holomorphic discs suggest being tied to 3d where projective lightcone is 1d.
- Integrability means trivial gravitational scattering!

Can we say something for general space-times in arbitrary dimensions, perhaps Einstein, but with non-trivial scattering?

- Yes with 'Ambitwistors'.

Ambitwistors in 4d

Ambitwistor spaces: spaces of complex null geodesics $\mathbb{A}$.



- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .

Ambitwistor Strings:

- Twistor-string for $N = 4$ Yang-Mills [Witten, Roiban, Spradlin, Volovich, 2003/4].
- $N = 8$ supergravity [Cachazo-Geyer, Cachazo-Skinner, CMS, 2012], [Skinner, 2013]
- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- Place original twistor-string in multi-dimensional context.
- In 4d, $\mathbb{A} = T^*\mathbb{PT} = T^*\mathbb{PT}^* \rightsquigarrow$ ambidextrous version. [Geyer, Lipstein & M 1404.6219.]

Provide string theories at $\alpha' = 0$ for field theory amplitudes.

Geometry of ambitwistor space

Holomorphic category, i.e. (M, g) complexification of $(M_{\mathbb{R}}, g_{\mathbb{R}})$.

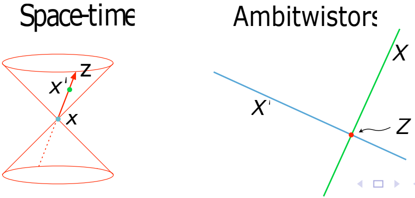
- $\mathbb{A} :=$ space of complex null geodesics with scale of P.
- $\mathbb{A} = T^*M|_{P^2=0}/\{D_0\}$ where $D_0 := P \cdot \nabla =$ geodesic spray.
- D_0 has Hamiltonian P^2 wrt symplectic form $\omega = dP_\mu \wedge dx^\mu$.
- Symplectic potential $\theta = P_\mu dx^\mu$, $\omega = d\theta$, descend to $\mathbb{A}$.

Projectivise: $P\mathbb{A} :=$ space of *unscaled* complex light rays.

- On $P\mathbb{A}$, $\theta \in \Omega^1_{P\mathbb{A}} \otimes L$ is a holomorphic contact structure.

Theorem (LeBrun 1983)

The complex structure on $P\mathbb{A}$ determines M and conformal metric g. The correspondence is stable under arbitrary deformations of the complex structure of $P\mathbb{A}$ that preserve θ.



Linearized LeBrun correspondence

θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:

$$\text{Deformations of complex structure } \leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)\,.$$

Analyze with double fibration

$$\begin{array}{ccc} & PT^*M|_{P^2=0} & \\ q \swarrow D_0 & & \searrow \\ P\mathbb{A}_S & & M\,. \end{array}$$

Key example: On flat space-time, set $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_{\mu\nu}$ then

$$\delta\theta = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}\epsilon_{\mu\nu}P^\mu P^\nu\,, \qquad \bar\delta(z) = \bar\partial\frac{1}{2\pi i z}$$

- Dolbeault form of Penrose's scattering Hamiltonian for null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$.
- Support on $k\cdot P = 0 \Rightarrow$ the *scattering equations*.

Ambitwistors in 4 dimensions

- In 4d, can solve $P^2 = 0$ with 2 cpt spinors:

$$P_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}\,, \qquad \alpha = 0, 1, \quad \dot{\alpha} = \dot{0}, \dot{1}\,.$$

- Replace $x^{\alpha\dot{\alpha}}$ by $(\mu^\alpha, \tilde{\mu}^{\dot{\alpha}})$

$$\mu^{\dot{\alpha}} = -ix^{\alpha\dot{\alpha}}\lambda_\alpha\,, \qquad \tilde{\mu}^\alpha = ix^{\alpha\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}$$

  constant under $x^{\alpha\dot{\alpha}} \to x^{\alpha\dot{\alpha}} + \alpha\lambda^\alpha\tilde{\lambda}^{\dot{\alpha}}$.

