A dispersion relation for conformal theories

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McGill University

on: 1703.00278 ‘analyticity in spin in conformal theories’
1711.02031 with Fernando Alday;
work in progress (w/ Anh-Khoi Trinh;
Yan Gobeil, Alex Maloney & Zahra Zahraee)

talk at « Algebraic methods in mathematical physics »
Centre de Recherches Mathématiques, Montréal, 16 Juillet 2018

Outline

1. Conformal field theories:
   -Numerical bootstrap and spectrum

2. Lorentzian inversion formula:
   - Why operators are analytic in spin
   - Group theory problem: from SO(3) to SO(d,2)

3. Applications:
   - Large-spin expansions, and extension to J= 0
   - CFTs dual to gravity: causality&bulk locality

We'll be interested in conformal field theories in d≥2

Any QFT defines correlators of local operators:

$$\langle 0|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle$$

CFT: scale and conformal invariant

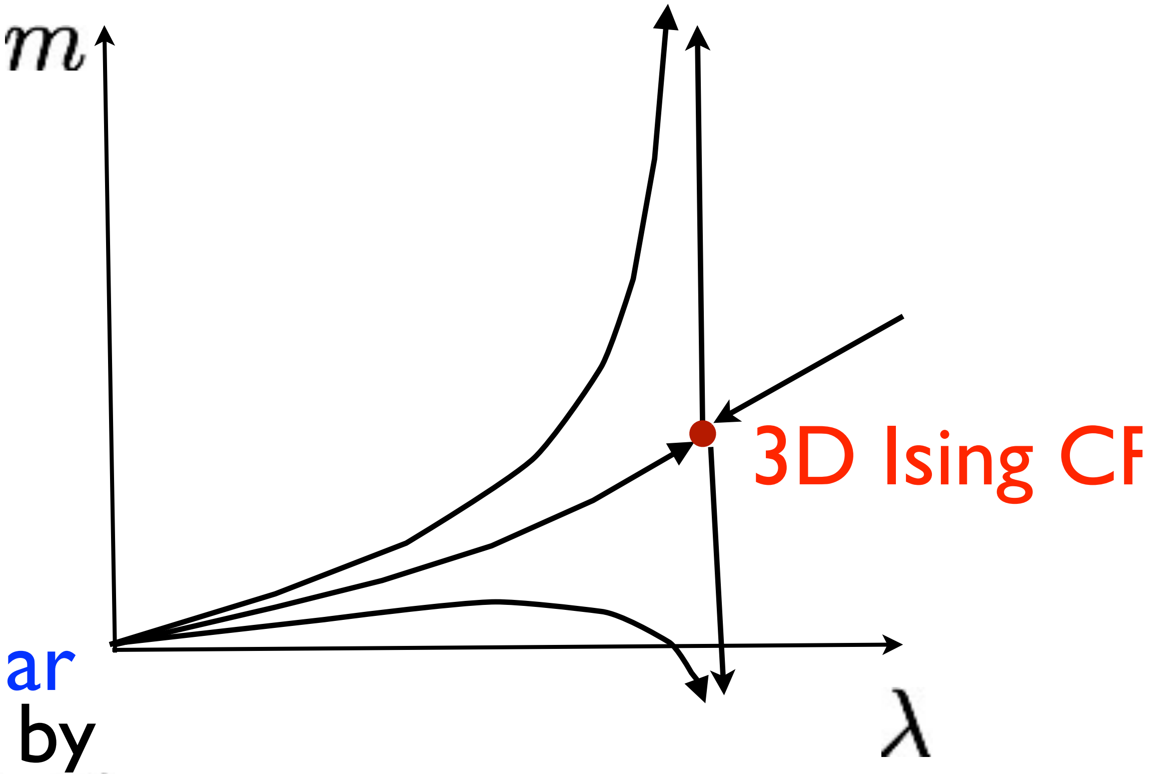
-Ubiquitous near second-order phase transitions;

-Short-distance limit of strong force (QCD);

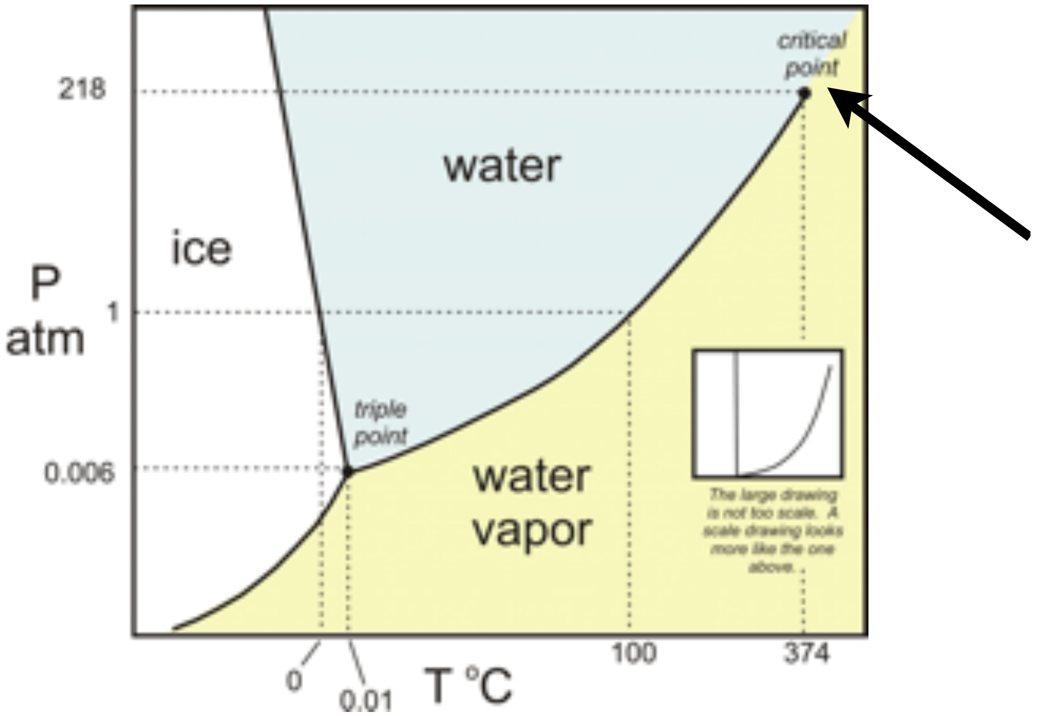
⇒ important mileposts in the space of all QFTs!

3D Ising Model: IR fixed point of Z2-symmetric scalar field theory

$$L = (\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4 \ \left[+ \kappa\phi^6 \ldots \right]$$



free scalar
perturbed by
$m^2\phi^2 + \lambda\phi^4$



[slide from Rychkov]

Lightest operators: $\Delta_\sigma = 0.5181489(10),$ Z2 odd

$\Delta_\epsilon = 1.412625(10),$ Z2 even

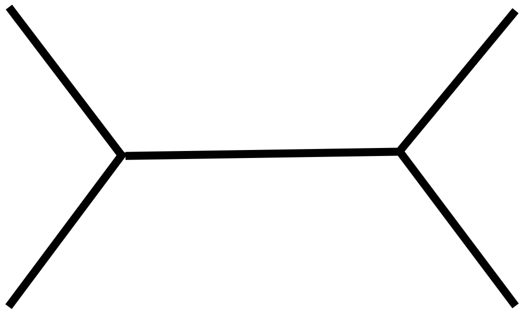
CFT: any 3 points can be mapped to 0,1,∞.
2- and 3-points correlators fixed by symmetry, up to #'s

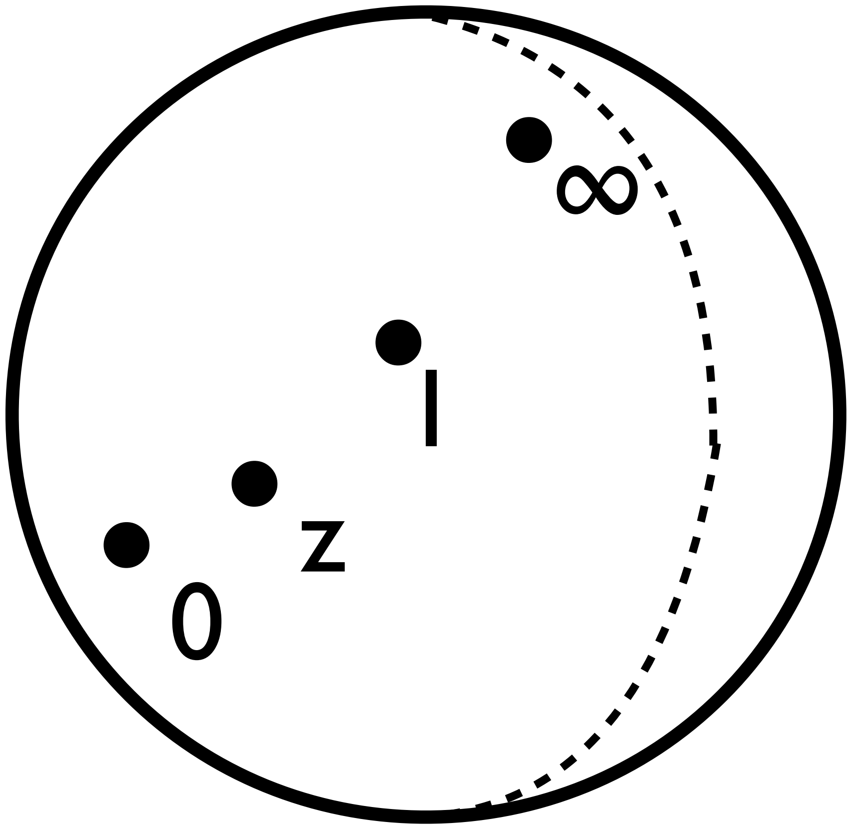
$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1-x_2|^{2\Delta_i}}$$ « dimensions »

$$\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle \propto f_{ijk}$$ « OPE coefficients »

