On Split Cuts from Elementary Disjunctions

Everything You Always Wanted to Know About BUT Were Afraid to Ask Egon

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Joint work with Matteo Fischetti & Andrea Tramontani

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Notation & Assumptions

- We consider:
$$\min\{c^T x : Ax \geq b, x \text{ integer}\} \tag{1}$$
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- The plan is derive the "strongest" cut, $\gamma x \geq \gamma_0$ violated by x^*, by using such a disjunction and doing it by the classical disjunctive approach of Balas:

$$P_0 \qquad\qquad P_1$$

$$\begin{array}{lrcl} (u) & Ax & \geq & b \\ (u_0) & -x_j & \geq & -\pi_0 \end{array} \qquad\qquad \begin{array}{lrcl} (v) & Ax & \geq & b \\ (v_0) & x_j & \geq & \pi_0 + 1 \end{array}$$

which is a valid cutting plane for the union of the two polyhedra P_0 and P_1.

Notation & Assumptions (cont.d)

- Such a cut can be separated by solving the so-called Cut Generating Linear Program:

$$
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\text{(CGLP)} & \min & \gamma x^* - \gamma_0 & & \\
 & & & & \\
\gamma & = & u^T A & - & u_0 e_j \\
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\gamma_0 & = & u^T b & - & u_0 \pi_0 \\
\gamma_0 & = & v^T b & + & v_0(\pi_0 + 1) \\
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- Lately, Balas & Perregaard (2002) developed an elegant and efficient way of solving the CGLP in the space of the original variables which represents a crucial speed-up.

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  1. solve the continuous relaxation
  2. for every fractional variable
     (a) consider the elementary disjunction associated with the corresponding row
     (b) strengthen the associated Gomory Mixed Integer cut (GMI) by performing a sequence of pivots to (possibly infeasible) alternative basis so as to implicitly solve the CGLP.

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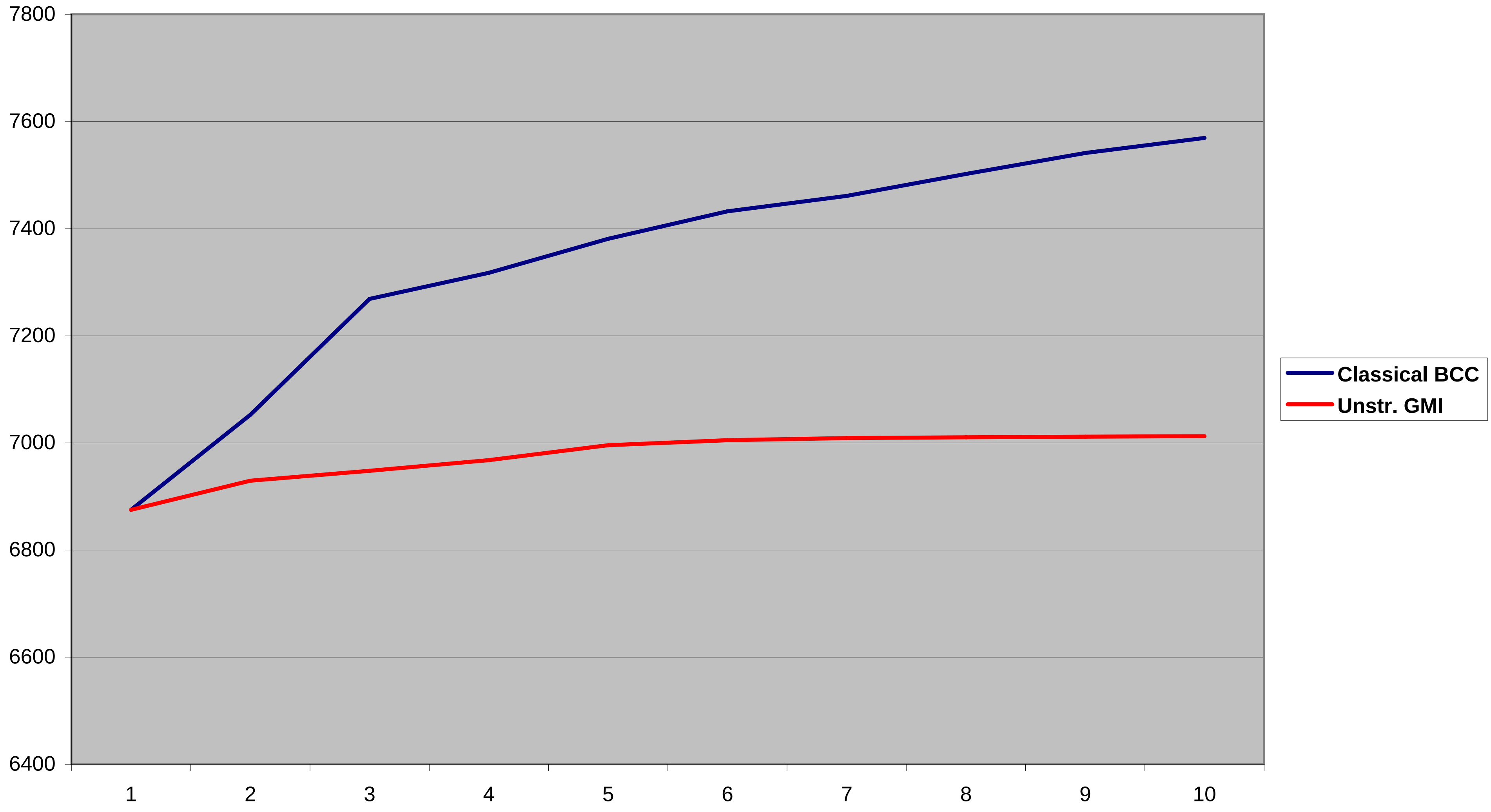
- Within 10 rounds of cuts, the indicators we report are:
  1. quality of the lower bound
  2. average cuts' density
  3. cuts' rank
  4. average cardinality of (u, v), i.e., how many constraints used on average to generate a cut

