LECTURE 2

SC AND E-DIMENSION

FORMAL DERIVATIVES

$K[X]^E$ HAS DERIVATIONS $\partial/\partial x$ ($x \in X$) SATISFYING:

i) $\partial/\partial x \cdot x = 1$

ii) $\partial/\partial x \cdot y = 0 \qquad y \in X, y \neq x$

iii) $\partial/\partial x \cdot k = 0 \qquad k \in K$

iv) $\partial/\partial x (E(f)) = (\partial/\partial x \cdot f) \cdot E(f)$

(AND IN FACT THE SET IS UNIQUE)

THIS DOES NOT NEED ANY TOPOLOGY ON K GIVING A MEANING TO DIFFERENTIATING FUNCTIONS, BUT FOR $K = \mathbb{R}$ OR $\mathbb{C}$

THE TERM AND FUNCTION INTERPRETATIONS AGREE.

IN GENERAL, GET A NOTION OF JACOBIAN MATRIX OF A SYSTEM.

DEFINE AN E-POLYNOMIAL MAP

$$K^n \longrightarrow K^m$$

AS ONE GIVEN BY AN m-TUPLE OF ELEMENTS OF

$$K[x_1, \ldots x_n]^E .$$

THEN THE NOTION OF THE FORMAL RANK OF SUCH A MAP AT $(\alpha_1, \ldots \alpha_n)$ IS CLEAR, USING THE JACOBIAN <u>MATRIX</u>.

FOR US THE MAIN CASE IS:

$$G : K^{n+m} \longrightarrow K^m$$

AN E-POLYNOMIAL MAP.

FOR $\bar{\beta} \in K^n$, $\tilde{\alpha} \in K^m$

$$G_{\bar{\beta}}(\tilde{\alpha}) = G(\bar{\beta}, \tilde{\alpha}) ,$$

SO $G_{\bar{\beta}} : K^m \rightarrow K^m$.

$$G_{\bar{\beta}}(\tilde{x}) = \tilde{0}$$

IS A SYSTEM, IN THE FAMILY

$$G(\bar{y}, \tilde{x}) = \tilde{0} \qquad \begin{array}{l} \tilde{x} : \text{UNKNOWNS} \\ \bar{y} : \text{PARAMETERS} \end{array}$$

NOW, FIX $\bar{\beta}$ AND CONSIDER THE $\bar{\alpha}$ WITH $G(\bar{\beta}, \bar{\alpha}) = \tilde{0}$ AND THE RANK OF THE JACO.

CALL THESE <u>NONSINGULARLY G-DEPENDENT ON $\bar{\beta}$</u>.

<u>N.B</u> IN THE ORDINARY POLYNOMIAL SITUATION THIS WOULD FORCE EACH COORDINATE OF $\bar{\alpha}$ TO BE <u>ALGEBRAIC</u> OVER $\bar{\beta}$ AND THE COEFFICIENTS OF G.

<u>KEY DEFINITION</u> (IN K)

α_1 IS E-ALGEBRAIC OVER $\bar{\beta}$ IF FOR SOME G AS ABOVE OVER $\mathbb{Q}$

α_1 IS THE FIRST COORDINATE OF AN $\bar{\alpha}$ NONSINGULARLY G-DEPENDENT ON $\bar{\beta}$

E-CLOSURE (IN K)

$$CL_E(\bar{\beta}) = \{\alpha : \alpha \text{ E-ALGEBRAIC OVER } \bar{\beta}\}$$

$$CL_E(B) = \bigcup_{\bar{\beta} \text{ IN } B} CL_E(\bar{\beta})$$

THEOREM $CL_E(B)$ IS AN E-SUBFIELD OF K.

THEOREM THE ASSOCIATED DEPENDENCE RELATION SATISFIES STEINITZ EXCHANGE (AND SO WE HAVE INVARIANT NOTION OF E-DIMENSION).

THEOREM ($K = \mathbb{R}$)
ASSUME SC, THEN π HAS DIMENSION 1
(CLEARLY e HAS DIMENSION 0)

NORMALIZED SYSTEMS

AN E-POLYNOMIAL USUALLY HAS ITERATED E

$$[\quad x + E(E(x^2+7)) \quad]$$

BUT SYSTEMS CAN, AT THE COST OF ADDING NEW VARIABLES, BE WRITTEN WITHOUT USING ITERATION.

THUS: $x + E(E(x^2+7)) = 0$

IS "EQUIVALENT TO"

$$\left.\begin{aligned} y - E(x^2+7) &= 0 \\ w - E(y) &= 0 \\ x + w &= 0 \end{aligned}\right\}$$

DEF: A NORMALIZED SYSTEM $G = \tilde{0}$ IS ONE WHERE ALL THE E-POLYNOMIALS ARE OF THE FORM

$F(x_1, \ldots x_n, E(x_1), \ldots E(x_n))$, F AN ORDINARY POLYNOMIAL

NORMALIZATION

THEOREM α_1 IS E-ALGEBRAIC OVER $\overline{\beta}$ $\Leftrightarrow$ FOR SOME NORMALIZED G α_1 IS FIRST COORDINATE OF AN $\tilde{\alpha}$ NONSINGULARLY G-DEPENDENT OVER $\overline{\beta}$.