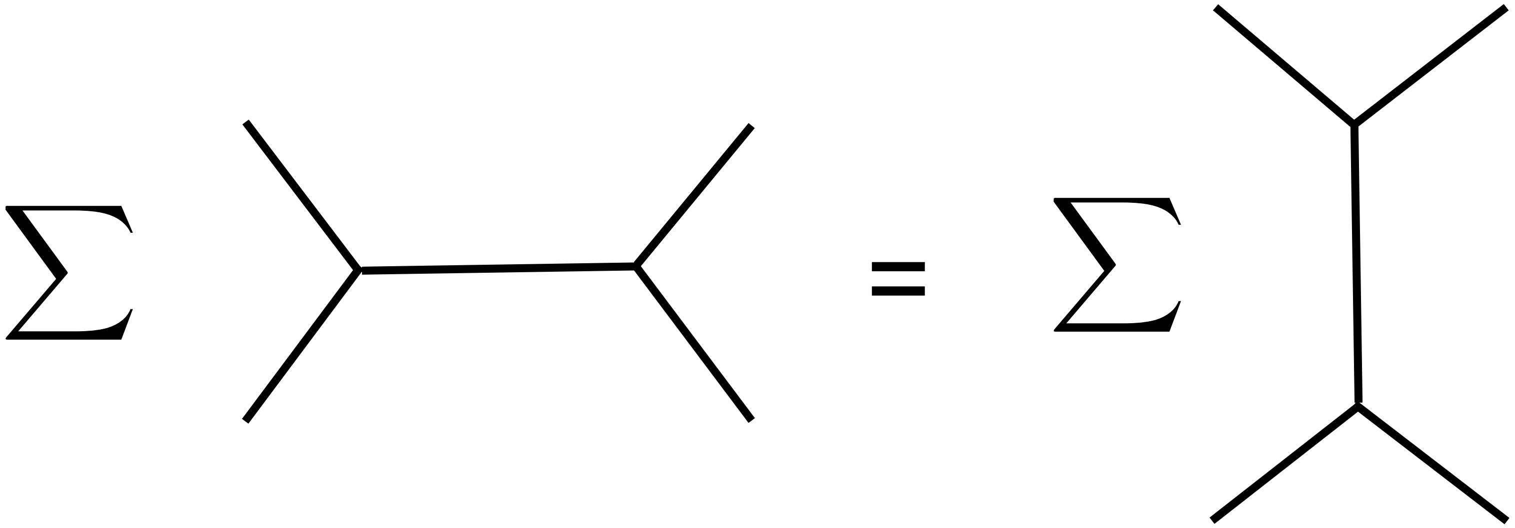
4-points determined by OPE:

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_k f_{12k}f_{34k}G_{J_k,\Delta_k}(z,\bar{z})$$





CFT = solution $\{\Delta_i, f_{ijk}\}$ to crossing equation

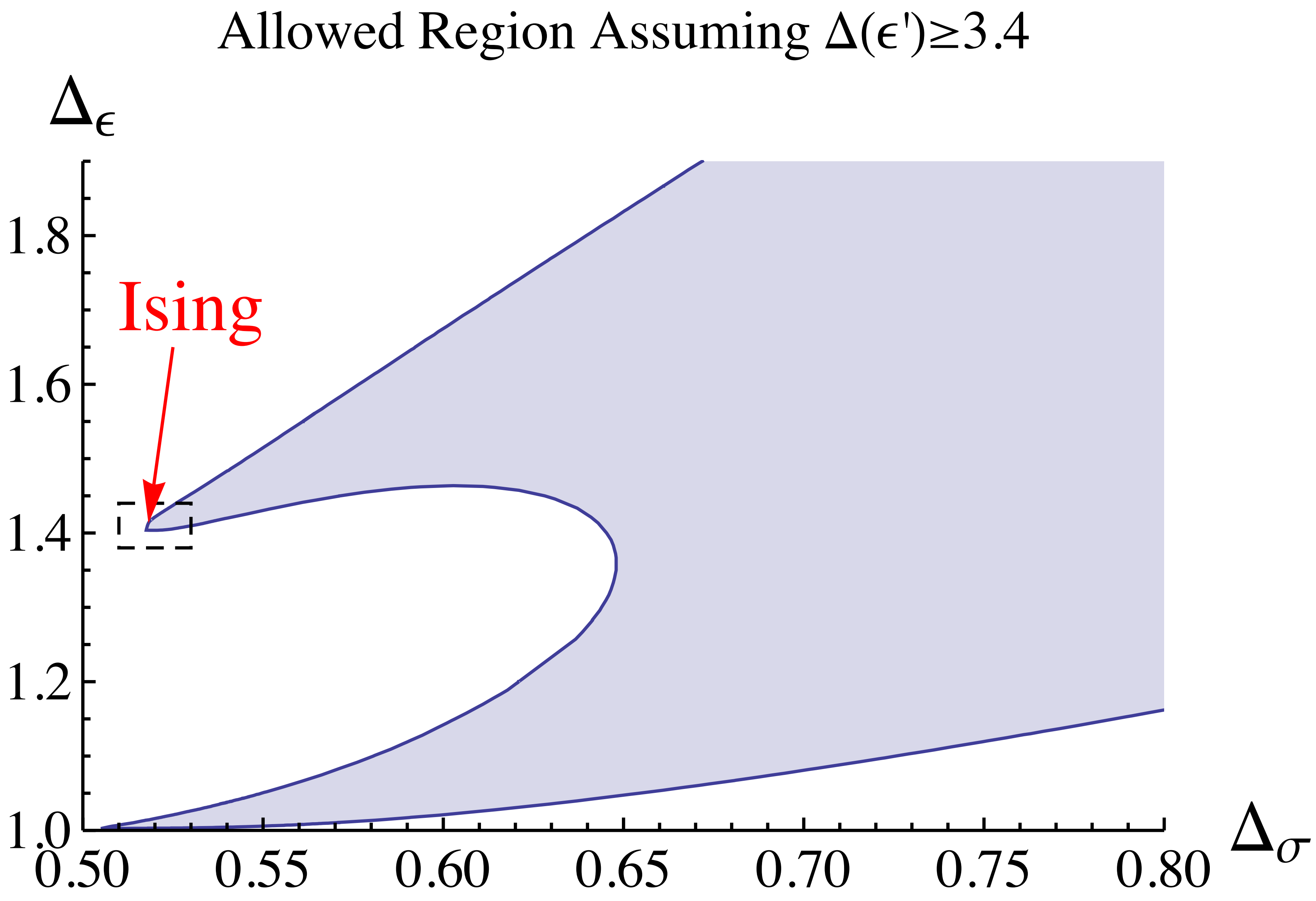


= infinite constraints on infinite unknowns!

‘bootstrap’

key for (unitary) d>2: coefficients are positive $(f_{ijk})^2 \geq 0$

[Rattazi, Rychkov, Tonni & Vichi '08



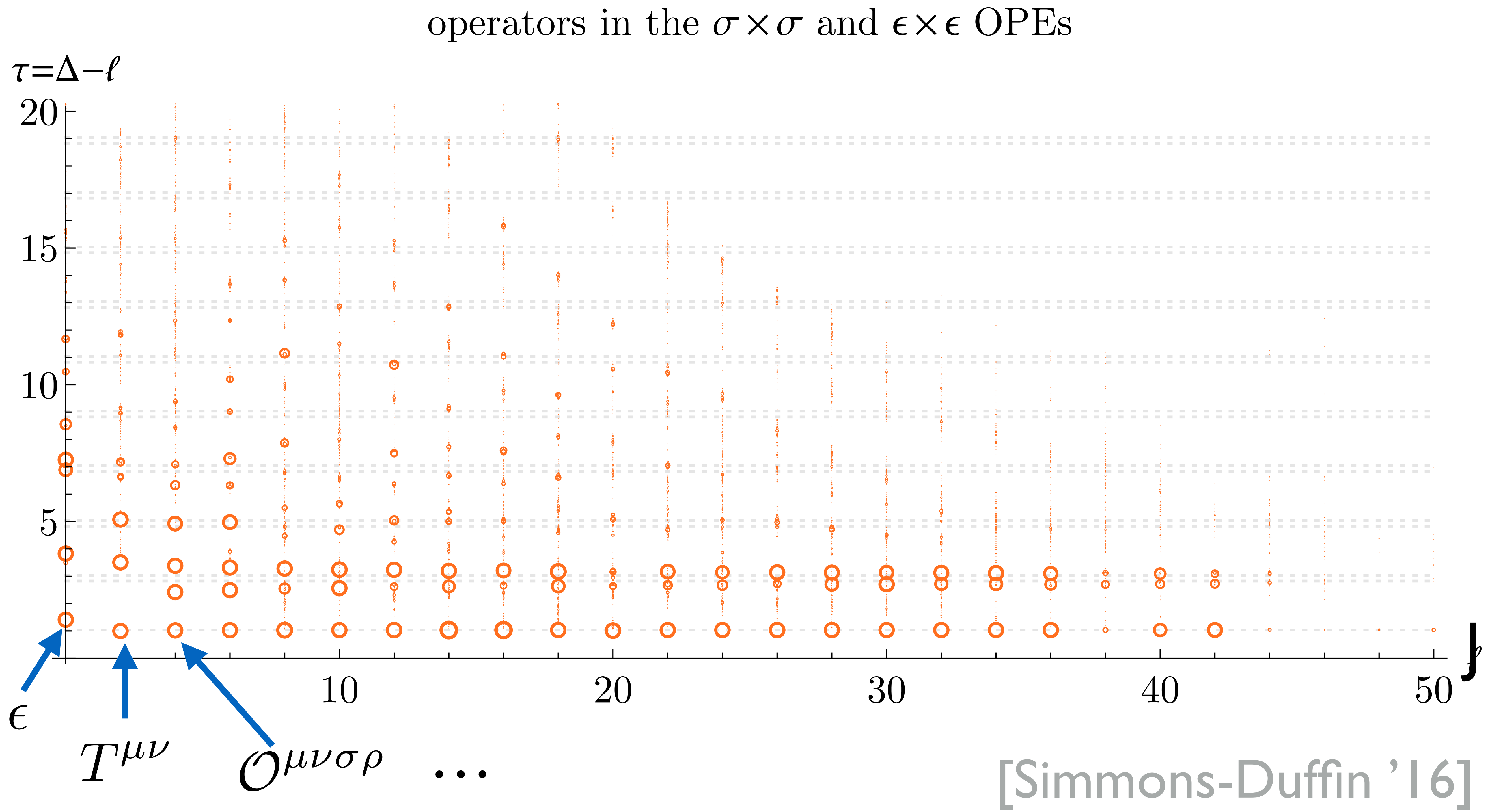
adding inequalities seems to isolate physical models!