Instance p0201: lower bound



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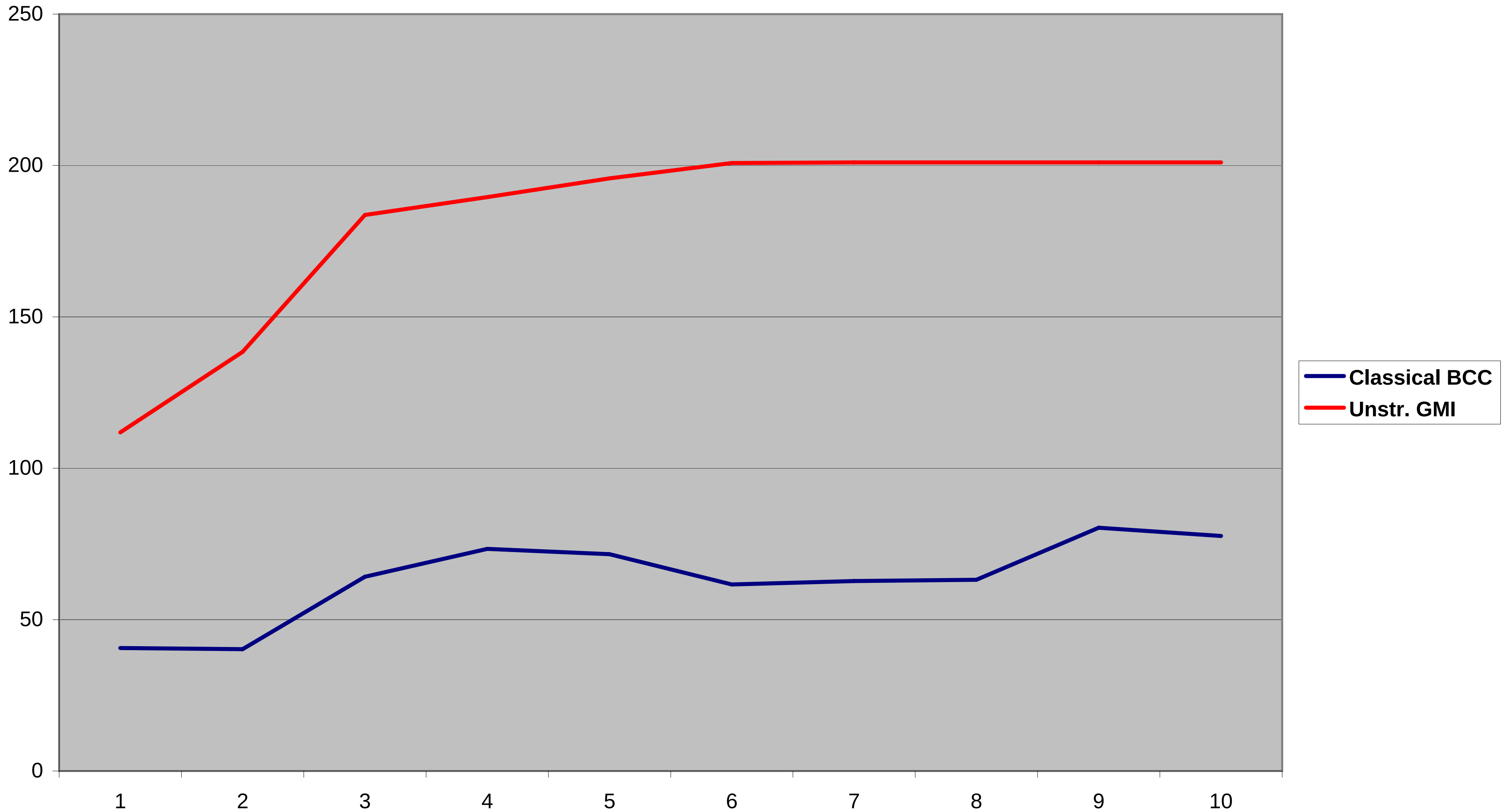
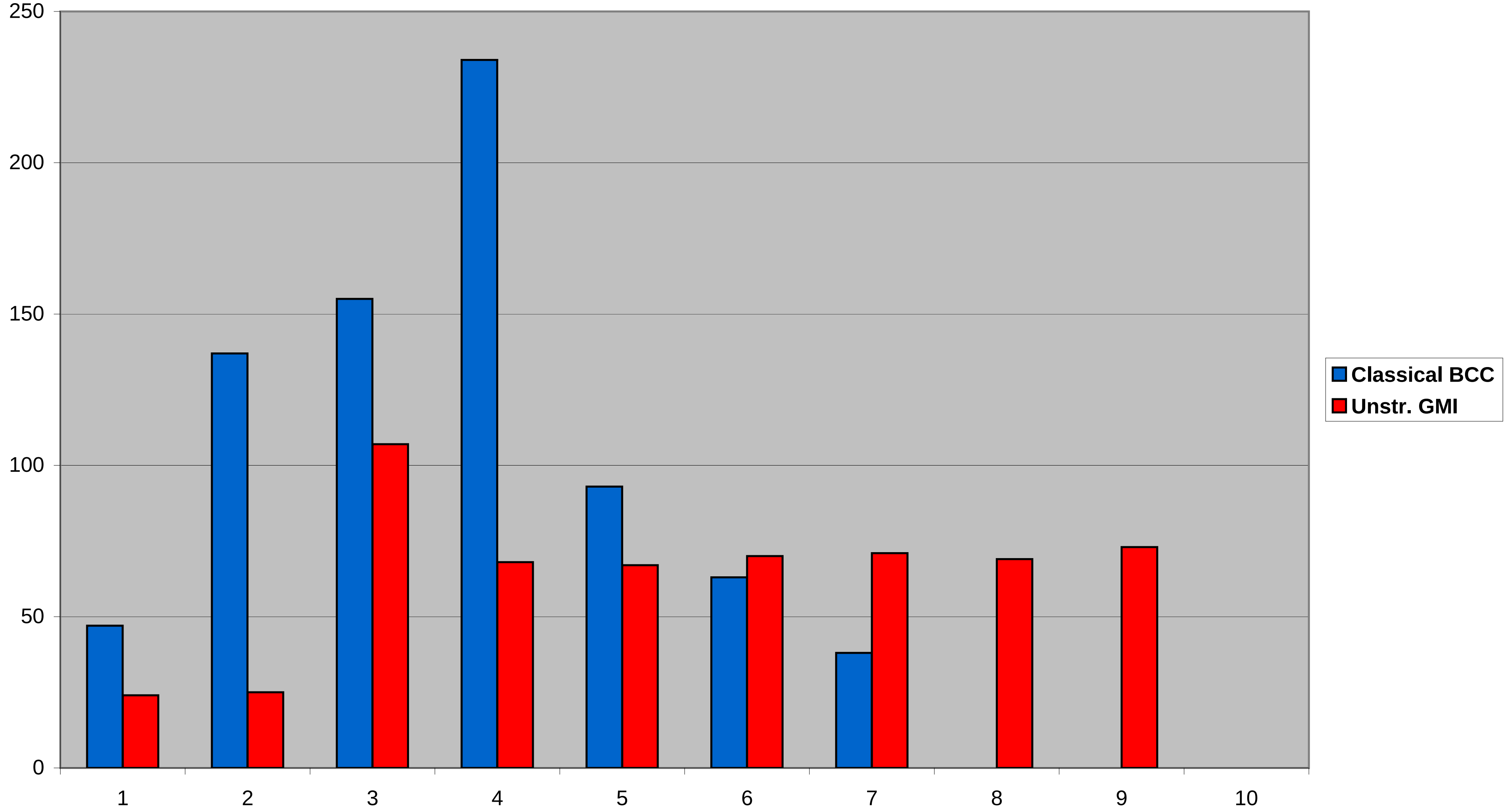


Instance p0201: cuts' rank



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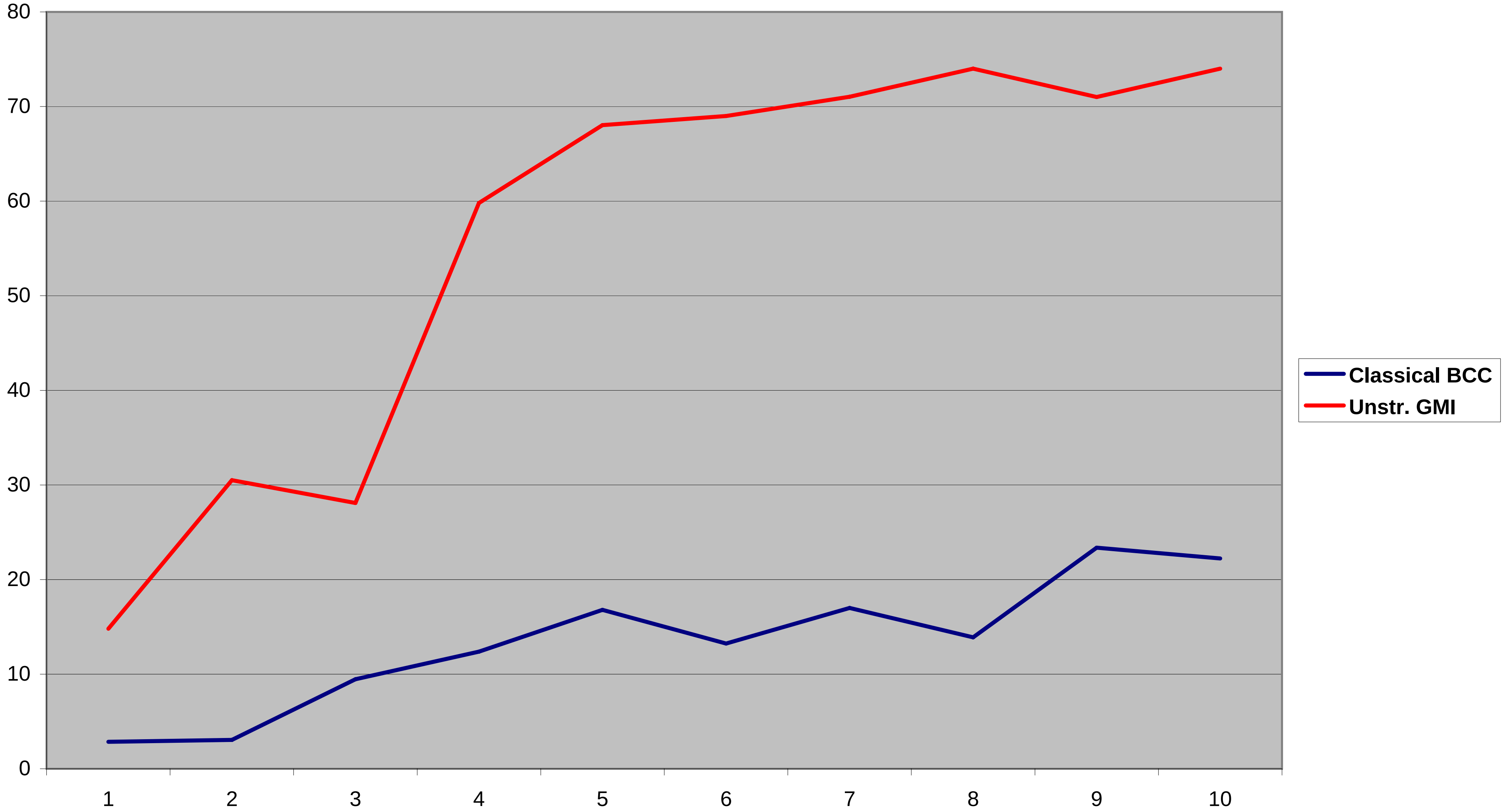

