LECTURE 3

"ANALYSIS IN $\mathbb{B}$"

WHEREAS OUR FIRST CLEAR CONCEPTION OF $\mathbb{C}$ IS GEOMETRICAL (WITH $\mathbb{R}$ IN THE PICTURE), AND E IS GIVEN ANALYTICALLY, WITH π, e HAVING DEFINITE LOCATIONS ON THE REAL LINE $\mathbb{R}$,

THINGS ARE VERY DIFFERENT FOR $\mathbb{B}$. IT HAS NO (OBVIOUS) GEOMETRY ATTACHED TO IT. RATHER IT IS OBTAINED BY (SET-THEORETIC) MODEL THEORY, A LIMIT CONSTRUCTION. THERE IS NO OBVIOUS NOTION OF <u>REAL LINE</u>, NO OBVIOUS TOPOLOGY, AND NO OBVIOUS WAY OF TELLING APART THE TWO GENERATORS OF KER (I.E. NO OBVIOUS WAY TO DISTINGUISH $\pm 2\pi i$)

[IMAGINE THAT WE HAD NO GEOMETRIC PICTURE OF THE FIELD $\mathbb{C}$, ONLY A MODEL-THEORETIC CONSTRUCTION. AND STEINITZ THEORY]

STILL, WE CAN COMPARE $\mathbb{C}$ AND $\mathbb{B}$ IN E-ALGEBRAIC TERMS. I GIVE SOME RESULTS DONE WITH PAOLA D'AQUINO AND GIUSY TERZO.

WARM-UP EXAMPLE

$$E(z) = p(z) \qquad p \in \mathbb{B}[z],\ p \neq 0.$$

LET $\alpha_2, \cdots \alpha_n \in \mathbb{B}$.

CONSIDER V GIVEN BY:

$$p(x_1) = y_1$$
$$x_2 \cdot (x_1 - \alpha_2) = 1$$
$$\vdots$$
$$x_n \cdot (x_1 - \alpha_n) = 1$$

EASY TO CHECK THAT IT RESPECTS FUNCTIONAL EQUATION AND IS FORMALLY SCHANUEL.

SO HAS A SOLUTION WITH $E(x_1) = y_1$ AND $x_1 \notin \{\alpha_2, \cdots \alpha_n\}$.

SO $E(z) = p(z)$ HAS ∞ MANY SOLUTIONS IN $\mathbb{B}$ (CF. LECTURE 1, PAGES 1-2).

SCHANUEL NULLSTELLENSATZ (SN).

FOR AN E-FIELD K, SN SAYS:
IF $F \in K[x_1, \ldots x_n]^E$ HAS NO ZERO IN K, THEN
$F = E(G)$, SOME
$G \in K[x_1, \ldots x_n]^E$.

SCHANUEL CONJECTURED THAT $\mathbb{C}$ SATISFIES SN.

<u>THEOREM</u> (HENSON-RUBEL, 1984)
$\mathbb{C}$ SATISFIES SN

<u>PROOF</u> HARD NEVANLINNA THEORY □

<u>THEOREM</u> (D'A-M-T, SHKOP) $\mathcal{B}$ (AND ALL THE UNIVERSAL E-FIELDS) SATISFY SN.

<u>PROOF</u> SERIOUS ABSTRACT ALGEBRA. □

CONSEQUENCES

THEOREM (LITTLE PICARD FOR $\mathbb{B}$: D'A-M-T).

IF $F \in \mathbb{B}[x]^E$, $F \notin \mathbb{B}$, THEN F OMITS AT MOST ONE VALUE

THERE IS EVEN A WEAK VERSION OF BIG PICARD.

THEOREM (D'A-M-T).

IF $F \in \mathbb{B}[x]^E$, $F \notin \mathbb{B}[x]$, F TAKES ALL BUT AT MOST ONE VALUE INFINITELY OFTEN.

THIS NEEDS FOR $\mathbb{B}$ AN IMPROVEMENT OF SN

NOT KNOWN FOR $\mathbb{C}$!!

SN^+

SN^+ SAYS: IF $F \in K[x]^E$ HAS ONLY FINITELY MANY ZEROS,

$F = E(G) \cdot P$,

P POLYNOMIAL, $G \in K[x]^E$.

THEOREM (D'A - M - T).

$\mathbb{B}$ SATISFIES SN^+.

IT IS "GENERAL NONSENSE" THAT IF $\mathbb{C} \models SC$ THEN

$\mathbb{C} \prec \mathbb{B}$. THE PRECEDING SHOW A WEAK "ELEMENTARITY" OF THE EMBEDDING

THEOREM (ASSUME $\mathbb{C} \models SC$)

IF $F \in \mathbb{C}[x_1, \cdots x_n]^E$ HAS NO ZERO IN $\mathbb{C}$, IT HAS NONE IN $\mathbb{B}$.