- Set

$$Z = (\lambda_\alpha, \mu^{\dot{\alpha}}) \in \mathbb{T}\,, \quad W = (\tilde{\mu}^\alpha, \tilde{\lambda}_{\dot{\alpha}}) \in \mathbb{T}^*$$

  where $Z \cdot W := \lambda_\alpha\tilde{\mu}^\alpha + \mu^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}}$ and incidence $\rightsquigarrow Z \cdot W = 0$.

- Thus:

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\}/\{Z \cdot \partial_Z - W \cdot \partial_W\}\,.$$

- Symplectic potential

$$\Theta := P \cdot dx = iW \cdot \mathrm{d}Z - iZ \cdot \mathrm{d}W.$$

Field equations on 4d ambitwistor space

Define *formal neighbourhoods* $\mathbb{PA}_{[k]}$ of $\mathbb{PA} \subset \mathbb{PT} \times \mathbb{PT}^*$ to extend objects off $\mathbb{PA}$ to $O((Z \cdot W)^{k+1})$ into ambient $\mathbb{PT} \times \mathbb{PT}^*$.

- 1. Off-shell YM fields $\leftrightarrow$ hol vector bundle $E \to \mathbb{PA}$.
  2. YM equs $\leftrightarrow$ E extends to $E \to \mathbb{PA}_{[3]}$. Isenberg, Green & Yasskin (1978)
- 1. Conformal gravity equs $\leftrightarrow$ $\mathbb{PA}$ admits extension to $\mathbb{PA}_{[5]}$.
  2. Einstein $\leftrightarrow$ extension to $\mathbb{PA}_{[6]}$, Baston & M. 1986, LeBrun 1991

Witten's (1978) approach:

- Can naturally super-symmetrize by extending $\mathbb{T} = \mathbb{C}^{4|\mathcal{N}}$.
- Formal neighbourhoods $\leftrightarrow$ susy via $\mathbb{PA}_{[k]} \leftrightarrow \mathcal{N} = k$.
- Yang-Mills field equs in terms of $\exists \mathcal{N} = 3$ SUSY for YM [Harnad, Hurtubise & Schnider 1986].
- Story unfinished for gravity.

Null geodesics and ambitwistor strings

Complexify: $(M_{\mathbb{R}}, g_{\mathbb{R}})$ real space-time dim. d, $\rightsquigarrow (M, g)$.
Let Σ be a Riemann surface.

Ambitwistor string action:

- Let $X : \Sigma \to M$, $P \in K \otimes X^* T^* M$

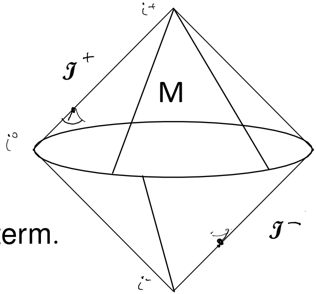
$$S = \int P \cdot \bar{\partial} X - e P^2/2\,.$$

  with $e \in \Omega^{0,1} \otimes T$, where $K = \Omega^{1,0}_{\Sigma}$ and $T = T^{1,0}\Sigma$.
- $e \rightsquigarrow P^2 = 0$,
- gauge: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

Solutions mod gauge are holomorphic maps to $\mathbb{A}$,
Ambitwistor space: $\mathbb{A} = T^* M|_{P^2=0}/\{\text{gauge}\}$.

Conformal scattering and the tree S-matrix

- Pose asymptotic data $g_{\rm in}$:
  resp. $\pm$ frequency at $\mathscr{I}^{\pm}$.
- Solve for g on M s.t. $\pm$ freq. parts at $\mathscr{I}^{\pm}$ agree with $g_{\rm in}$.
- S-matrix is functional of $g_{\rm in}$



$$\mathcal{S}[g_{in}] = S_{EH}[g] := \frac{1}{\kappa^2}\int_M R\, d\, vol + \text{bdy term.}$$

- generating fn $g^{\mp}|_{\mathscr{I}^{\pm}} = \partial\mathcal{S}/\partial g^{\pm}_{\rm in}$.