In Summary

Table 1: 10 iterations of cuts. At each iteration one cut is generated from any fractional variable. No strengthening in the cut computation.

Unstrengthened GMI vs. "Classical" BCC approach						
	Unstrengthened GMI			"Classical" BCC		
Instance	n.cuts	gap%	$\#(u,v)$	n.cuts	gap%	$\#(u,v)$
bell3a	137	70.74	59.49	71	70.74	43.72
bell5	202	28.18	31.20	178	94.29	11.75
blend2	156	28.73	11.70	192	30.51	8.10
flugpl	93	15.15	7.57	92	18.36	5.85
gt2	191	98.71	14.52	196	93.46	10.28
lseu	152	32.94	14.34	196	41.33	9.17
*m.share1	68	0.00	1.00	74	0.00	1.39
mod008	104	12.09	10.40	139	17.05	12.41
p0033	103	58.33	5.72	113	67.86	4.81
p0201	574	18.58	56.03	767	93.82	13.43
rout	445	8.52	135.39	434	24.26	68.07
*stein27	235	0.00	19.74	252	0.00	6.53
vpm1	255	36.95	9.03	263	55.84	5.39
vpm2	424	42.08	71.72	403	74.96	17.27

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avg.	236.333	37.583	35.593	253.667	56.873	17.521

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- However, nothing is perfect! Normalization (2) is dependent on the scaling of the constraints. In the second set of experiments we simply multiplied by 1,000 any generated cut before adding it to the current relaxation.