PROOF EASY. $\square$.

CONNECTION TO SC

$\tilde{\alpha}$, $\overline{\beta}$, G AS ABOVE, NORMALIZED.

$G: K^{n+m} \longrightarrow K^m$

SINCE G POLYNOMIAL, AND SYSTEM $G_{\overline{\beta}}$ IS NONSINGULAR

THE TRANSCENDENCE DEGREE OF $(\alpha_1, \ldots \alpha_m, E(\alpha_1), \ldots E(\alpha_m))$ IS $\leq 2m - m = m$.

IF SC, MUST $= m$,

UNLESS THE α_i ARE LINEARLY DEPENDENT OVER $\mathbb{Q}$.

THE <u>REAL</u> DIMENSION OF π.

LET $\alpha_1 = \pi$, AND LET

$\tilde{\alpha} = (\alpha_1, \ldots \alpha_m)$ BE A NONSINGULAR NORMALIZED SYSTEM (m EQUATIONS) OVER $\mathbb{Q}$.

THEN IF SC EITHER THE α_j ARE LINEARLY DEPENDENT OVER $\mathbb{Q}$, OR THE TRANSCENDENCE DEGREE OF

$$(\alpha_1, \cdots \alpha_m, E(\alpha_1), \cdots E(\alpha_m)) = m.$$

BY WORKING WITH LEAST POSSIBLE m CAN ASSUME

$$\mathbb{Q}\text{-DIM}(\tilde{\alpha}) = m.$$

NOW CONSIDER

$(\alpha_1, \cdots \alpha_m, i\pi)$, WHOSE

$\mathbb{Q}$-DIM IS $m+1$.

BUT $\text{TRANS. DEG}_{\mathbb{Q}}(\alpha_1, \cdots \alpha_m, i\pi, E(\alpha_1), \cdots E(\alpha_m), E(i\pi))$

$= m$

CONTRADICTING SC FOR $\mathbb{C}$!

THE COMPLEX DIMENSION OF π.

THE PRECEDING SHOWS THAT THE REAL DIMENSION OF π IS 1.

HOWEVER, CONSIDER :

$$\left.\begin{aligned} y^2 + 1 &= 0 \\ E(yx) - 1 &= 0 \end{aligned}\right\}$$

(π, i) IS A NONSINGULAR ZERO IN $\mathbb{C}$.

SO π HAS COMPLEX DIMENSION ZERO.

NUMBER OF ZEROS OF NORMALIZED NONSINGULAR SYSTEM.

$\mathbb{R}$: $< \infty$ HOVANSKI 1980

$\mathbb{C}$: $\leq \aleph_0$ EASY.

SO E-CLOSURE HAS VERY DIFFERENT PROPERTIES IN THE TWO CASES.

STRONG EXTENSIONS

TO AVOID MAINLY NOTATIONAL TROUBLES, WE WORK UNTIL FURTHER NOTICE WITH E-FIELDS SATISFYING SC.

DEF: $K \rightarrowtail L$ IS STRONG ($K \triangleleft L$) IF FOR $\tilde{\alpha}$ IN K

$$\mathrm{DIM}_K(\tilde{\alpha}) = \mathrm{DIM}_L(\tilde{\alpha}).$$

(NO DIMENSION DROP).

EXERCISE STRONG MAPS FORM A GOOD CATEGORY.

N.B NOT OBVIOUS THAT AN E-FIELD SATISFIES SC (BUT CAN BE DONE BY VARIANT OF $[X]^E$ CONSTRUCTION).

PERIODS

KER = GROUP OF PERIODS

$= \{x : E(x) = 1\}$.

(SP) ("STANDARD PERIODS")

KER IS INFINITE CYCLIC.

SP $\Rightarrow$ $K^{*} \supseteq$ ALL ROOTS OF UNITY.

COR $K \rightarrowtail L$ BOTH SP

$\Rightarrow$ SAME PERIODS.

ZILBER'S BASIC CATEGORY

OBJECTS K SATISFYING SC AND SP

MORPHISMS STRONG EMBEDDINGS

<u>AIM</u> TO CONSTRUCT, AND UNDERSTAND, "UNIVERSAL" OR "EXISTENTIALLY CLOSED" OBJECTS K IN THE CATEGORY, SATISFYING "HILBERT NULLSTELLENSATZ", I.E. IF A SYSTEM $G \,\tilde{=}\, 0$ OVER K IS SOLVED IN $K \triangleleft L$, THEN SOLVED IN K.

<u>FACT</u> THESE EXIST, AND ARE COFINAL IN CATEGORY, FOR GENERAL NONSENSE REASONS.

THEY ARE ALGEBRAICALLY CLOSED FIELDS.