Consistency alone determines critical exponents!

spin & $\mathbb{Z}_2$	name	Δ	OPE coefficient
$\ell = 0,\ \mathbb{Z}_2 = -$	σ	$0.518154(15)$	
$\ell = 0,\ \mathbb{Z}_2 = +$	ϵ	$1.41267(13)$	$f^2_{\sigma\sigma\epsilon} = 1.10636(9)$
	ϵ'	$3.8303(18)$	$f^2_{\sigma\sigma\epsilon'} = 0.002810(6)$
$\ell = 2,\ \mathbb{Z}_2 = +$	T	3	$c/c_{\text{free}} = 0.946534(11)$
	T'	$5.500(15)$	$f^2_{\sigma\sigma T'} = 2.97(2) \times 10^{-4}$

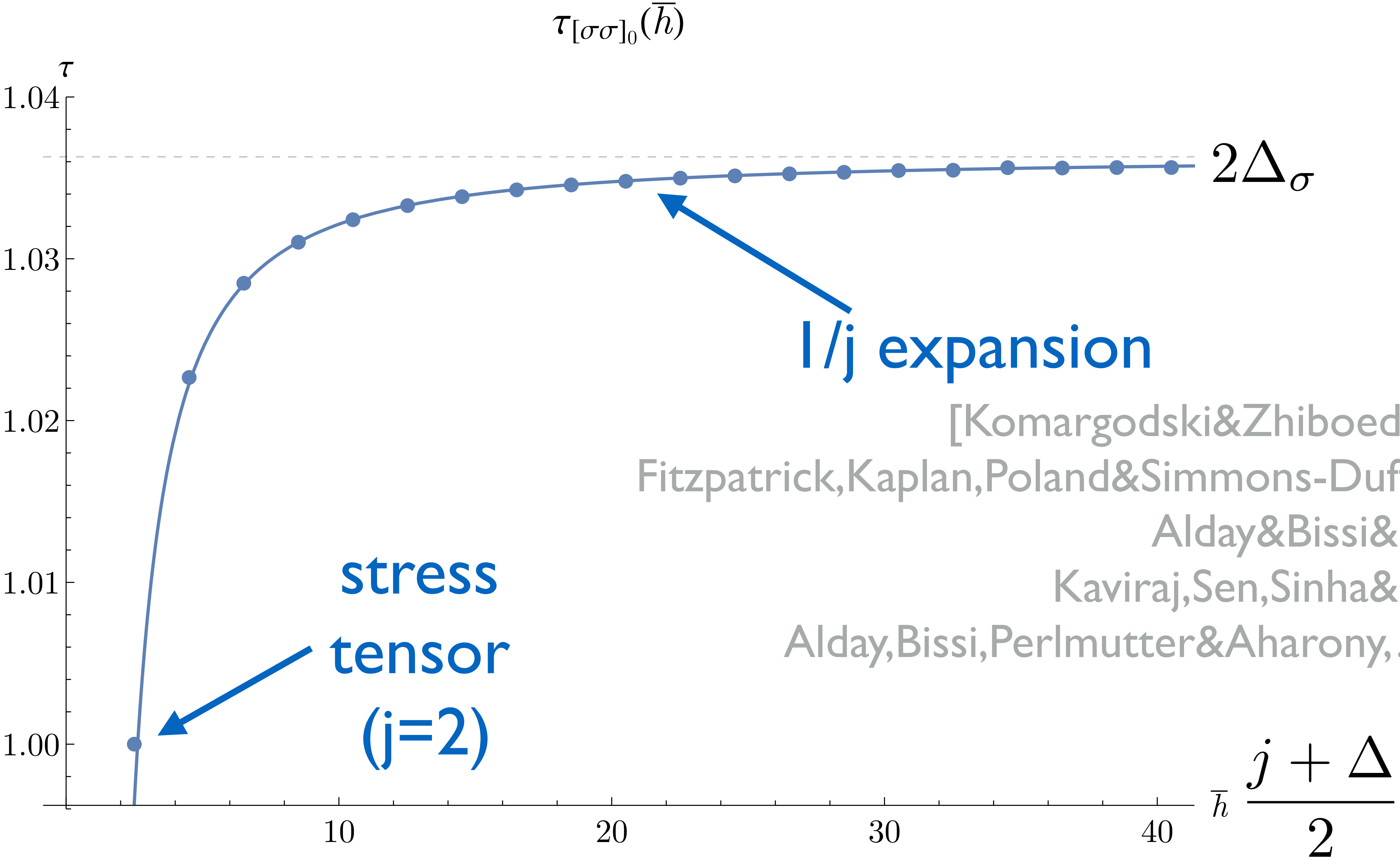
[El-Showk,Paulos,Poland,
Rychkov,Simmons-Duffin&Vichi '14]

Numerical methods also probe the spectrum



[Simmons-Duffin '16]

Lowest trajectory in 3D Ising



1/j expansion

[Komargodski&Zhiboedov,
Fitzpatrick,Kaplan,Poland&Simmons-Duffin,
Alday&Bissi&…,
Kaviraj,Sen,Sinha&…,
Alday,Bissi,Perlmutter&Aharony,…]

stress tensor (j=2)

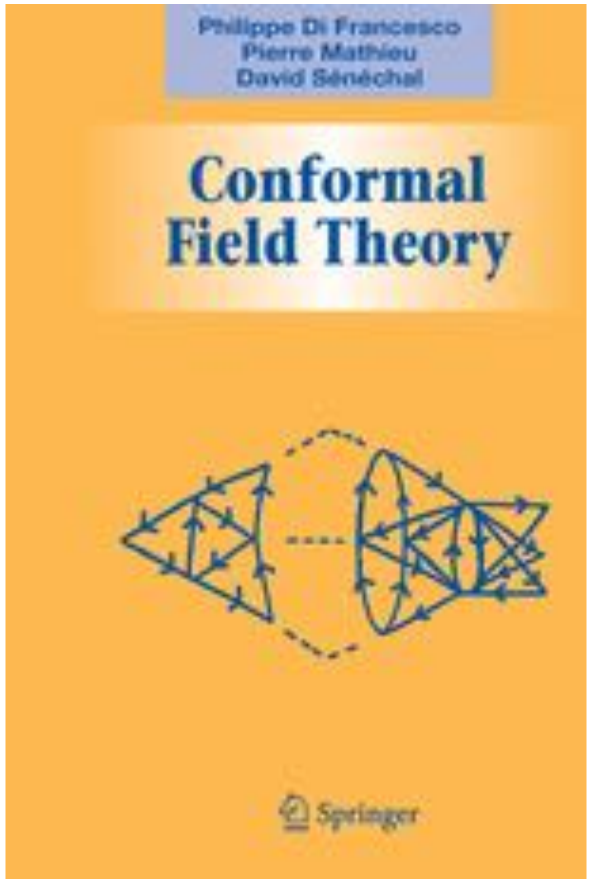
[Alday&Zhiboedov ’15;
Plot from Simmons-Duffin ’16]