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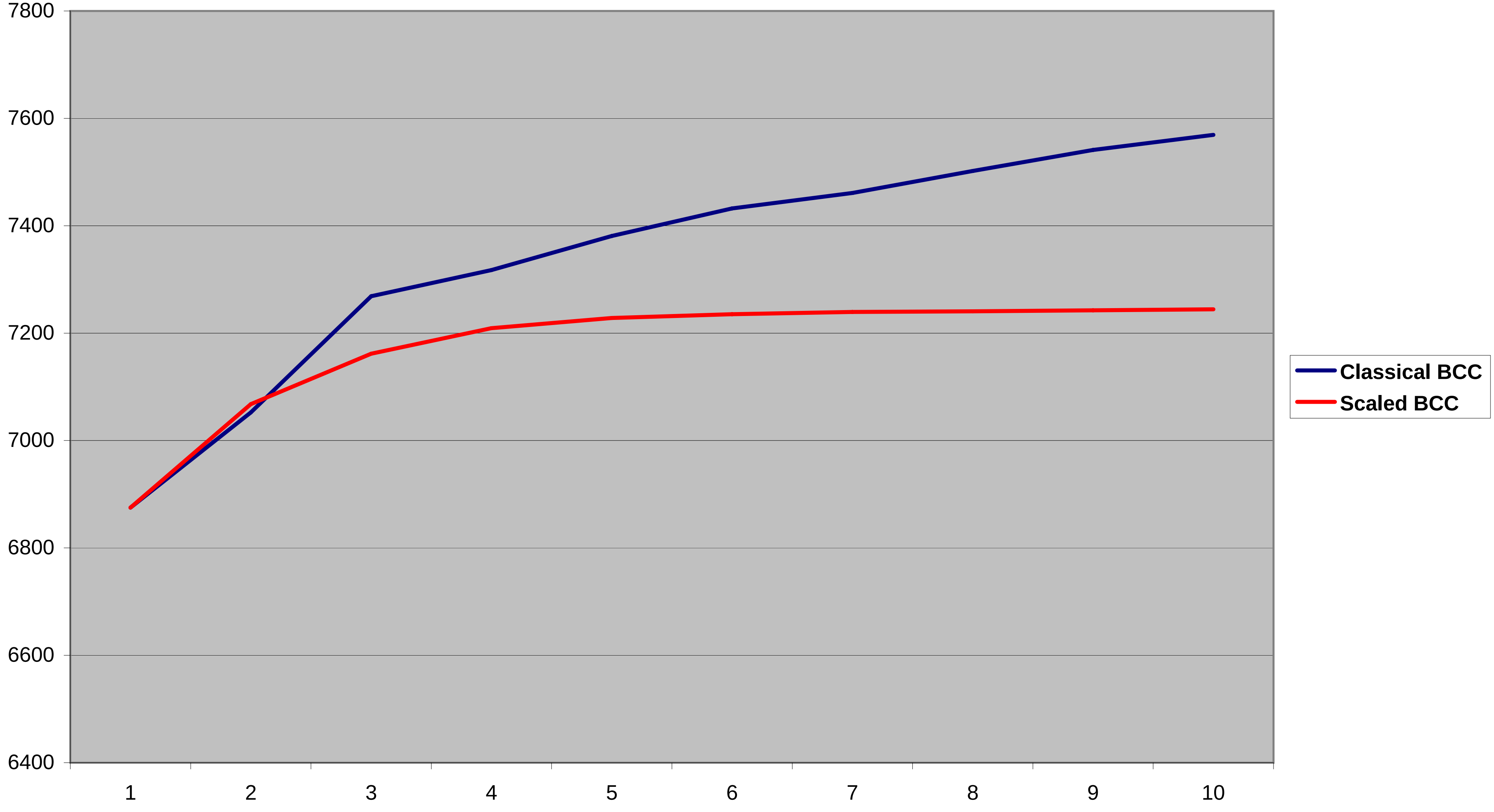
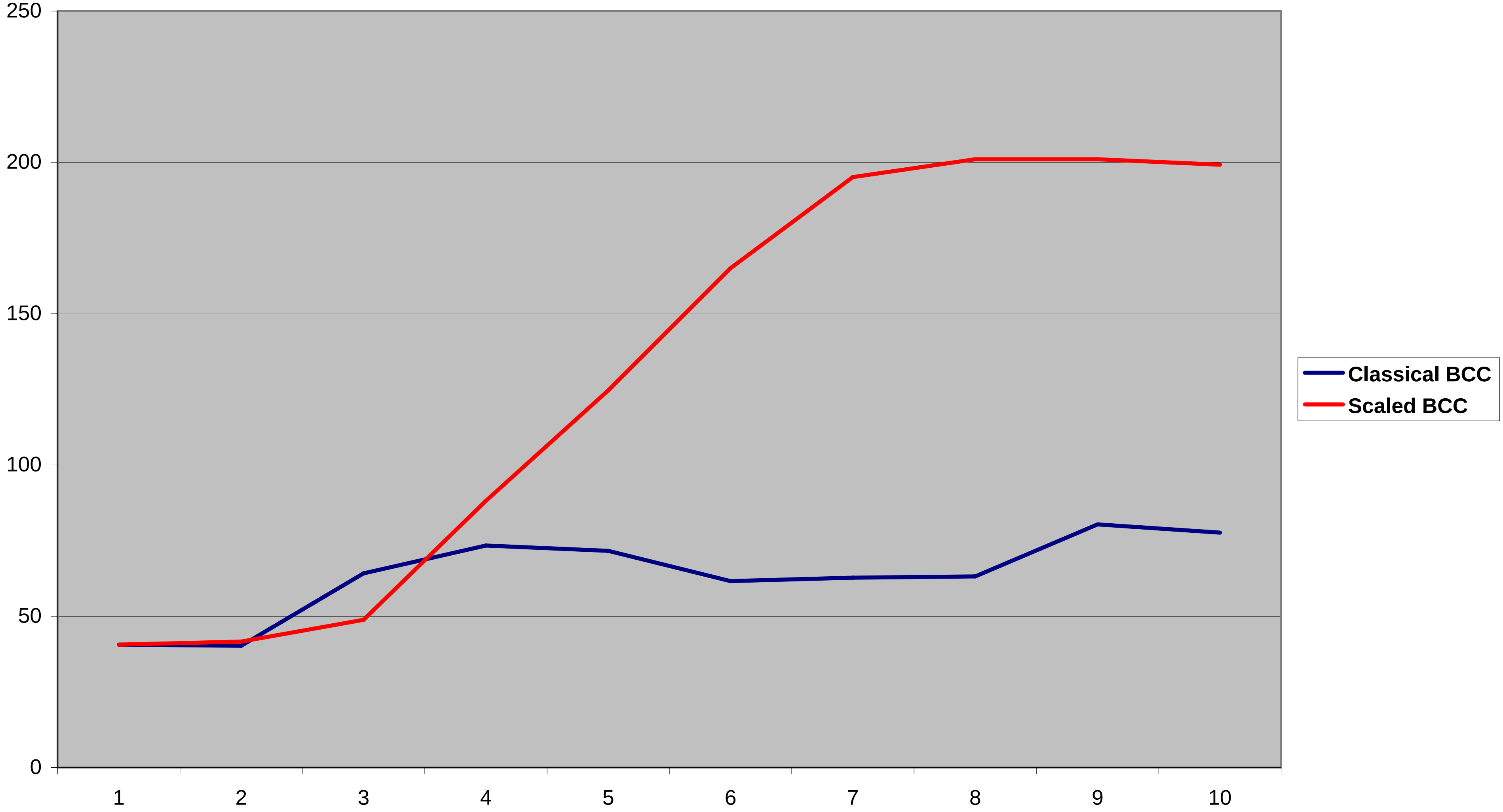
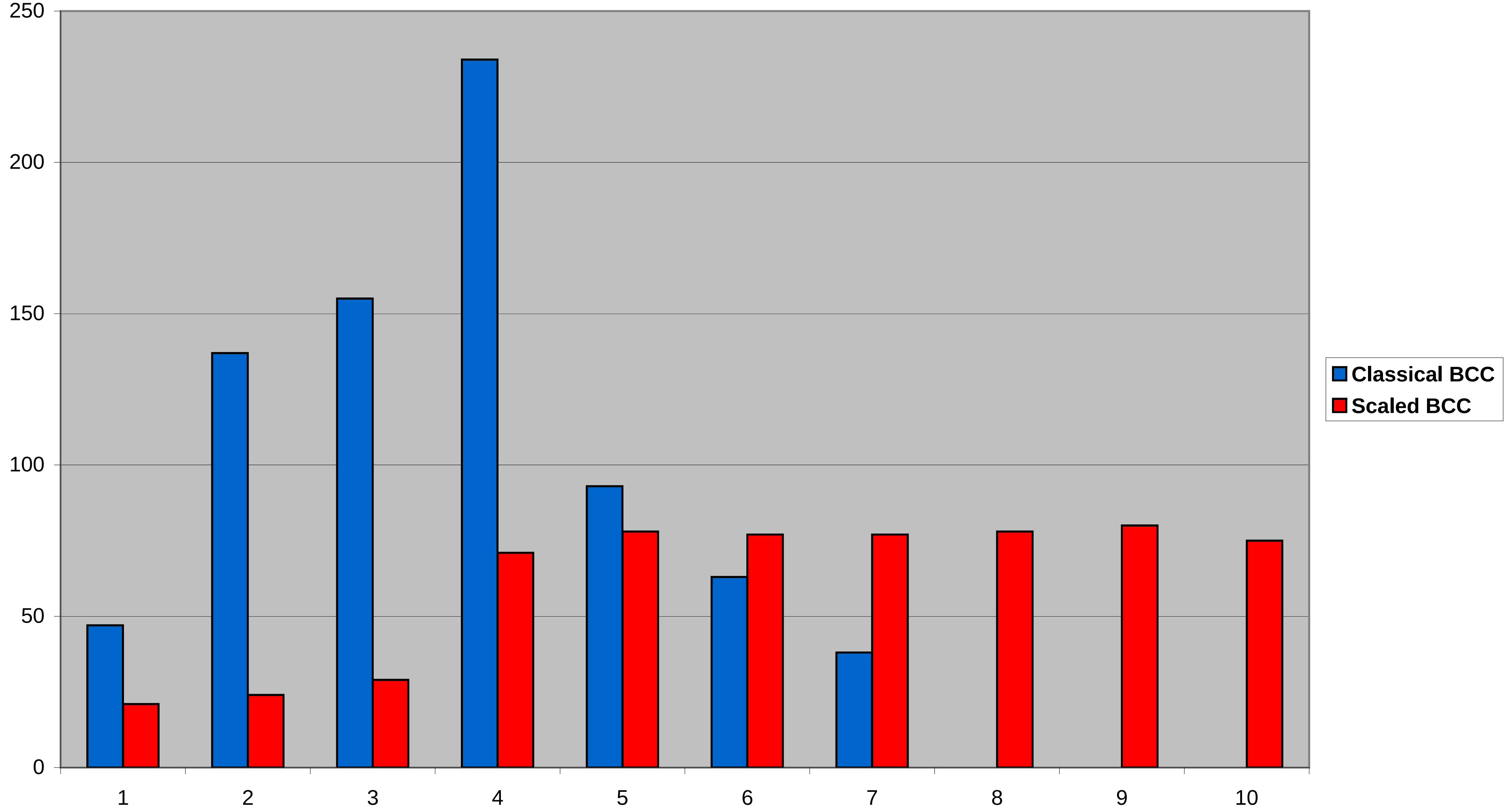