Usually evaluate S-matrix perturbatively

- Pose data $g_{\rm in} = \sum_{i=1}^{n} \epsilon_i g_i|_{\mathscr{I}}$, and solve for g on M.
- (tree) S-matrix is

$$\mathcal{M}(g_1, \dots, g_n) = \text{ Coeff of } \prod_i \epsilon_i \text{ in } S_{EH}[g]$$

[Usually use Fourier modes for g_j: $g_{j\mu\nu} = \xi_{j\mu}\xi_{j\nu}e^{ik_j\cdot x}$.]

Strings and the S-matrix

Key proposal field theory S-matrix

$$\mathcal{S}[g_{in}] = \int D[XP] \mathrm{e}^{iS^{g_{in}}_{\text{string}}}$$

The miracle is that (perturbatively at least) the RHS is computed directly from $g_{in} = \sum_i \epsilon_i g_i$

$$S^{g_{in}}_{\text{string}} = S^{g_{flat}}_{\text{string}} + \sum_i \epsilon_i \int V_i$$

Where V_i are *vertex operators* represented here e.g. by $\delta\theta = \bar{\delta}(k \cdot P)\mathrm{e}^{ik \cdot x}\epsilon_{\mu\nu}P^\mu P^\nu$. Thus

$$\mathcal{M}(g_1, \ldots, g_n) = \int D[XP \ldots] V_1 \ldots V_n \, \mathrm{e}^{iS^{flat}_{string}} =: \langle V_1 \ldots V_n \rangle$$

Twistor and 4d ambitwistor strings

with Yvonne Geyer & Arthur Lipstein arxiv:1404.6219

Contact structure Θ on $\mathbb{PA}$ gives action

$$S = i\int_\Sigma W\cdot\bar\partial Z - Z\cdot\bar\partial W + aZ\cdot W\,.$$

- a Lagrange multiplier $\rightsquigarrow Z\cdot W = 0$, gauge quotients phase.
- At $\mathcal{N}=4$ similar to Witten/Berkovits twistor-string.

Now consider ambidextrous model perhaps with $\mathcal{N}<4$

- Take $Z, W \in K^{1/2}$, (reduce to $Z\cdot W = 0$ globally).
- Use wave fns from both $\mathbb{PT}$ and $\mathbb{PT}^* \rightsquigarrow$ YM vertex ops

$$V_i = \int_\Sigma f_i(Z) J\cdot t_i\,, \qquad f_i \in H^1(\mathbb{PT},\mathcal{O})$$

$$\widetilde{V}_i = \int_\Sigma \tilde{f}_i(W) J\cdot t_i, \qquad \tilde{f}_i \in H^1(\mathbb{PT}^*,\mathcal{O})$$

J being worldsheet current algebra and $t_i \in \mathfrak{g}$ Lie algebra.

New Amplitude formulae in 4d

N^{k}MHV amplitude

$$\mathcal{A}(1,\dots,n) = \langle V_1 \dots \tilde{V}_k V_{k+1} \dots V_n \rangle = \int D[ZW]\, \tilde{V}_1 \dots \tilde{V}_k V_{k+1} \dots V_n\, \mathrm{e}^{iS}\,.$$

- For momentum eigenstates $k_i = \lambda_i \tilde{\lambda}_i$ take:

$$V_i = \int \frac{\mathrm{d}s_i}{s_i} \bar{\delta}^2(\lambda_i - s_i \lambda(\sigma_i)) \mathrm{e}^{is_i[\mu\,\tilde{\lambda}_i]} J \cdot t_i$$

$$\widetilde{V}_i = \int \frac{\mathrm{d}s_i}{s_i} \bar{\delta}^2(\tilde{\lambda}_i - s_i \tilde{\lambda}(\sigma_i)) \mathrm{e}^{is_i\langle\tilde{\mu}\,\lambda_i\rangle} J \cdot t_i$$