IT IS A FAMILIAR PHENOMENON IN MODEL THEORY THAT SUCH GENERAL NONSENSE DOES NOT GUARANTEE INFORMATIVE AXIOMS FOR THE STRUCTURES OBTAINED.

CLASSICALLY IT WAS DONE FOR CATEGORY OF MODELS OF UNIVERSAL AXIOMS WITH EMBEDDINGS AS MORPHISMS.

SUCCESSES:

1. FIELDS $\longrightarrow$ ALG. CLOSED FIELDS

2. ORDERED FIELDS $\longrightarrow$ REAL CLOSED FIELDS

3. FIELDS WITH AUTOMORPHISM $\longrightarrow$ ACFA

"FAILURES"

4. GROUPS $\longrightarrow$ EXIST. CLOSED GROUPS (NOT AN ELEMENTARY CLASS)

ZILBER'S AXIOMS

THEY ARE EXTREMELY NATURAL, AND FAIRLY EASY TO WORK WITH. THEY BELONG TO A TRADITION OF ROBINSONIAN "SUCCESSES".

FROM THE DISCUSSION OF NORMALIZATION, A NATURAL AIM IS TO FIND CONDITIONS ON A VARIETY

$$V \subseteq K^m \times (K^*)^m$$

SUCH THAT IN SOME STRONG EXTENSION OF K IT MEETS THE m-DIMENSIONAL GRAPH $\mathcal{E}_m$ OF E.

$$\mathcal{E}_m = \{ (\alpha_1, \dots \alpha_m, E(\alpha_1), \dots E(\alpha_m) \}$$

$$\underset{\text{DEF}}{=} \{ (\tilde{\alpha}, \widetilde{E(\alpha)}) \}$$

IT IS NATURAL TO CONSIDER GENERIC POINTS

$$(x_1, \dots x_m, y_1, \dots y_m) \text{ OF } V$$

AND LOOK FOR CONSTRAINTS ON EASY SUCCESS.

THE CONSTRAINTS

1. FUNCTIONAL EQUATION

- IF GENERICALLY

$$\lambda_1 \cdot x_1 + \cdots + \lambda_m \cdot x_m = c \in K,$$

$\lambda_1, \ldots \lambda_m \in \mathbb{Q}$, THEN

WANT $y_1^{\lambda_1} \cdots y_m^{\lambda_m} = E(c)$

GENERICALLY. (ENOUGH FOR $\lambda_j \in \mathbb{Z}$)

- IF GENERICALLY

$$y_1^{\lambda_1} \cdots y_m^{\lambda_m} = d \in K^*$$

$\lambda_j \in \mathbb{Z}$, THEN GENERICALLY

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = \mathrm{LOG}(d)$$

GENERICALLY, FOR ONE OF THE LOGS OF d.

(THERE ARE LOGS IN ZILBER'S MODELS)

II FORMALLY SCHANUEL

LET $\tilde{x} = (x_1, \ldots x_m)$

$\tilde{y} = (y_1, \ldots y_m)$.

LET A BE A $j \times m$ MATRIX OVER $\mathbb{Z}$, WITH ROWS

$\rho_1, \ldots \rho_j$

Then THE $\rho_i \cdot \tilde{x}^T$ MATCH THE

$$\tilde{y}^{\rho_i} = y_1^{a_{i1}} y_2^{a_{i2}} \cdots y_m^{a_{im}}$$

WHERE $\rho_i = (a_{i1}, \ldots a_{im})$.

THEN, GENERICALLY,

$$\mathrm{TRANSDEG}_{\mathbb{Q}}(\rho_1\tilde{x}^T, \ldots \rho_j\tilde{x}^T, \tilde{y}^{\rho_1}, \ldots y^{\rho_j})$$

$$\geq \mathrm{LINDIM}_{\mathbb{Q}}(\rho_1\tilde{x}^T, \ldots \rho_j \cdot \tilde{x}^T).$$

ZILBER AXIOMS

ROUGHLY (IN INTERESTS OF NOTATION),

FOR EACH V, IF THE TWO CONSTRAINTS ARE MET,

V MEETS $\mathcal{E}_m$.

THEOREM. THESE ARE THE AXIOMS FOR THE UNIVERSAL MODELS IN ZILBER'S CATEGORY

YOU MAY ASK MORE, THAT V MEETS $\mathcal{E}_m$ IN A GENERIC POINT.

SUCH MODELS EXIST AND ARE COFINAL, THE STRONGLY UNIVERSAL, OR SIMPLY, FOR US, ZILBER $\mathcal{E}$-FIELDS

CCC

K HAS CCC IF THE CLOSURE OF A COUNTABLE SET IS COUNTABLE.

"STEINITZ FOR ZILBER FIELD".

THERE IS A UNIQUE CCC ZILBER FIELD OF CARDINAL $2^{\aleph_0}$. !!

ZILBER'S CONJECTURE.

THIS FIELD, $\mathbb{B}$ IS ISOMORPHIC TO $\mathbb{C}$.