Instance p0201: average cuts' density



Instance p0201: cuts' rank



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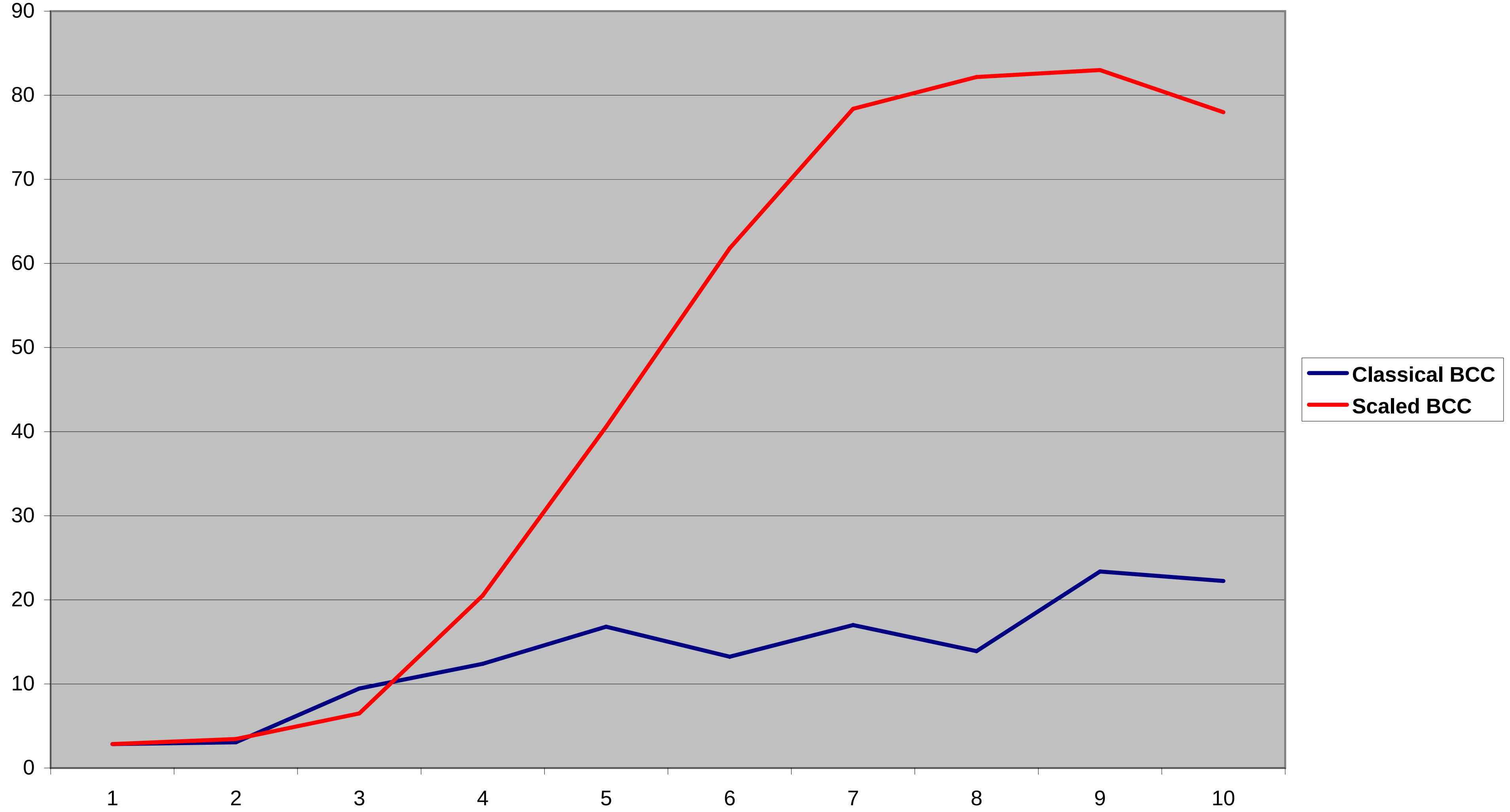

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Table 2: 10 iterations of cuts. At each iteration one cut is generated from any fractional variable. No strengthening in the cut computation.

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mod008	139	17.05	12.41	104	3.90	10.21
p0033	113	67.86	4.81	94	57.09	6.40
p0201	767	93.82	13.43	610	49.91	45.72
rout	434	24.26	68.07	435	13.03	152.66
*stein27	252	0.00	6.53	248	0.00	22.39
vpm1	263	55.84	5.39	244	47.59	8.50
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Nothing is perfect: Example 1

$$
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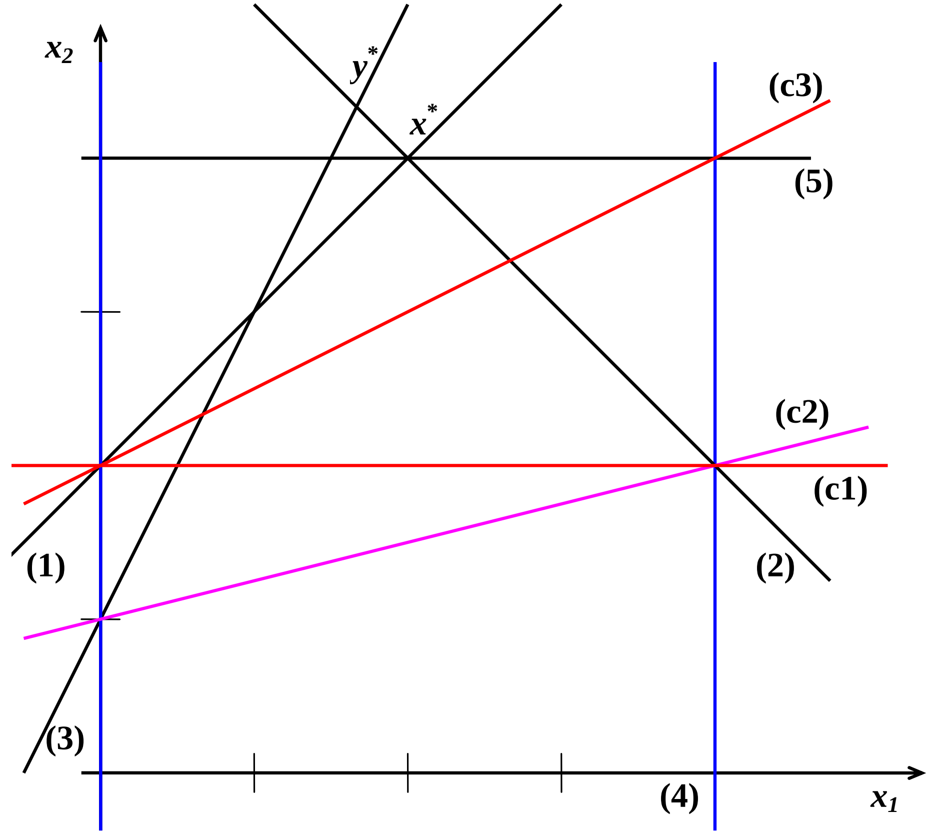
\min & -x_1 & -2x_2 & & \\
(1) & 4x_1 & -4x_2 & \geq & -2 \\
(2) & -2x_1 & -2x_2 & \geq & -3 \\
(3) & 8x_1 & -4x_2 & \geq & -1 \\
(4) & -x_1 & & \geq & -1 \\
(5) & & -k\ x_2 & \geq & -k \quad (k > 0) \\
 & x_1, & x_2 & \geq & 0
\end{array}
$$

Cuts from the disjunction $x_1 \leq 0$ OR $x_1 \geq 1$:

$$
\begin{array}{lrrcl}
(c1) & & 2x_2 & \leq & 1 \\
(c2) & -x_1 & +4x_2 & \leq & 1 \\
(c3) & -x_1 & +2x_2 & \leq & 1
\end{array}
$$



$(c1)$: corresponds to the basic solution of the CGLP (u_1, v_2, u_0, v_0), of value $z_1 = -\frac{2}{11}$, optimal for $k \leq 8$

$(c2)$: corresponds to the basic solution of the CGLP (u_3, v_2, u_0, v_0), of value $z_2 = -\frac{1}{6}$, never optimal

$(c3)$: corresponds to the basic solution of the CGLP (u_1, v_5, u_0, v_0), of value $z_3 = -\frac{k}{4+5k}$, optimal for $k \geq 8$

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  1. in the space $(\gamma, \gamma_0, u, v, u_0, v_0)$ the cone has 117 extreme rays which results into 117 vertices once normalization (2) is applied.

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  4. in other words, working on the extended space $(\gamma, \gamma_0, u, v, u_0, v_0)$ makes points in the interior of the polyhedron become vertices but this is independent of the normalization itself.