- Take exponentials into action to give:

$$S[Z,W] = \int_\Sigma W \cdot \bar{\partial} Z + \sum_{i=1}^{k} s_i [\mu \tilde{\lambda}_i] \delta^2(\sigma - \sigma_i) + \sum_{r=k+1}^{n} s_r \langle \tilde{\mu}\, \lambda_r \rangle \delta^2(\sigma - \sigma_r)$$

Gives sources

$$\bar{\partial}\lambda = \sum_r s_r \lambda_r \delta^2(\sigma - \sigma_r)\,, \qquad \bar{\partial}\tilde{\lambda} = \sum_i s_i \tilde{\lambda}_i \delta^2(\sigma - \sigma_i)$$

with solutions $\mu = \tilde{\mu} = 0$ and

$$\lambda(\sigma) = \sum_{r=k+1}^{n} \frac{\lambda_r}{(\sigma\,\sigma_r)}\,, \qquad \tilde{\lambda}(\sigma) = \sum_{i=1}^{k} \frac{\tilde{\lambda}_i}{(\sigma\,\sigma_i)}\,, \qquad \sigma_\alpha = \frac{1}{s}(1,\sigma)$$

- For Yang-Mills obtain amplitude

$$\mathcal{A}(1,\ldots,n) = \int \prod_{i=1}^{n} \frac{\mathrm{d}^2\sigma_i}{(\sigma_i\,\sigma_{i+1})} \prod_{i=1}^{k} \bar{\delta}^2(\tilde{\lambda}_i - \tilde{\lambda}(\sigma)) \prod_{i=k+1}^{n} \bar{\delta}^2(\lambda_i - \lambda(\sigma_i))$$

- Essentially the formula obtained by Witten (2005).
- The delta functions $\Rightarrow$ scattering equations refined by the MHV degree.

Gravity

Adapt Skinner model:

- Introduce skew *infinity twistors*

$$\langle Z_1 Z_2\rangle = I_{\alpha\beta} Z_1^\alpha Z_2^\beta\,, \quad [W_1\; W_2] = I^{\alpha\beta} W_{1\alpha} W_{2\beta}\,.$$

- include Fermions $(\rho, \tilde\rho) \in \mathbb{T}\times\mathbb{T}^*$ spinors on Σ.
- gauge currents from an $SL(1|2)$.
- Vertex ops from $h \in H^1(\mathbb{PT}, \mathcal{O}(2))$ and conjugates are

$$V_h = [W, \partial]h + \rho\cdot\partial[\tilde\rho, \partial]h, \qquad \widetilde{V}_{\tilde h} = \langle Z, \tilde\partial\rangle\tilde h + \tilde\rho\cdot\tilde\partial\langle\rho, \tilde\partial\rangle\tilde h\,.$$

- Amplitude formulae: replace $\prod \frac{1}{(i\, i{+}1)}$ by $\det{}'\,\mathcal{H}$

$$\mathbb{H}_{ij} = \frac{\langle i\, j\rangle}{(i\, j)}, \quad i,j \le k\,, \qquad \widetilde{\mathbb{H}}_{ij} = \frac{[i\, j]}{(i\, j)}\,, \quad i,j > k\,,$$

$$\mathcal{H}_{ij} = \begin{pmatrix} \mathbb{H} & 0 \\ 0 & \widetilde{\mathbb{H}} \end{pmatrix}, \qquad \mathcal{H}_{ii} = -\sum_{l\neq i} \mathcal{H}_{il}\,.$$

$$\mathcal{M}(1,\ldots,n) = \int \det{}'\mathcal{H} \prod_{i=1}^n \mathrm{d}^2\sigma_i \prod_{i=1}^k \bar\delta^2(\tilde\lambda_i - \tilde\lambda(\sigma)) \prod_{i=k+1}^n \bar\delta^2(\lambda_i - \lambda(\sigma_i))$$

Summary

- Formulae proved by recursion.
- Valid for any amount of SUSY unlike original twistor-strings.
- Higher dimensional analogues yield 10d supergravities
- successfully compute loop effects.
- Suggest surprising new structures: Colour/Kinematic dualities.
- String field theory should be a geometric formulation of (super-)gravities in ambitwistor space.

Happy Birthday Claude!