In stat.mech, hard to understand such a continuous curve

we’ll argue it’s origin is: 3D Ising = unitary Minkowski CFT

Euclidean 3D CFT ⇒ Lorentzian 2+1D CFT

Wick
rotation



Philippe Di Francesco
Pierre Mathieu
David Sénéchal

Conformal Field Theory

Springer

+

The Analytic S-Matrix

R.J. EDEN
P.V. LANDSHOFF
D.I. OLIVE
J.C. POLKINGHORNE

Cambridge University Press

=

New tool:
CFT dispersion relation

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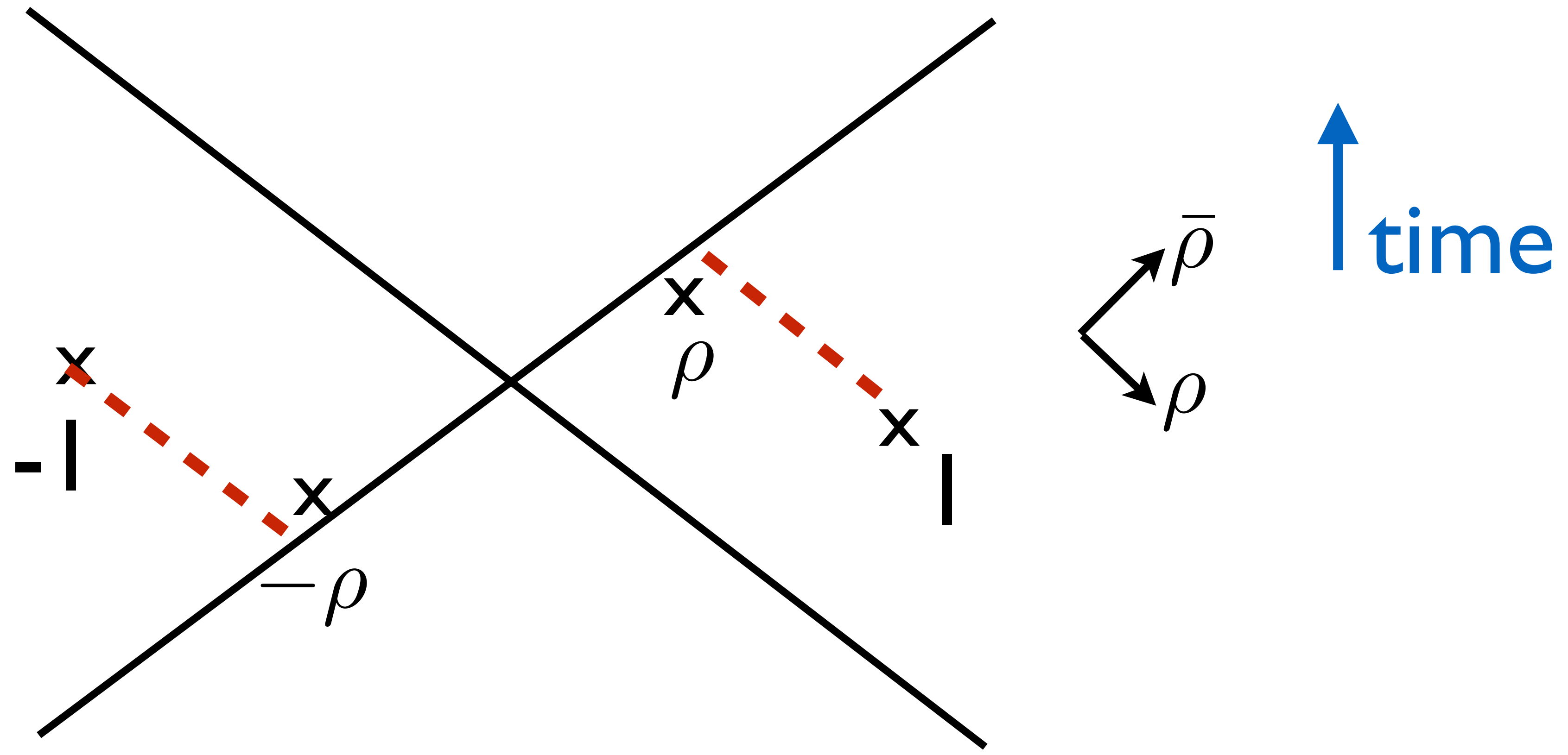
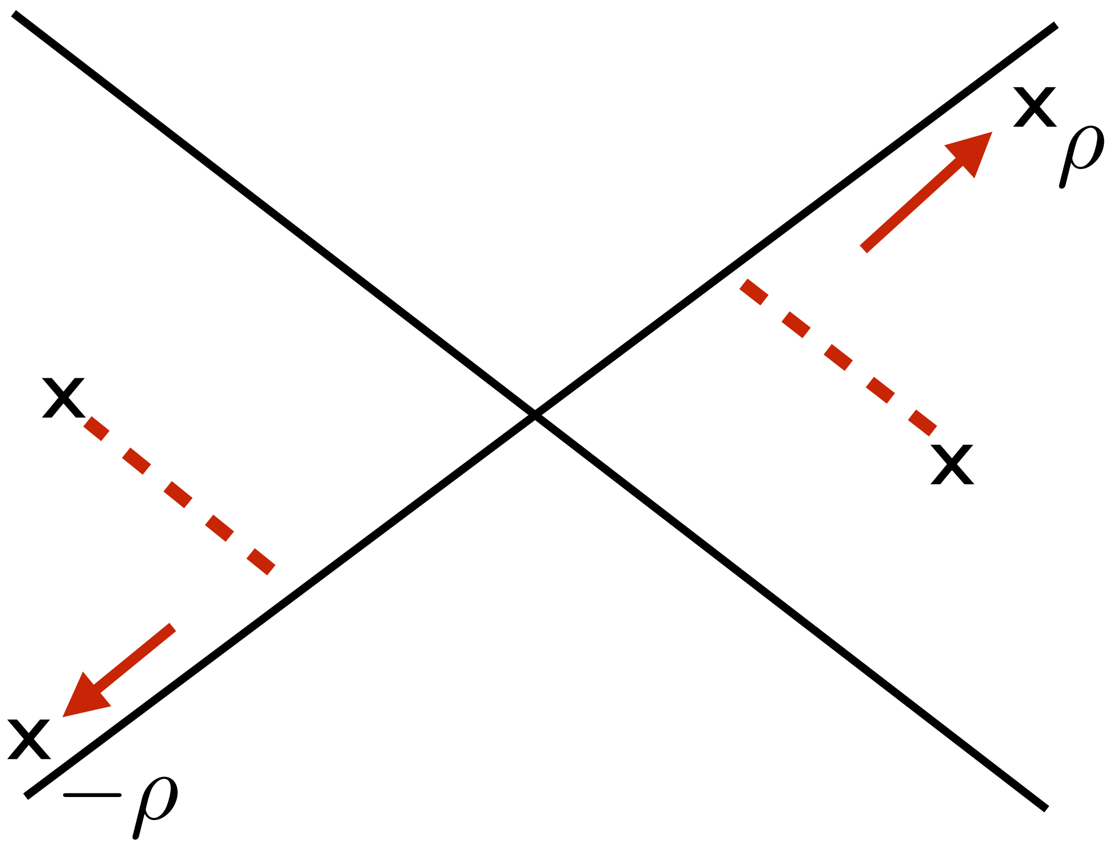
Consider four points in space-time at $(1, -1, \rho, -\rho)$

[Rychkov& Hogervorst '13]



When all spacelike, OPE ~ $\sum f^2 \rho^{\#} \bar{\rho}^{\#}$ converges

Now boost $(\rho, -\rho)$



large boost = 'Regge limit'

OPE diverges, yet
correlator stays bounded [cross-channel OPE]

That gives 'analyticity in spin'

Toy example: amplitude f(E) that's:

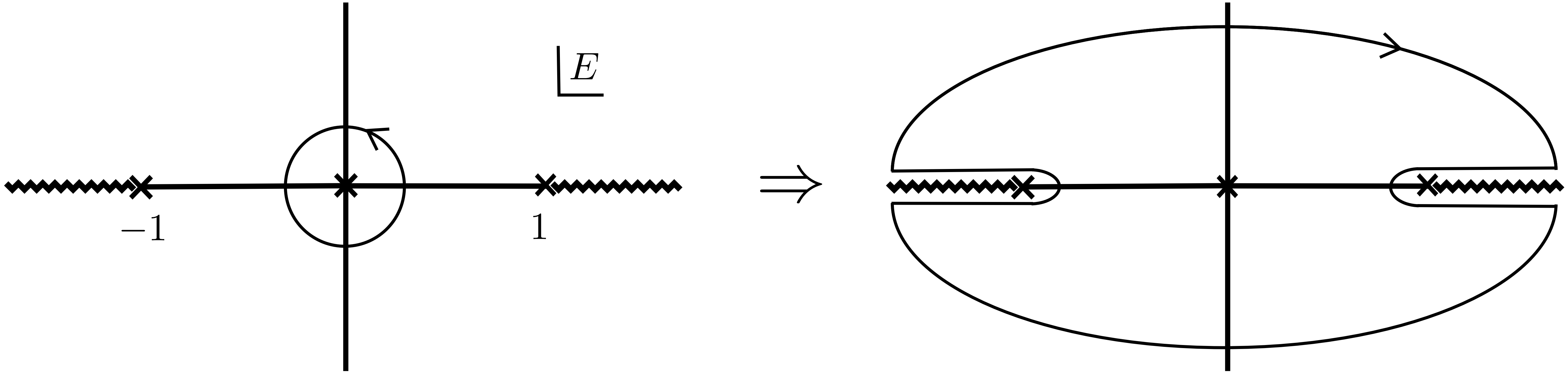
1. Analytic in cut plane
2. |f(E)| bounded at large |E|
3. Has Taylor series at small E

$$f(E) = \sum_{J=0}^{\infty} f_J E^J$$

(=Correlator with $E = \sqrt{\bar{\rho}/\rho} = \exp(\text{boost})$)

Q: What does this 'nice behavior' imply for series?

⇒ can write coefficients as integral over branch cut



$$f_J \equiv \frac{1}{2\pi i}\oint_{|E|<1}\frac{dE}{E}E^{-J}f(E)$$

$$= \frac{1}{2\pi}\int_1^\infty \frac{dE}{E}E^{-J}\,\big(\text{Disc}\,f(E) + (-1)^J\text{Disc}\,f(-E)\big) \qquad (J > 0),$$

⇒ coefficients f_J are analytic in J (& bounded at large Im J)

[for odd/even J separately]

Resumming series gives Kramers-Kronig relation:

$$f(E) = f(\infty) + \int_{|E'|>1} \frac{dE' \, \mathrm{Disc}\, f(E')}{2\pi(E' - E - i0)}$$

Ex: Re(f) ~ phase velocity of light
Im(f) ~ absorption by medium

Absorptive part determines propagation

Causality: 'no instantaneous action at a distance'.
Forces mediated by exchanging excitations.

3D Ising spectrum: ‘experimental’ evidence for a CFT Kramers-Kronig relation!

Froissart-Gribov formula: analyticity for SO(3) partial waves

SO(3) partial waves: $a_j(s) = \int_{-1}^{1} d\cos\theta\, P_j(\cos(\theta))\mathcal{M}(s,t(\cos\theta))$

+

disp. relation: $\mathcal{M}(s,t) = \int \frac{dt'}{\pi(t-t')} \mathrm{Im}\, \mathcal{M}(s,t')$

$+(t \leftrightarrow u)$

=

analyticity in spin $a_j(s) = \int_{1}^{\infty} d\cosh\eta\, Q_j(\cosh(\eta))\mathrm{Im}\,\mathcal{M}$

$+(-1)^j(t \leftrightarrow u)$

[Froissart-Gribov ~60]

foundation of Regge theory

Generalizing it to CFT

Euclidean | Lorentzian

Taylor series:

E^J $\longrightarrow$ E^{-J}

Rotation symmetry:

SO(2) $\cos(j\theta) \longrightarrow e^{-j\eta}$ SO(1,1)

SO(3) $P_j(\cos\theta) \longrightarrow Q_j(\cosh\eta)$ SO(2,1)

[Froissart-Gribov ~'60]

Conformal symmetry:

SO(d+1,1) $G_{j,\Delta}(z,\bar{z}) \longrightarrow$???????? SO(d,2)

dispersion relation
for CFT data

A very physical problem:

‘reconstruct CFT data from absorptive part’

mapped to group theory!

« conformal blocks » G = solutions to quadratic (and quartic) Casimir eqs.