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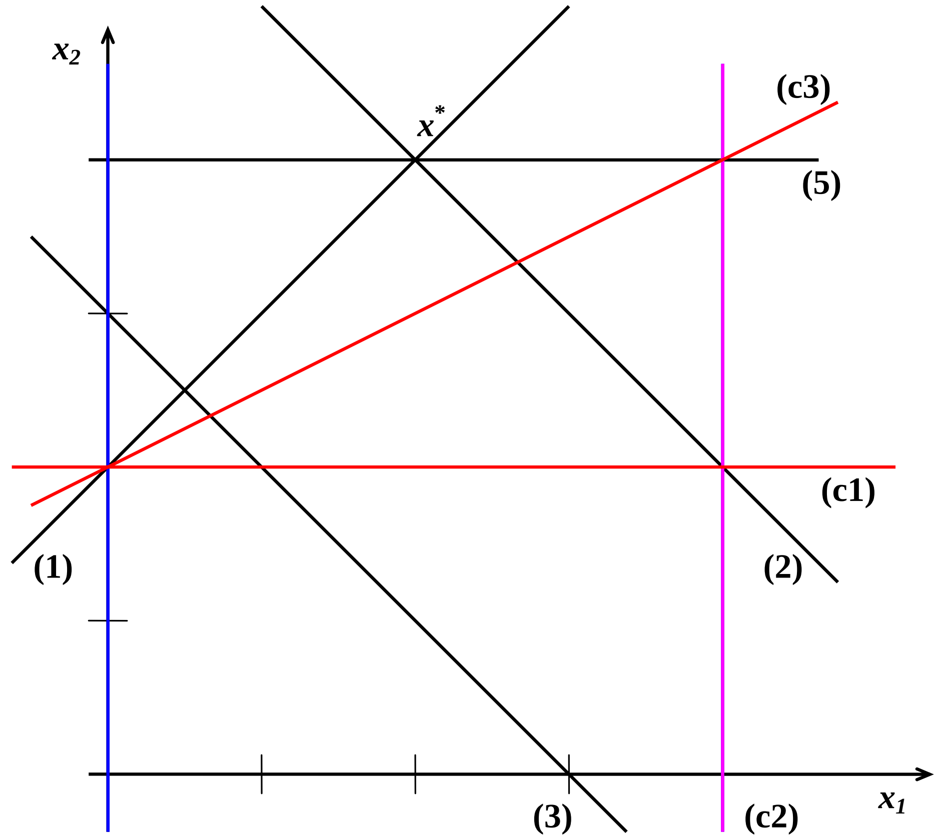
  5. however, the normalization changes the ranking of these vertices in terms of violation and this can result in very bad choices in terms of separated cuts.

Nothing is perfect: Example 2

$$\begin{array}{lrrcr} \min & -x_1 & -2x_2 & & \\ (1) & 2x_1 & -2x_2 & \geq & -1 \\ (2) & -2x_1 & -2x_2 & \geq & -3 \\ (3) & 4x_1 & +4x_2 & \geq & 3 \\ (4) & -x_1 & & \geq & -1 \\ (5) & & -x_2 & \geq & -1 \\ & x_1, & x_2 & \geq & 0 \end{array}$$

Cuts from the disjunction $x_1 \leq 0$ OR $x_1 \geq 1$:

$$\begin{array}{lrrcr} (c1) & & 2x_2 & \leq & 1 \\ (c2) & x_1 & & \geq & 1 \\ (c3) & -x_1 & +2x_2 & \leq & 1 \end{array}$$



$(c1)$: corresponds to the basic solution of the CGLP (u_1, v_2, u_0, v_0), of value $z_1 = -\frac{1}{6}$

$(c2)$: corresponds to the basic solution of the CGLP (u_1, u_3, u_0, v_0), of value $z_2 = -\frac{1}{22}$

$(c3)$: corresponds to the basic solution of the CGLP (u_1, v_5, u_0, v_0), of value $z_3 = -\frac{1}{10}$

$P_0 = \emptyset \Rightarrow x_1 \geq 1$ is a valid cut, but is not the best one for the CGLP

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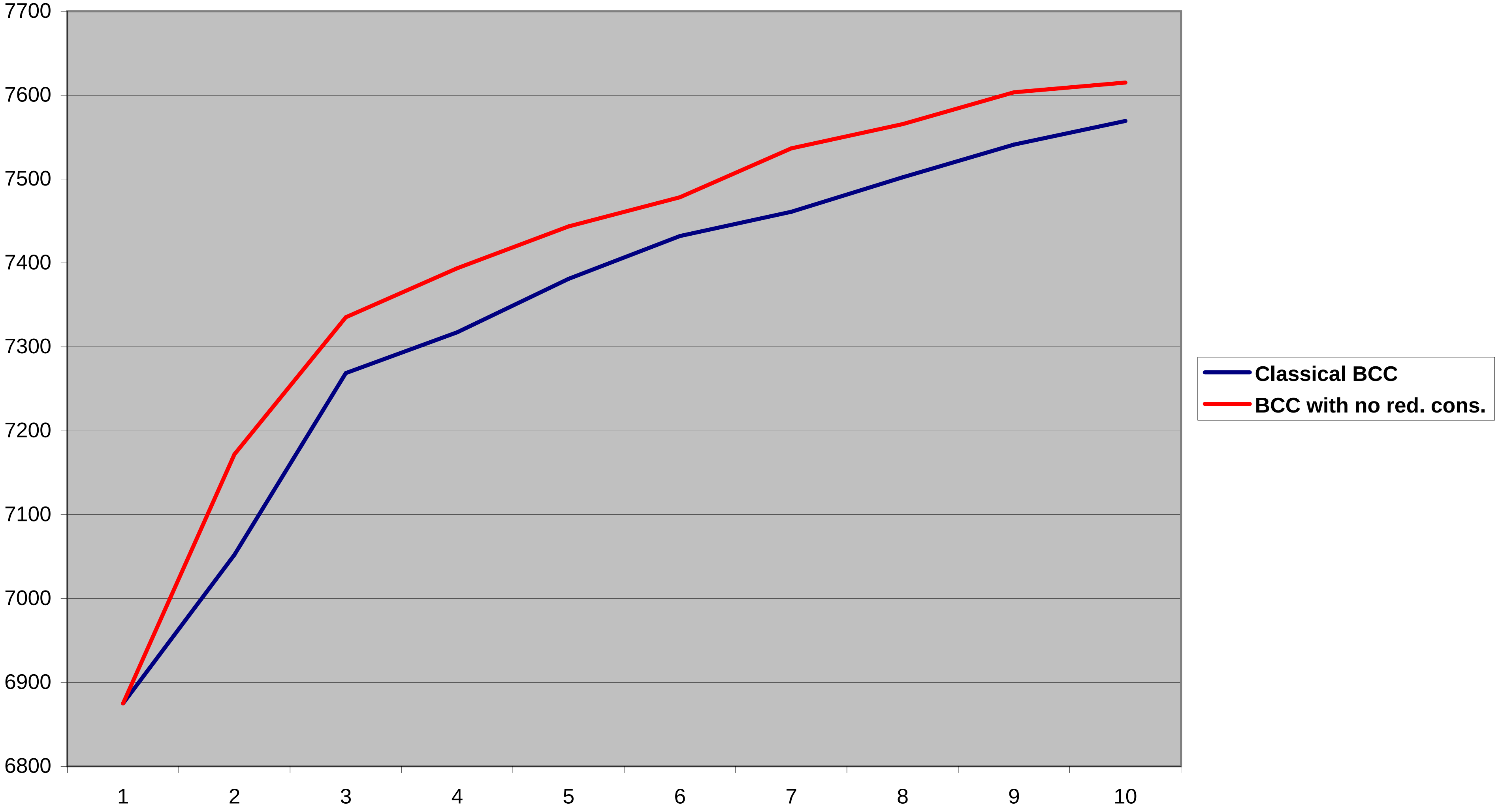
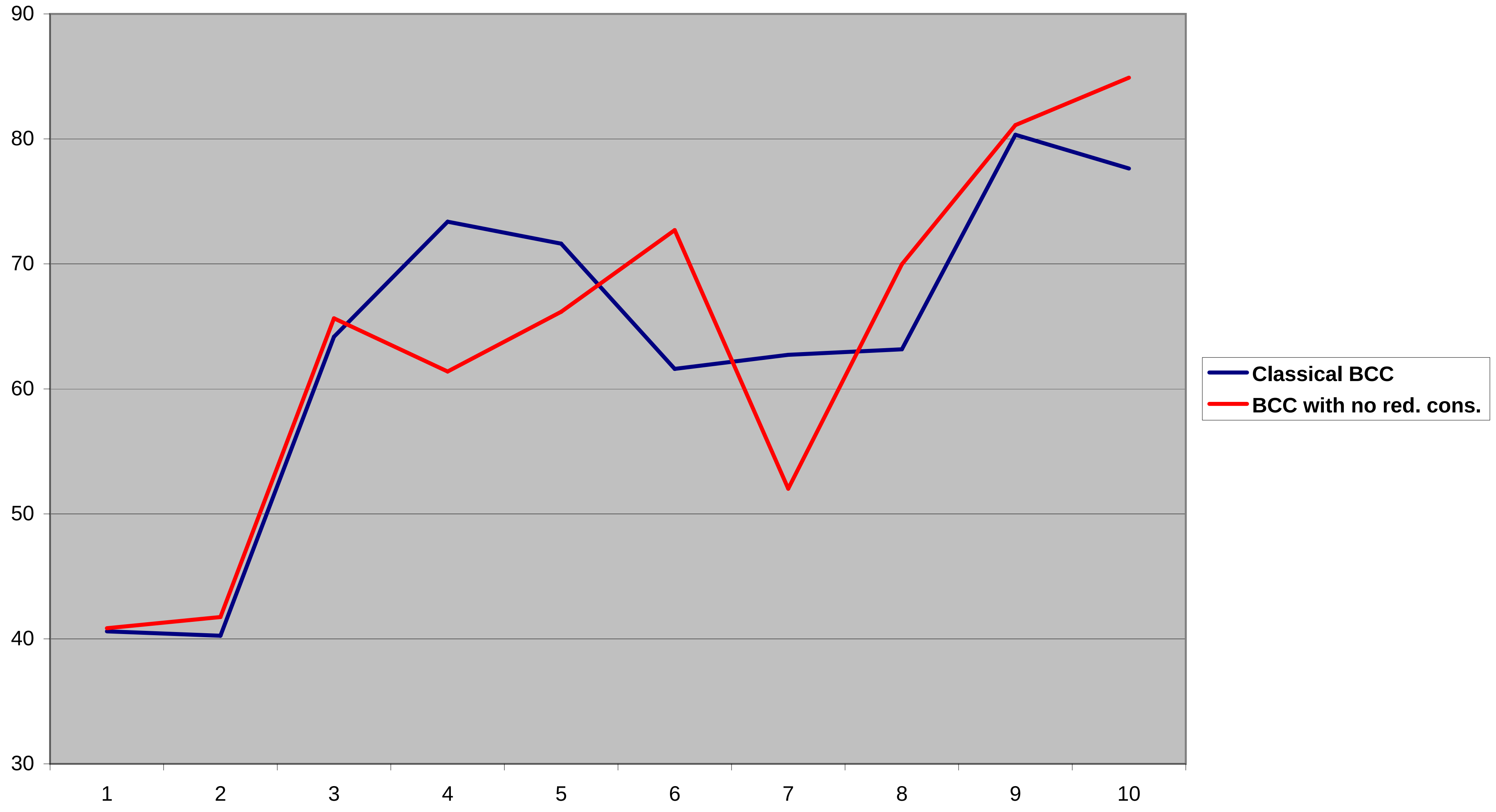
- In the third set of experiments we eliminated redundant constraints in a trivial way (i.e., by solving LPs) before solving the CGLP. To get a full picture, we did not project the separation problem on the support of x^* (to be discussed later).

