EXPONENTIAL FIELDS

SOME QUESTIONS FROM COMPLEX ANALYIS.

1 IF $p(z) \in \mathbb{C}[z]$ HOW MANY ZEROS DOES

$$\exp(z) = p(z)$$

HAVE IN $\mathbb{C}$?

II IF $\lambda_1, \ldots \lambda_m, \mu_1, \ldots \mu_n$
$c_1, \ldots c_m, d_1, \ldots d_n$
$\in \mathbb{C}$, WHEN DOES THE SYSTEM

$$c_1 \exp(\lambda_1 z) + \cdots + c_m \exp(\lambda_m z)$$
$$= 0$$
$$= d_1 \exp(\mu_1 z) + \cdots d_n \exp(\mu_n z)$$

HAVE INFINITELY MANY SOLUTIONS IN $\mathbb{C}$?

ANSWERS

I. <u>INFINITELY MANY</u>

UNLESS $p = 0$ [HADAMARD, OR EARLIER]

[<u>BUT</u> ONLY <u>FINITELY</u> MANY <u>REAL</u> SOLUTIONS, IF $p \in \mathbb{R}[z]$.
HARDY, OR EARLIER]

<u>II</u>. UNKNOWN, BUT THERE IS A NICE CONJECTURE OF H. SHAPIRO (50 YEARS AGO) IN TERMS OF COMMON FACTORS.

MAJOR PROJECT

UNDERSTAND ANY UNDERLYING "EXPONENTIAL ALGEBRA"

— E.G. ANALOGUES OF POLYNOMIAL IDEALS, VARIETIES, NULLSTELLENSATZ.

COMMUTATIVE EXPONENTIAL RINGS (E-RINGS)

UNITAL RINGS R, WITH $E: R \to R$ SUCH THAT

$$E(x+y) = E(x) \cdot E(y)$$
$$E(0) = 1$$

SO VALUES OF E ARE UNITS

PERIODS $\{x : E(x) = 1\}$

IN $\mathbb{C}$, $2\pi i \mathbb{Z}$

IN $\mathbb{R}$, $\{0\}$.

[EASY, BY MODEL THEORY, TO CONSTRUCT R WITH BIZARRE GROUP OF PERIODS.
IN INFINITARY LOGIC, CAN EXPRESS THAT GROUP OF PERIODS IS INFINITE CYCLIC]

UNIVERSAL ALGEBRA E-RINGS FORM AN EQUATIONAL CLASS, SO FREE E-RING ON X IS WELL-DEFINED. BUT WE NEED A PERSPICUOUS (COMPUTABLE) CHARACTERIZATION OF
— NO A PRIORI GUARANTEE OF THIS.

A BASIC CONSTRUCTION

R GIVEN, WITH A PARTIAL E DEFINED ON AN ADDITIVE SUBGROUP D (SO $E : D \to R$), AND GIVEN Δ WITH

$$R = D \oplus \Delta.$$

GO TO MULTIPLICATIVE GROUP t^{Δ} ($t^{\delta_1} \cdot t^{\delta_2} = t^{\delta_1 + \delta_2}$), AND THE GROUP RING $R \rightarrowtail R[t^{\Delta}]$.

EXTEND E TO R BY ADDITIVITY, VIA

$$E(\delta) = t^{\delta}, \quad \delta \in \Delta.$$

NOW SPLIT

$$R[t^{\Delta}] = R \oplus t^{\Delta \setminus \{0\}}$$

AND ITERATE ω TIMES.

SPECIAL CASE

$$R = \mathbb{Z}[X], \; D = (0), \; \Delta = R.$$

RESULT: $[X]^E$, FREE E-RING ON X.

E-FIELDS

THESE ARE CHARACTERISTIC 0 FIELDS K WITH $E : K \to K$ GIVING AN E-RING STRUCTURE.

FEW EXAMPLES WERE SEEN IN NATURE PRIOR TO 1980.

- $\mathbb{R}$ AND $\mathbb{C}$ WITH THE ANALYTICALLY DEFINED EXP
- GONSHOR-KRUSKAL SURREAL EXPONENTIAL (IN GONSHOR'S BOOK)
- LE-SERIES (DAHN-WOLTER, V. DEN DRIES-MACINTYRE-MARKER)
- ÉCALLE TRANSSÉRIES
- P-ADIC CASES (REALLY ONLY <u>RINGS</u>)

E-POLYNOMIALS

LET K BE AN E-FIELD. LET
$R = K[X] = K \oplus (X)$
[(X) = IDEAL GENERATED BY X].
DO PREVIOUS CONSTRUCTION TO GET

$K[X]^E$ (A FREE OBJECT)

THE RING OF E-POLYNOMIALS OVER K IN X.