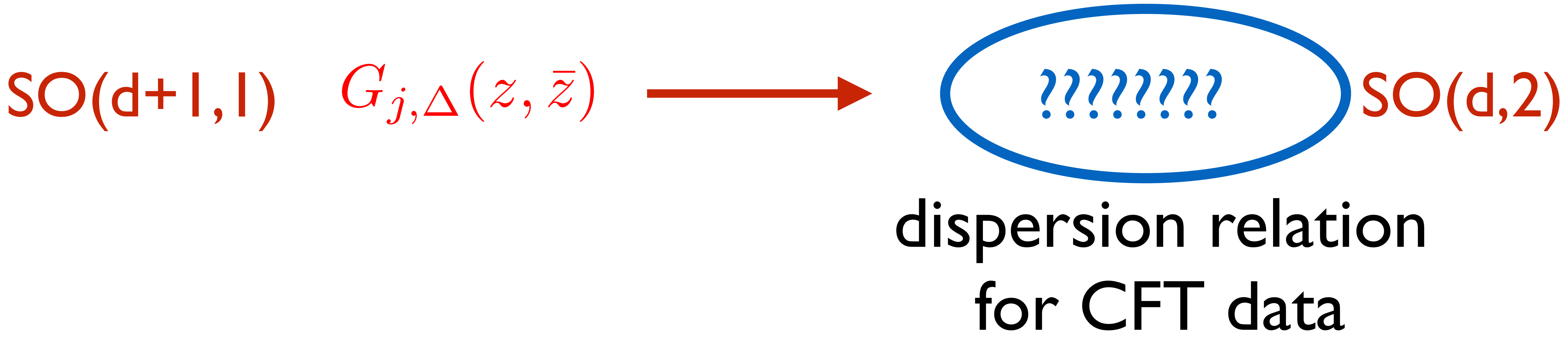
Ex, in 4D:

$$G_{J,\Delta}(z,\bar{z}) = \frac{z\bar{z}}{\bar{z}-z}\big[k_{\Delta-J-2}(z)k_{\Delta+J}(\bar{z}) - k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z})\big]$$

$$k_\beta(z) = \bar{z}^{\beta/2}\,{}_2F_1(\beta/2+a,\beta/2+b,\beta,z)\,.$$

(No closed form for non-even D, but good series expansions)

Group theory: Can you fill the missing box?



SO(d+1,1) $G_{j,\Delta}(z,\bar{z})$ $\longrightarrow$???????? SO(d,2)

dispersion relation
for CFT data

Hint: solution is some SO(d,2) Weyl reflection:

$$j \longleftrightarrow 2-d-j,$$

$$\Delta \longleftrightarrow d-\Delta,$$

$$\Delta \longleftrightarrow j+d-1$$

Calculation: split harmonic function F into pieces which vanish in each Regge limit:

$$2\cos(j\theta) = e^{ij\theta} + e^{-ij\theta}$$

$$\Rightarrow \quad F_{j,\Delta}(z,\bar{z}) = F^{(+)}_{j,\Delta} + F^{(-)}_{j,\Delta}$$

$\sim e^{-|\theta|}$ $\quad$ $\sim e^{-|\theta|}$

$(\theta \to +i\infty)$ $\quad$ $(\theta \to -i\infty)$

8 constraints, 4 parameters,
fingers crossed…

Lorentzian inversion formula

$$c(J,\Delta) = \int_{\diamond} \left[G_{\Delta+1-d,j+d-1}\right] \times \left[\mathrm{dDisc}\, G\right]$$

causal
diamond

s-channel
OPE coefficients

block with
J and Δ
exchanged

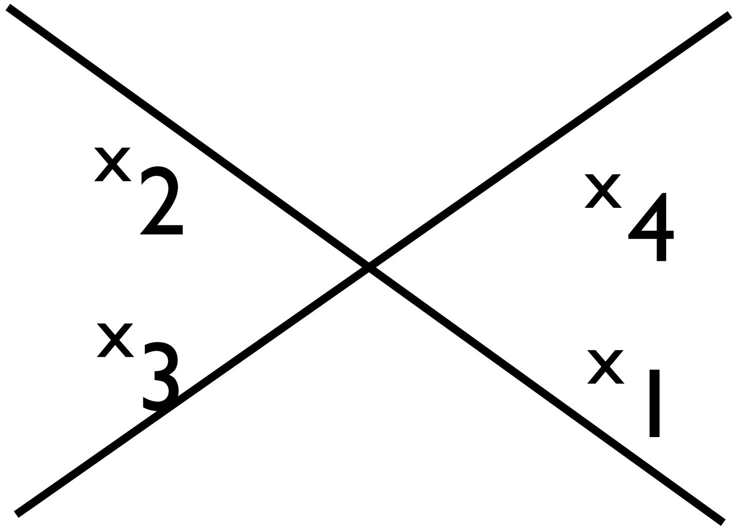
absorptive
part

[SCH ’17]

converges for J>1 (boundedness in Regge limit)

[see also: Simmons-Duffin, Stanford& Witten;
Kravchuk& Simmons-Duffin ‘18]

What's 'absorptive part'?



$$\langle 0|T\phi_1\cdots\phi_4|0\rangle \equiv G = G_E + i\mathcal{M}$$

$$\langle 0|\bar{T}\phi_1\cdots\phi_4|0\rangle \equiv G^* = G_E - i\mathcal{M}^*$$

$$\langle 0|\phi_2\phi_3\phi_1\phi_4|0\rangle \equiv G_E$$

$$\mathrm{dDiscG} \equiv G_E - \tfrac{1}{2}G - \tfrac{1}{2}G^* = \text{“}\mathrm{Im}\,\mathcal{M}\text{”}$$

equal to double-commutator:

$$\mathrm{dDisc}\,G \equiv \tfrac{1}{2}\langle 0|[\phi_2,\phi_3][\phi_1,\phi_4]|0\rangle$$

Positive & bounded

cf: [Maldacena, Shenker&Stanford 'bound on chaos']
[Hartman,Kundu&Tajdini 'proof of ANEC']

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Large-spin expansion

Analyticity in spin: organizing principle for CFT spectrum

Simplest at large-J: integral pushed to corner $(z, \bar{z}) \to (0, 1)$

large spin in s-channel $\leftrightarrow$ low twist in t-channel

$\Rightarrow$ Solve crossing in asymptotic series in 1/J

[Komargodski&Zhiboedov,
Fitzpatrick,Kaplan,Poland&Simmons-Duffin,
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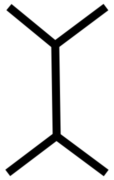
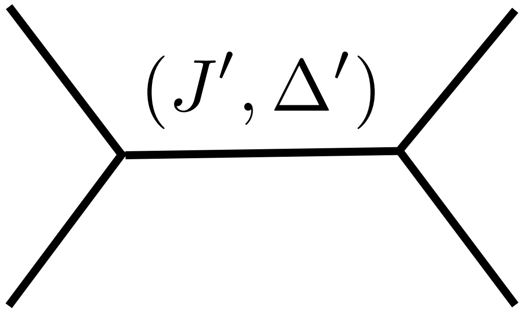
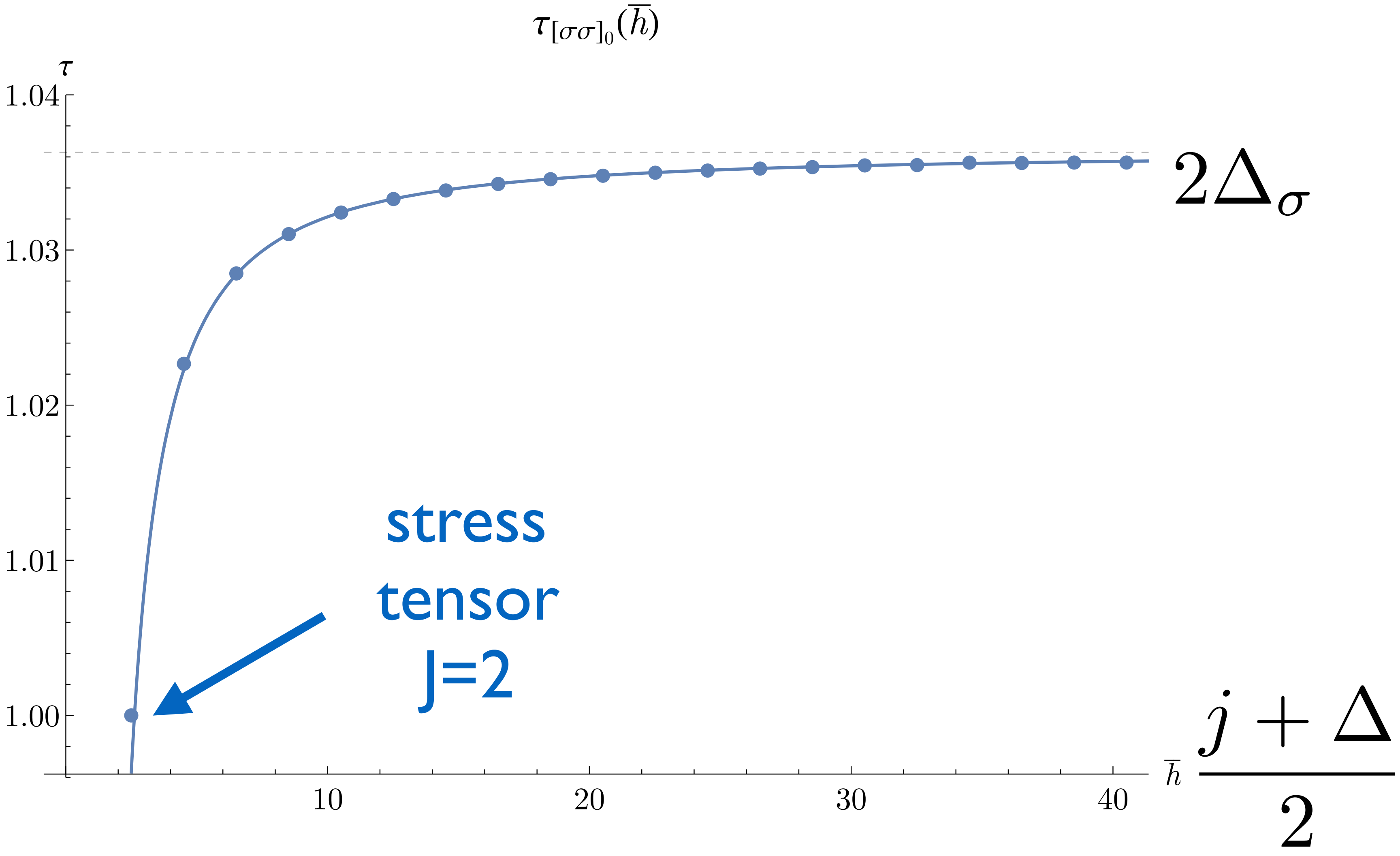
Insert cross-channel OPE in inversion formula:

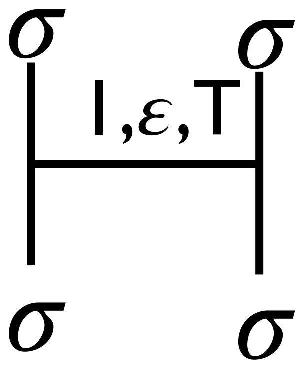
Coefficient [] = $\sum_{(J',\Delta')} \int$ 

$$\sim \int (1-\bar{z})^{\tau'} \sim J^{-\tau'}$$

large J dominated by lowest twists $\tau' = \Delta' - J'$

Large-spin limit in 3D Ising





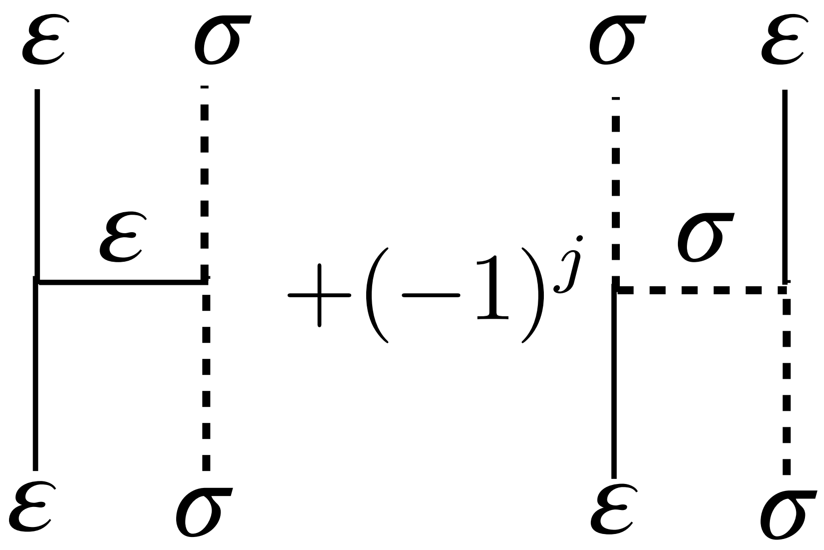
New: - 1/J expansion obtained from convergent integral

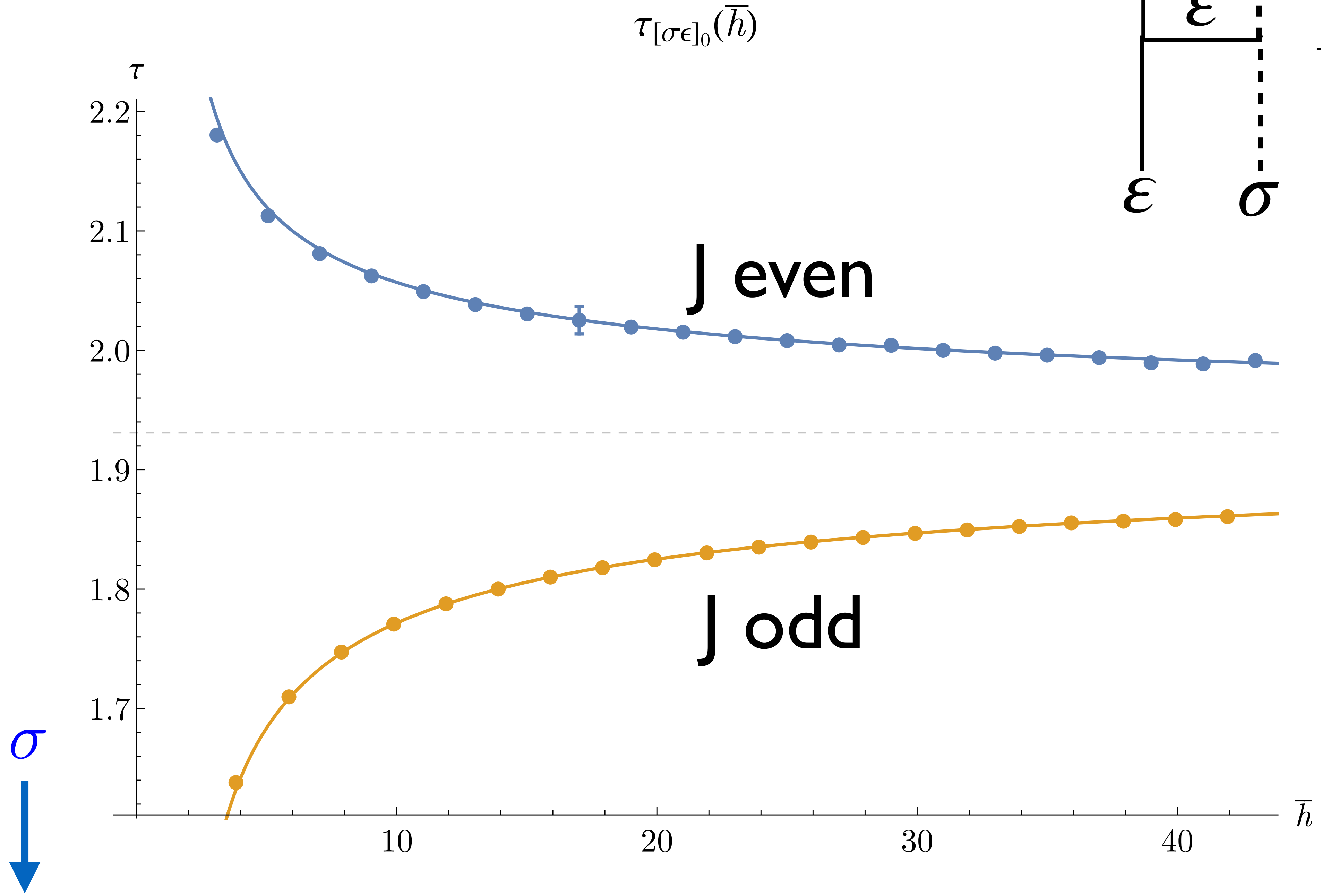
- No 'stick-outs'
- any op. with J>1 must be on a trajectory

`Best Before J> 1`

What about J=0?

Z_2-odd operators



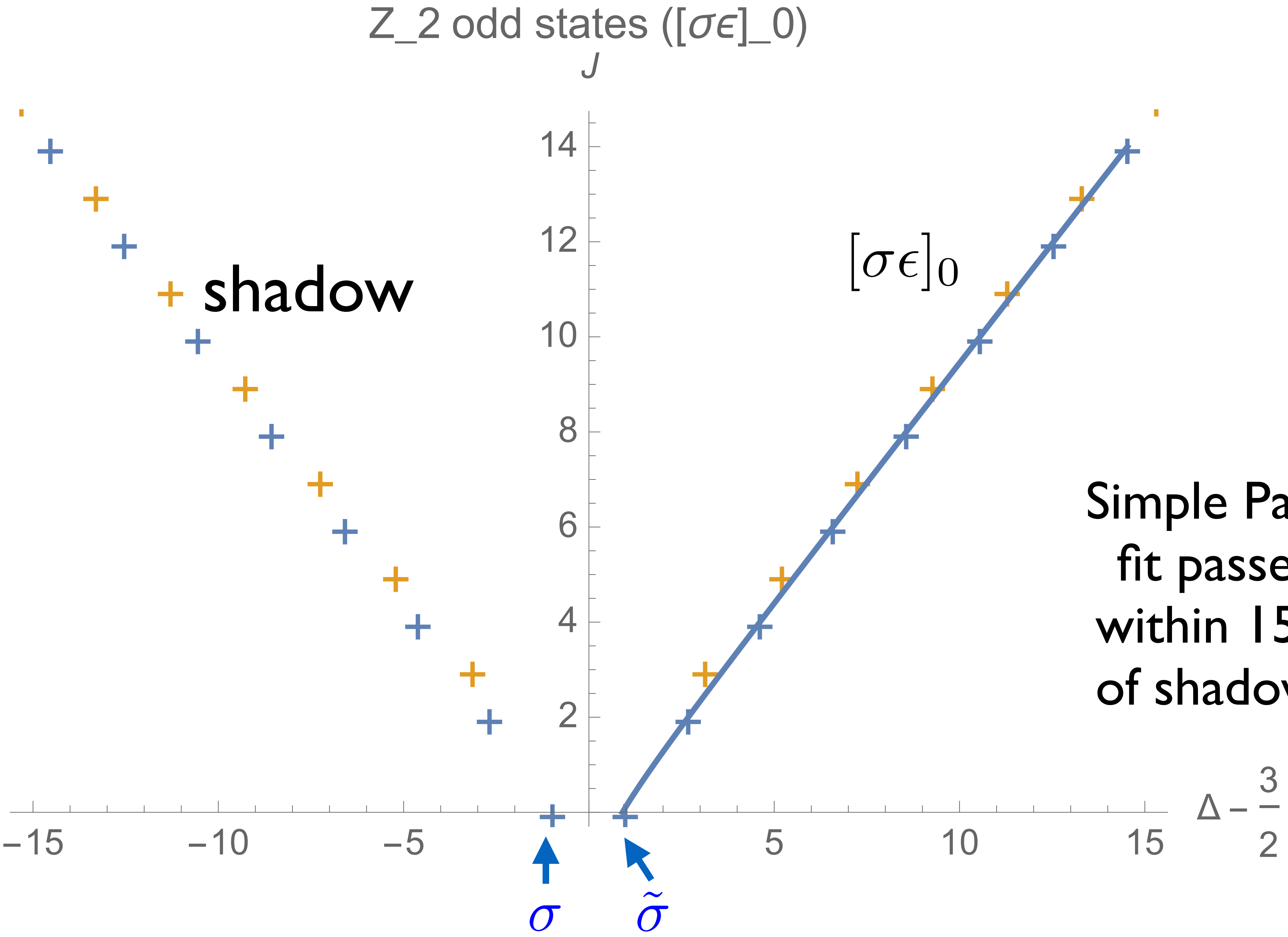


Works great for J>1, but seems hopeless for J=0!