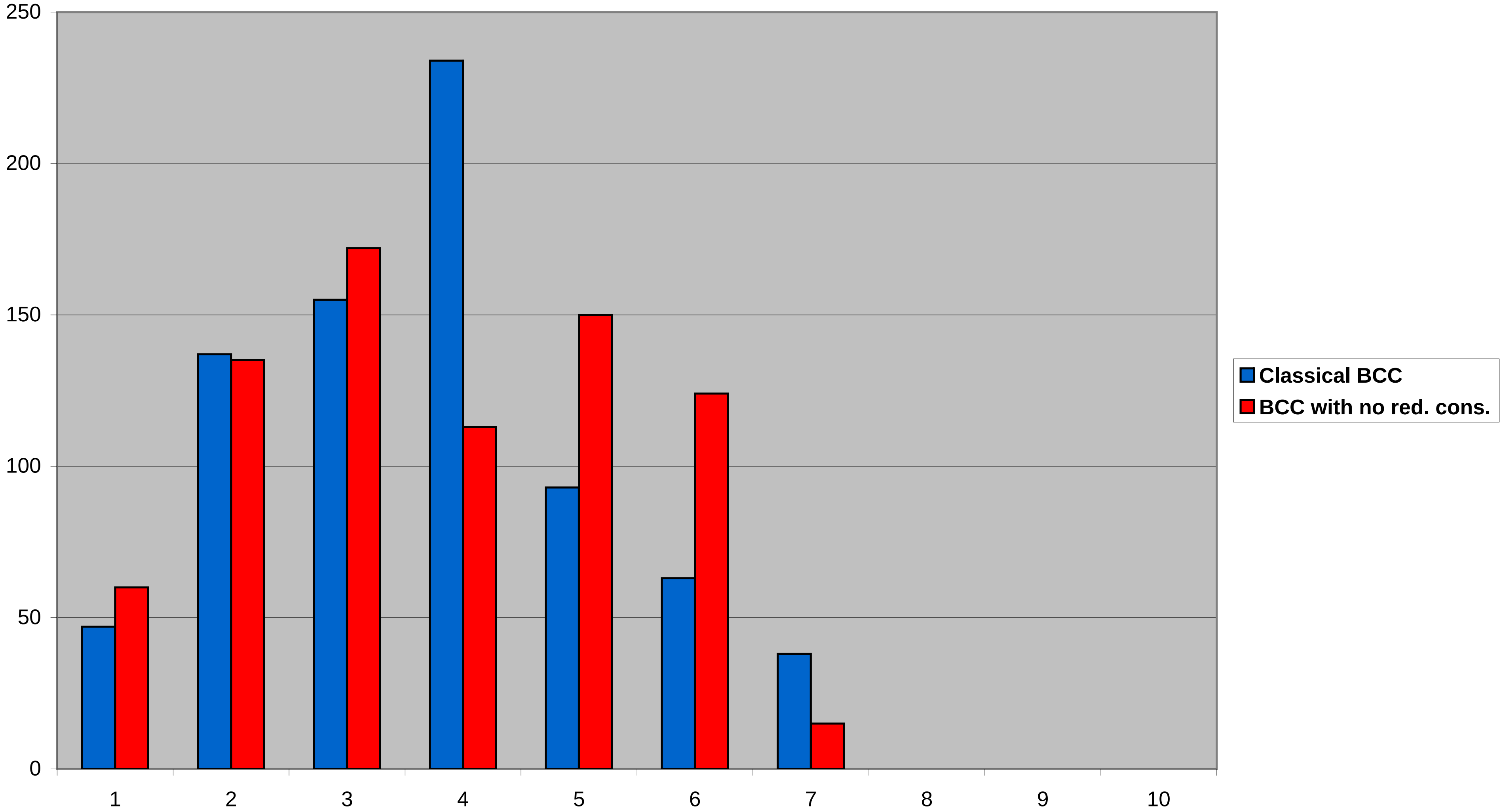
Instance p0201: lower bound



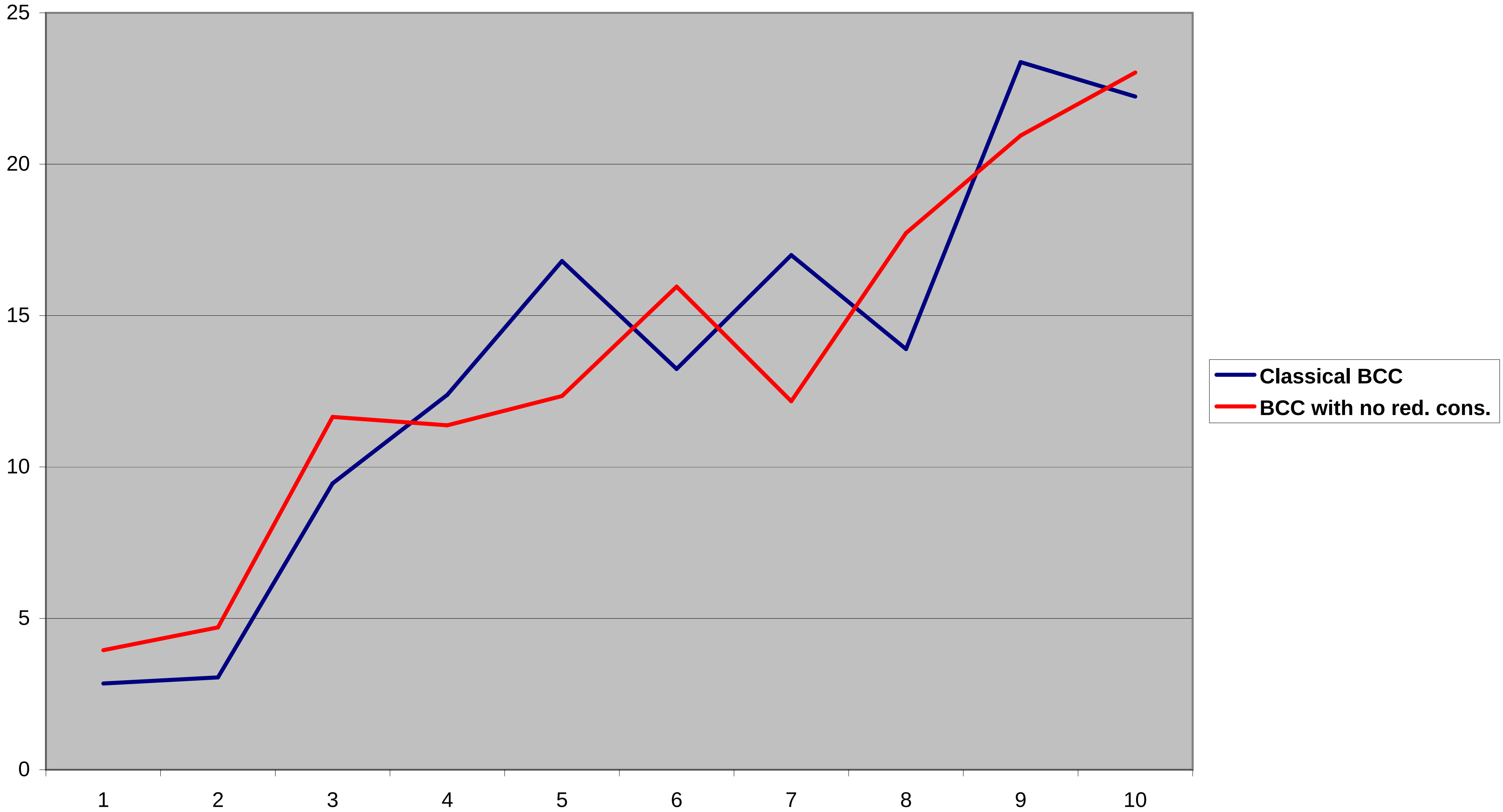
Instance p0201: average cuts' density



Instance p0201: cuts' rank



Instance p0201: average cardinality of (u, v)



In Summary

Table 3: 10 iterations of cuts. At each iteration one cut is generated from any fractional variable. No strengthening in the cut computation.

“Classical” BCC approach vs. “No redundancy” BCC approach with no projection						
	“Classical” BCC			“No redundancy” BCC		
Instance	n.cuts	gap%	$\#(u,v)$	n.cuts	gap%	$\#(u,v)$
bell3a	71	70.74	64.65	54	70.74	66.19
bell5	188	94.12	16.83	189	93.54	15.80
blend2	197	30.49	71.42	212	30.63	119.90
flugpl	93	18.34	6.45	90	18.83	6.48
gt2	218	94.13	58.11	167	93.68	63.16
lseu	171	42.46	23.86	184	45.10	30.96
*m.share1	77	0.00	55.99	77	0.00	56.00
mod008	107	15.46	304.18	107	15.48	304.19
p0033	116	57.25	8.75	126	70.32	10.99
p0201	692	92.53	23.40	757	98.31	37.44
rout	349	29.46	189.07	384	31.93	202.18
*stein27	251	0.00	7.29	249	0.00	6.46
vpm1	267	50.62	11.13	282	54.55	11.10
vpm2	390	74.73	24.23	376	76.47	22.82

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avg.	238.250	55.861	66.840	244.000	58.298	74.267

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- This seems to be particularly crucial for the variable bounds and we defined an extended support of x^* by avoiding projecting out variables at the bound whose bound constraints are in turn redundant.

Working on the support: computation

Table 4: 10 iterations of cuts. At each iteration one cut is generated from any fractional variable. No strengthening in the cut computation.

"Classical" BCC approach vs. "No redundancy" BCC approach with cuts separated projected on the support

	"Classical" BCC				"No redundancy" support				"No redundancy" extended support			
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bell3a	71	70.74	69.25	43.72	88	70.74	69.32	44.82	54	70.74	65.61	44.60