GET NATURAL E-RING MORPHISM

$K[X]^E \longrightarrow K^{K^X}$.

<u>EXERCISE</u> INJECTIVE IF

$K = \mathbb{C}$ OR $\mathbb{R}$.

SO WE CONFLATE THE TERM AND FUNCTION INTERPRETATION OF E-POLYNOMIAL OVER K.

THE SCHANUEL CONDITION

AN E-DOMAIN R SATISFIES SC
IF FOR ALL $\lambda_1, \cdots \lambda_n \in R$

TRANSCENDENCE DEGREE OVER $\mathbb{Q}$

OF $\lambda_1, \cdots \lambda_n, E(\lambda_1), \cdots E(\lambda_n)$

$\geq$ LINEAR DIMENSION OVER $\mathbb{Q}$

OF $\lambda_1, \cdots \lambda_n$.

["NO EXOTIC RELATIONS"].

THEOREM 1 $[X]^E$ SATISFIES SC.

THEOREM 2 IF K SATISFIES SC,

SO DOES $K[X]^E$.

THEOREM 3 IF $\mathbb{C} \models$ SC,

THEN THE E-SUBRING OF $\mathbb{R}$
GENERATED BY ϕ IS

$\cong [\phi]^E$.

THE HUGE PROBLEM

SCHANUEL'S CONJECTURE:

$\mathbb{C} \models SC.$

IN MY LECTURES I WILL DESCRIBE HOW SC LINKS THE LOGIC AND ANALYSIS OF THE CLASSICAL EXPONENTIAL FIELDS.

THESE LINKS HAVE BEEN DETECTED MAINLY BY MODEL THEORISTS, WITH THE MOTIVATION BEING A LONG-OPEN PROBLEM OF <u>TARSKI</u> ON THE LOGIC OF THE

REAL EXPONENTIAL FIELD

LOGICAL QUESTIONS

I FOCUS ON

- SYSTEMS OF EQUATIONS (AND MAYBE ORDER CONDITIONS)
- DEFINABLE SETS AND FUNCTIONS

[THE FIRST CONCERNED MAINLY WITH DECISION PROBLEMS, THE SECOND WITH GEOMETRY AND TOPOLOGY].

WE WORK OVER E-FIELDS, OR ORDERED E-FIELDS.

LET K BE SUCH, AND L AN E-SUBFIELD.

WE DEFINE THE NOTIONS OF SYSTEMS AND DEFINITIONS

<u>OVER</u> (THE PARAMETERS) <u>L</u>.

IF L IS THE E-FIELD GENERATED BY $\mathbb{Q}$, WE TALK OF <u>PURE</u> SYSTEMS AND DEFINITIONS

$F \in K[X]^E$ IS OVER L IF IT IS IN THE SMALLEST E-SUBRING OF $K[X]^E$ CONTAINING $L[X]$.

<u>BASIC SETS OVER L</u> FOR SOME $X = \{x_1, \cdots x_n\}$, DEFINED BY:

$$F(x_1, \cdots x_n) = 0$$

(OR ALSO > 0 IN ORDERED CASE)

$F \in K[X]^E$, OVER L.

<u>CONSTRUCTIBLE SETS OVER L</u>

BOOLEAN COMBINATIONS OF BASIC SETS.

<u>EXISTENTIAL (OR SOLVABILITY) SETS OVER L</u>

SIMPLE PROJECTIONS OF BASIC SETS.

FINALLY,

DEFINABLE SETS OVER L :

GET THEM BY STARTING WITH BASIC SETS , AND CLOSING UNDER BOOLEAN OPERATIONS AND PROJECTION

METHODOLOGICAL POINT I DID THIS ABSTRACTLY TO AVOID SYNTACTICAL ENNUI , BUT TO FORMULATE THINGS ALGORITHMICALLY YOU HAVE TO WORK WITH DEFINITIONS, NOT DEFINED SETS

SYSTEMS OVER L

FINITE INTERSECTION OF CONDITIONS

$F_1 = 0$

$F_2 \neq 0$

$F_3 > 0$ (IF RELEVANT)

<u>L-DEFINABLE FUNCTION</u>:

ONE WHOSE GRAPH IS L-DEFINABLE.

<u>L-DEFINABLE POINT</u>:

ONE WHOSE SINGLETON SET IS L-DEFINABLE

(USED IN LATER LECTURES)

RESULTS FOR $\mathbb{R}$ AND $\mathbb{C}$

1. (TARSKI). THE SET $\mathbb{Z}$ IS (PURE) DEFINABLE IN THE E-FIELD $\mathbb{C}$.