What's analytic in spin is generating function c(J,Δ), whose poles encode spectrum:

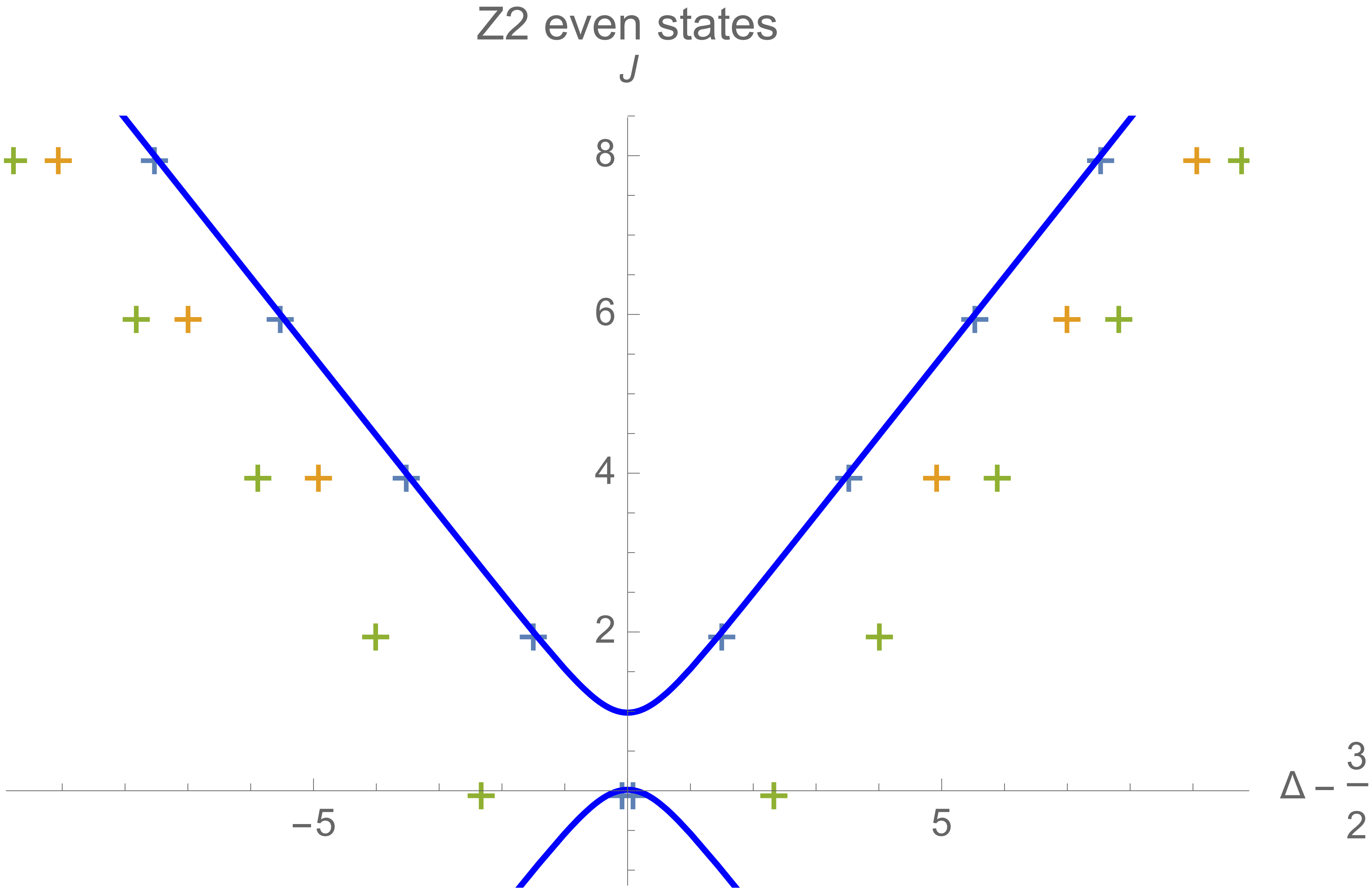
$$c(J, \Delta') \to \frac{f^2_{OO \to J,n}}{\Delta - \Delta_n(J)}$$

Has shadow symmetry: $\Delta \to d - \Delta$



Simple Padé
fit passes
within 15%
of shadow!

[Brower, Polchinski, Strassler& Tan]



(fit accounts for possible square-root branch point)

Conjectures:

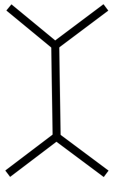
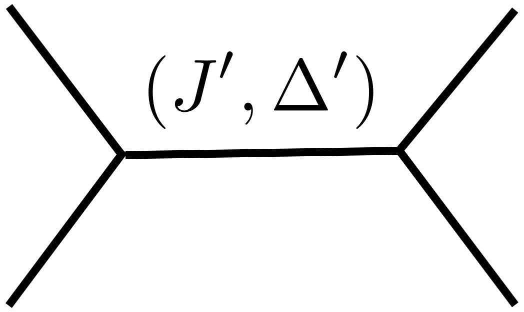
In 3D Ising:

1. the shadow of σ is on the $[\sigma\varepsilon]_0^+$ trajectory, that of ε on (a branch of) $[\sigma\sigma]_0$ [cf: Zhiboedov+Turiaci '18; Alday '18]

2. Residue of $[\sigma\varepsilon]_0^-$ has a fine-tuned zero at J=1

3. Intercept $J^*<1$: dDisc→0 in Regge limit (corrollary: spectrum is regular (non-chaotic))

$$\lim_{\Delta\to\infty} \frac{\langle a \sin^2(\pi\gamma)\rangle_\Delta}{\langle a\rangle_\Delta} \to 0$$

Coefficient [] = $\sum_{(J',\Delta')}\int$ 

To check conjectures numerically:

- Evaluate ‘block times block’ integral accurately
  ⇒ either numerically, or use in (1-z) series ✓
- Sum over (known!) cross-channel operators
  ⇒ control truncation errors

[work in progress: Classens-Howe, Gobeil, Maloney& Zahraee]

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Recall our Lorentzian inversion formula:

$$c(J,\Delta) = \int_{\diamond} \left[G_{\Delta+1-d,j+d-1}\right] \times \left[\mathrm{dDisc}\, G\right]$$

s-channel
OPE coefficients

absorptive
part

‘absorptive part’ especially simple in AdS/CFT!

Theories with AdS gravity duals have:

- Large-N expansion (small $\hbar$ in AdS)
- Few light single-traces, all with small spin ≤2 (up to a very high dimension $\Delta_{\text{gap}} \gg 1$) [HPPS '09]

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simple consequence for dDisc:

$$\mathrm{dDisc}\, G = \sum_{J',\Delta'} \sin^2\!\left(\tfrac{\pi}{2}(\Delta' - 2\Delta)\right) \left(\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\right)^{\Delta'+J'} \left(\frac{1-\sqrt{\bar{\rho}}}{1+\sqrt{\bar{\rho}}}\right)^{\Delta'-J'}$$

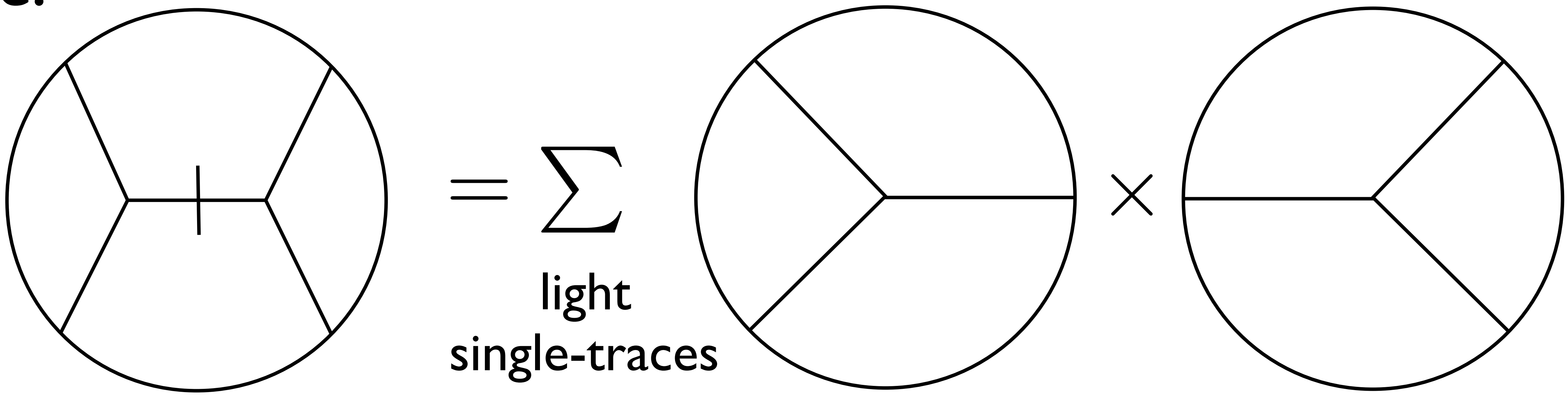
kills double-traces

kills heavy

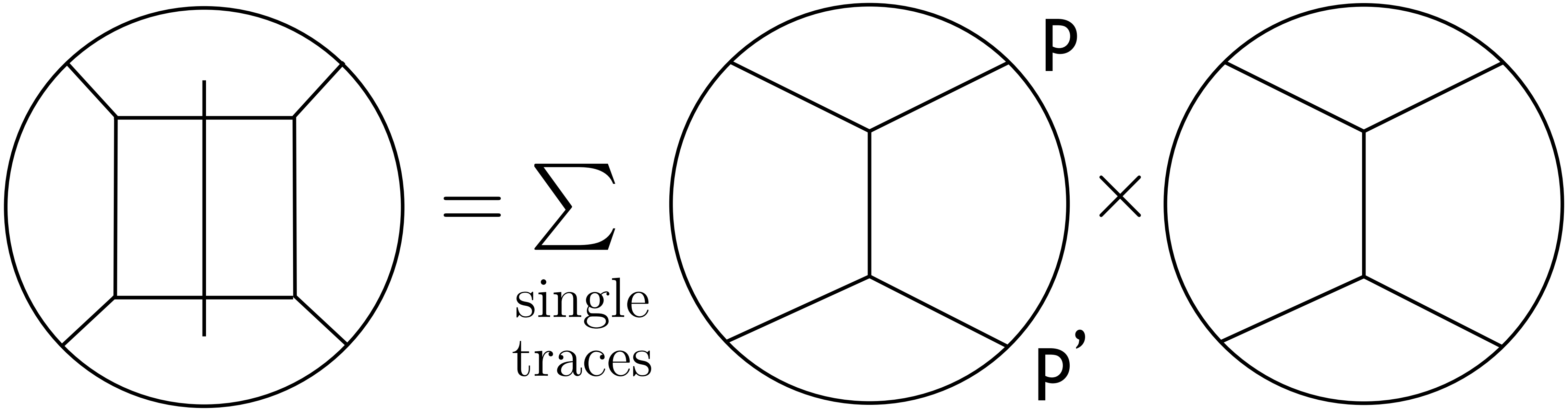
theories with local AdS dual ⇔ dDisc saturated by few light primaries