bell5	178	94.29	72.69	11.75	207	94.62	72.88	13.32	180	94.29	71.64	11.99
blend2	192	30.51	53.06	8.10	200	30.99	53.54	10.84	193	30.53	53.99	8.34
flugpl	92	18.36	86.11	5.85	93	18.94	86.11	5.89	93	18.86	86.29	5.95
gt2	196	93.46	18.30	10.28	191	94.13	18.14	10.58	187	93.88	20.00	13.10
lseu	196	41.33	29.44	9.17	191	40.16	27.08	12.28	178	43.45	29.41	9.08
*m.share1	74	0.00	11.94	1.39	130	0.00	13.39	2.56	77	0.00	12.59	1.69
mod008	139	17.05	4.51	12.41	136	17.70	4.42	12.17	157	19.13	5.85	14.43
p0033	113	67.86	55.76	4.81	106	70.32	55.76	5.74	146	70.29	58.84	5.89
p0201	767	93.82	45.02	13.43	873	81.59	43.43	25.83	769	100.00	48.93	13.39
rout	434	24.26	42.19	68.07	355	6.56	38.11	58.23	353	30.88	69.46	140.29
*stein27	252	0.00	93.70	6.53	252	0.00	93.70	6.68	251	0.00	93.61	7.13
vpm1	263	55.84	62.14	5.39	275	50.18	62.25	6.30	259	57.63	65.18	6.60
vpm2	403	74.96	64.74	17.27	377	75.30	65.08	18.10	373	75.84	67.15	17.71

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avg.	253.667	56.873	50.268	17.521	257.667	54.269	49.677	18.675	245.167	58.793	53.529	24.281

Conclusions and Future Work

- We got some insights about the use of normalizations in the separation of disjunctive cuts.
- We have shown that such normalizations – even the good ones – are not fully safe.
- We have shown that redundant constraints hurt in the separation of disjunctive cuts.

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- We have shown that such normalizations – even the good ones – are not fully safe.
- We have shown that redundant constraints hurt in the separation of disjunctive cuts.

- Even after the elimination of redundant constraints one might separate non supporting cuts. Can we do better?
- Can we come up with a better normalization (equivalently, a different objective function) such that the cheating effect of redundant constraints can be mitigated?
- Can we remove redundant constraints efficiently, e.g., in the framework of Balas & Perregaard?
- Can we separate directly on the (γ, γ_0) space?