<u>PROOF:</u> $\mathbb{Z}$ IS THE COMPLEMENT OF THE PROJECTION TO x_1-SPACE OF

$$E(x_2) = 1 \wedge E(x_1 \cdot x_2) \neq 1 .$$

<u>COR:</u> THE E-FIELD $\mathbb{C}$ IS AT LEAST AS UNDECIDABLE AS ARITHMETIC.

[DETERRED LOGICIANS, UNTIL <u>ZILBER</u>, IN LAST DECADE, FROM FURTHER STUDY OF DEFINABLE SETS IN $\mathbb{C}$]

RECALL THAT $\mathbb{C}$, QUA FIELD, IS <u>DECIDABLE</u>

A REFINEMENT

LACZKOVICH (1990's, FOLLOWING AN IDEA OF MINE) SHOWED:

$\mathbb{Z}$ IS EXISTENTIALLY DEFINABLE IN THE E-FIELD $\mathbb{C}$.

COR: "HILBERT'S 10^{TH} PROBLEM" FOR THE E-FIELD $\mathbb{C}$ IS UNDECIDABLE.

NOTE 1 THERE IS, HOWEVER, MORE TO BE SAID FOR SYSTEMS CONSISTING OF A SINGLE POLYNOMIAL.

NOTE 2 ZILBER DETECTED A CRITERION CONJECTURALLY SUFFICIENT FOR SYSTEMS TO BE SOLVABLE

(MORE IN LATER LECTURES)

ANOTHER NEGATIVE RESULT

(AJM, 1991). IN THE COMPLEX CASE, THE COMPLEMENT OF AN EXISTENTIAL SET NEED NOT BE EXISTENTIAL.

(IN PURE FIELD CASE EXISTENTIAL = CONSTRUCTIBLE)

BUT WE DO NOT KNOW IF THINGS GET WORSE IN TERMS OF A HIERARCHY THEOREM.

POSITIVE RESULTS FOR $\mathbb{R}$

THEOREM (WILKIE 1991)

DEFINABLE = EXISTENTIAL

(EASY TO SEE ≠ CONSTRUCTIBLE)

COR. EXISTENTIAL CLOSED UNDER COMPLEMENT.

COR VERY GENERAL FINITENESS THEOREMS FOR GEOMETRY AND TOPOLOGY OF DEFINABLE SETS

[ELABORATIONS BY VAN DEN DRIES, MILLER, MARKER, MACINTYRE USED IN LIE THEORY BY SCHMID - VILONEN AND IN DIOPHANTINE GEOMETRY BY PILA].

N.B. PROOF NOT EFFECTIVE

BUT

THEOREM (MACINTYRE – WILKIE 1992) EFFECTIVE IF $\mathbb{R}$ SATISFIES SC.

THEOREM IF $\mathbb{R}$ SATISFIES SC, ALGORITHM FOR TESTING SOLVABILITY OF SYSTEMS OVER $\mathbb{Q}$.

[IN NEITHER CASE HAS SC BEEN REMOVED]

THEOREM IF $\mathbb{C}$ SATISFIES SC, π IS NOT $\mathbb{Q}(e)$ DEFINABLE IN $\mathbb{R}$

[SC CLEARLY RELEVANT HERE, AS WITHOUT IT WE DON'T KNOW IF π ALGEBRAIC OVER e]

ASSUMING SC FOR $\mathbb{R}$, THE DOMINATING PROBLEM OF TARSKI WAS SOLVED.

<u>THEOREM</u> (MACINTYRE - WILKIE 1992)

IF $\mathbb{R}$ SATISFIES SC, THEN THERE IS AN ALGORITHM FOR TESTING TRUTH OF FIRST-ORDER SENTENCES ABOUT THE E-FIELD $\mathbb{R}$

(INCLUDES PROBLEMS IN HARDY'S 1911 ASYMPTOTICS OF LE FUNCTIONS).

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BURIED IN WILKIE'S PROOF IS THE IDEA FOR DEFINING IN A REAL SETTING A NOTION OF <u>E-ALGEBRAIC</u> IN MODELS OF SC. THERE IS A RELATED NOTION OF E-DIMENSION, DISCOVERED IN A DIFFERENT FORMALISM BY ZILBER. ALL THIS WILL BE EXPLAINED IN LATER LECTURES.

ZILBER'S E-FIELDS

THESE ARE CONSTRUCTED <u>UNCONDITIONALLY</u> TO SATISFY SC. IN ADDITION, AND IT IS FROM THIS THAT THEIR MARVELLOUS PROPERTIES COME, THEY HAVE INFINITE CYCLIC GROUP OF PERIODS,

<u>AND</u>

ARE UNIVERSAL (SATISFY A KIND OF E-NULLSTELLENSATZ) RELATIVE TO EMBEDDINGS WHICH DO NOT CAUSE "DIMENSION DROP"

(TO BE EXPLAINED NEXT TIME)