Cutting rules for dDisc in AdS/CFT:

tree:



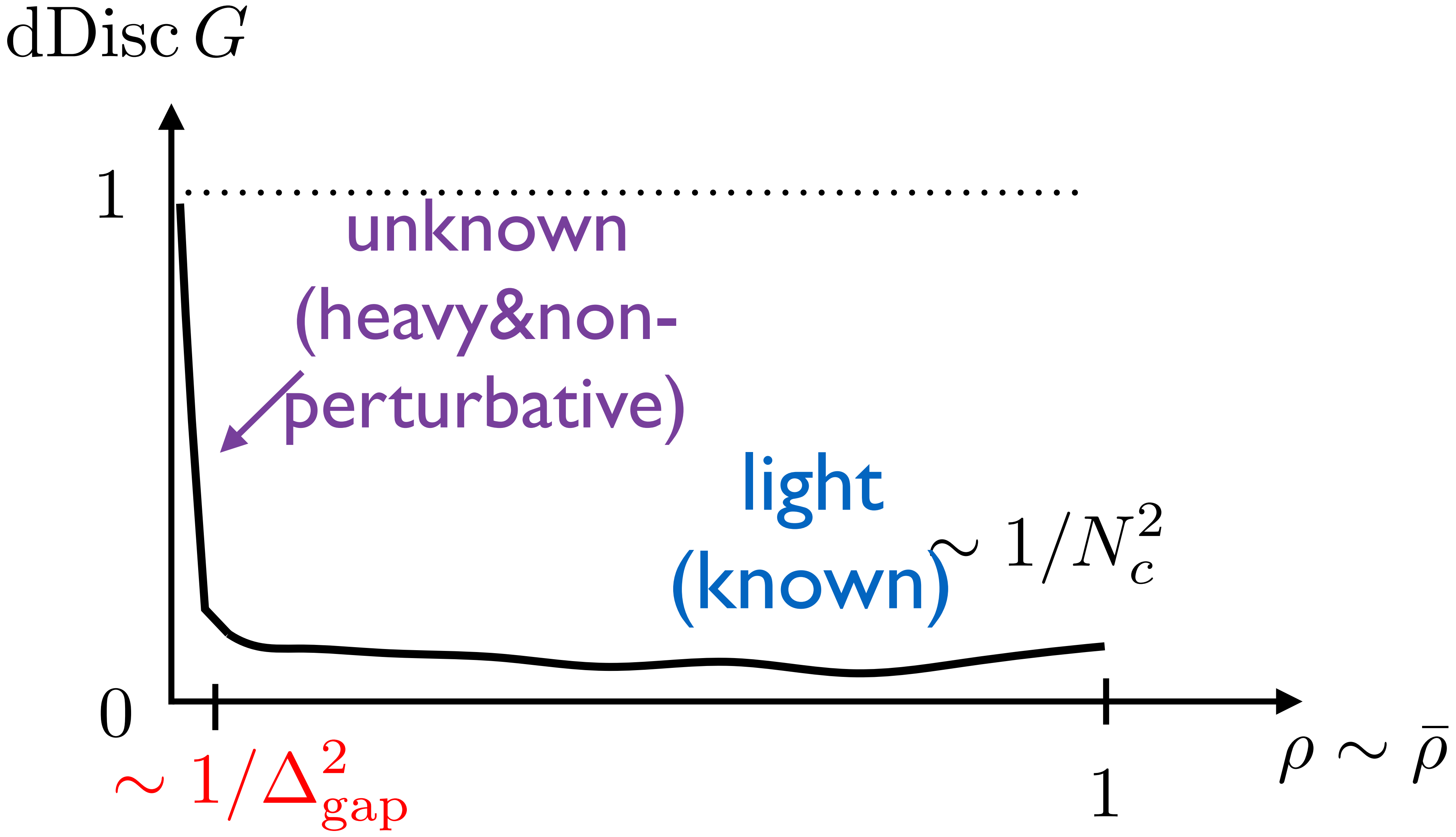
one-loop:



Excitations = particles in AdS!

[Alday& SCH '17]

Nonperturbative picture:



$$c_{j,\Delta} = \int F_{j,\Delta}\,\mathrm{dDisc}\,G \;= c_{j,\Delta}\Big|_{\text{light}} + c_{j,\Delta}\Big|_{\text{heavy}}$$

'minimal solution'

correction small for j>2

[see also: Alday, Bissi&Perlmutter; Li, Meltzer&Poland]

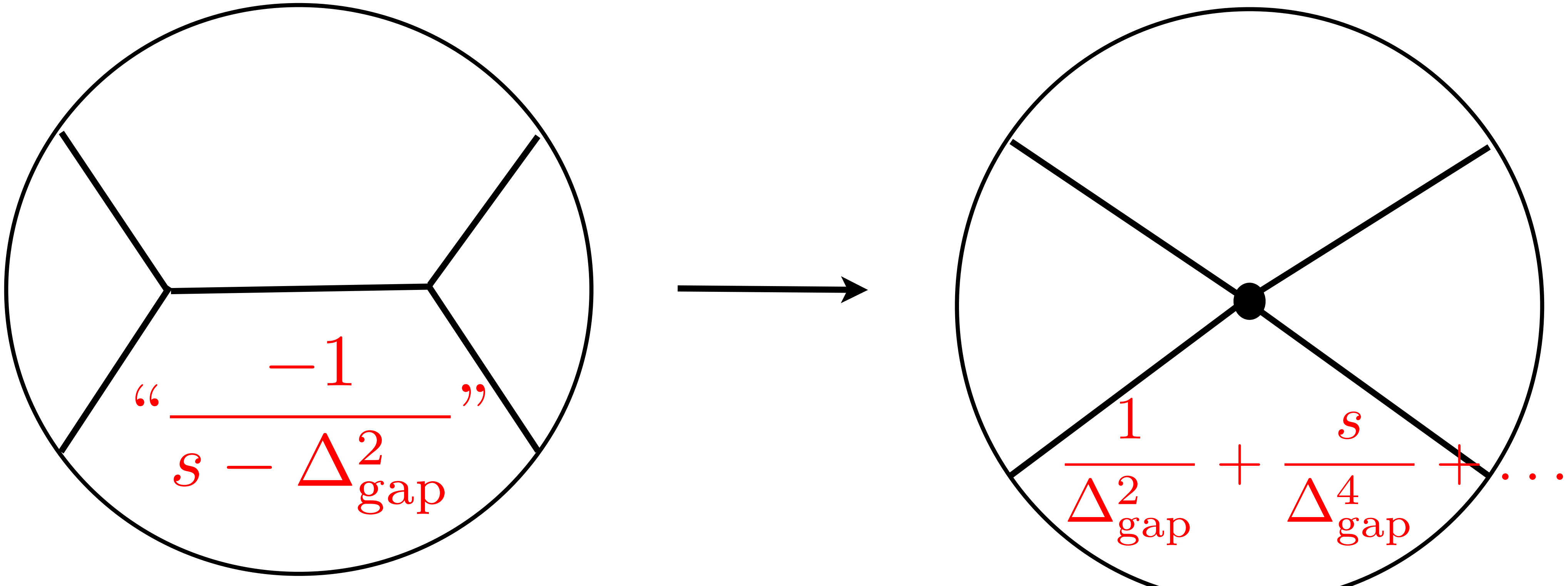
‘Heavy’ part depends on nonperturbative UV completion.

It’s weighed by $\sim (\rho\bar{\rho})^{J/2}$. Use positivity + boundedness:

$$\boxed{\left|c(j,\tfrac{d}{2}+i\nu)_{\text{heavy}}\right| \leq \frac{1}{c_T}\frac{\#}{(\Delta_{\text{gap}}^2)^{j-2}}}$$

Establishes EFT power-counting in AdS: HPSS conjecture

[for 4pt correlators]



$$\text{“}\frac{-1}{s-\Delta_{\text{gap}}^2}\text{”} \longrightarrow \frac{1}{\Delta_{\text{gap}}^2}+\frac{s}{\Delta_{\text{gap}}^4}+\ldots$$

for strongly coupled N=4 SYM dual to $AdS_5 x S_5$:

Mixing problem between different S_5 spherical harmonics has revealed amazingly simple eigenvalues:

$$\gamma = -\frac{\pi}{c}\frac{\Delta^{(8)}}{(J_{\text{Jeff}}+1)_6} + O(1/c^2)$$

[Aprile,Drummond,Heslop&Paul '18]

We confirm this conjecture, and make a new one:

SO(4,2)xSO(6)∈ SO(10,2) symmetry unifies all harmonics

[Anh-Khoi Trinh& SCH, to appear]

⇒Amazing structures await in non-planar N=4 SYM!

Summary

- Dispersion relation for OPE coefficients:

$$c(j,\Delta) \equiv \int_0^1 d\rho d\bar{\rho}\, g_{\Delta,j}\, \mathrm{dDisc}\, G$$

s-channel cross-channels

- -Organizes spectrum into analytic families,
  works for all operators in 3D Ising?
  -Efficient cutting rules for AdS/CFT

- *Open directions:*
  - interplay with numerical bootstrap?
  - why/when does it work for J≤1?
  - apply to more strongly coupled CFTs;
    trees and loops in AdS: AdS_5xS_5 hidden symmetries?