BUT: NOT CLEAR THAT IF $F \in \mathbb{C}[x]^E$ HAS $< \infty$ ZEROS IN $\mathbb{C}$ THEN SAME HOLDS IN $\mathbb{B}$.

ALGORITHMIC ASPECTS

LACZKOVICH'S PROOF WORKS FOR $\mathbb{R}$ TO GIVE UNDECIDABILITY FOR TESTING SOLVABILITY OF SYSTEMS OVER $\mathbb{Q}$.

HOWEVER, THERE IS ONE IMPORTANT DECIDABILITY RESULT.

THEOREM THE SET OF SYSTEMS OVER $\mathbb{Q}$ OF THE FORM

$$F = 0$$, FOR A SINGLE F ,

SOLVABLE IN $\mathbb{R}$, IS DECIDABLE.

IF $\mathbb{C} \models SC$,

$\mathbb{C}$ AND $\mathbb{R}$ AGREE ON SOLVABILITY OF SUCH SYSTEMS

DEFINITIONS, AUTOMORPHISMS, π

CONSIDER THE FIELD $\mathbb{C}$ AS GIVEN, WITH $\mathbb{R}$ GIVEN, ALLOWING THE LOCALLY COMPACT TOPOLOGY TO BE DEFINED.

(SET-THEORETIC) FACT $\mathbb{C}$ HAS $2^{2^{\aleph_0}}$ FIELD AUTOMORPHISMS, BUT ONLY TWO ARE LEBESGUE MEASURABLE, NAMELY THE IDENTITY AND COMPLEX CONJUGATION $z \mapsto \bar{z}$.

BOTH OF COURSE RESPECT E.

OPEN PROBLEM DOES ANY OTHER FIELD AUTOMORPHISM RESPECT E ?

DEEP RESULT $\mathfrak{B}$ HAS $2^{2^{\aleph_0}}$ E-FIELD AUTOMORPHISMS.

HALF-OPEN PROBLEM DOES $\mathfrak{B}$ HAVE AN E-FIELD AUTOMORPHISM OF ORDER 2 ?

("YES" ANNOUNCED PRIVATELY BY VICENZO MANTOVA (ENS PISA).

$\mathbb{R}$ IN $\mathbb{C}$?

FIELD-THEORETICALLY ONE CANNOT SEE $\mathbb{R}$ IN $\mathbb{C}$.

CERTAINLY NOT FIRST-ORDER DEFINABLE, AS $\mathbb{R}$ NOT CONSTRUCTIBLE.

BUT SITUATION MUCH WORSE. WE LOOK FOR $K \subseteq \mathbb{C}$ SO $\mathbb{C} = K(i)$. SUCH K ARE EXACTLY THE FIXED FIELDS OF INVOLUTIONS $\sigma \in \mathrm{AUT}(\mathbb{C})$, AND THEY ARE ALL REAL CLOSED (ARTIN-SCHREIER). MOREOVER, ISOMORPHISM TYPE OF K DETERMINED BY CONJUGACY CLASS OF σ, ALWAYS OF CARDINAL $2^{2^{\aleph_0}}$. MOREOVER, THERE ARE $2^{2^{\aleph_0}}$ CONJUGACY CLASSES OF INVOLUTIONS

$\mathrm{Fix}(\sigma)$ MAY WELL BE NONARCHIMEDEAN. INDEED UP TO $\cong$ IT CAN BE ANY REAL-CLOSED FIELD OF CARDINAL $2^{\aleph_0}$

<u>DEEP THEOREM</u> : <u>NO</u> REAL-CLOSED FIELD IS E-DEFINABLE IN $\mathbb{S}3$

(ZILBER – A "STABILITY" THEOREM).

* WE DO NOT KNOW THE ANALOGUE FOR $\mathbb{C}$. *

ZILBER SHOWED AN ENORMOUS CONSTRAINT ON DEFINABILITY IN $\mathbb{S}3$

<u>THEOREM.</u> ANY SET DEFINABLE IN $\mathbb{S}3$ IS COUNTABLE OR COCOUNTABLE

UNKNOWN FOR $\mathbb{C}$.

SINE, COSINE, π

NOW WE ASSUME ABOUT K ONLY STANDARD PERIODS, AND $i \in K$.

AS IN LECTURE 1, $\mathbb{Z}$ IS DEFINABLE.

IN $\mathbb{C}$, i NOT DEFINABLE, BECAUSE OF COMPLEX CONJUGATION.

DEFINE SINE AND COSINE ON K BY:

$$\text{COSINE}(z) = \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\text{SINE}(z) = \tfrac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

THESE FUNCTIONS ARE DEFINABLE OVER $\mathbb{Q}$, AND USUAL LAWS HOLD.

LET $\{\alpha, -\alpha\}$ BE THE SET OF GENERATORS OF KER

LEMMA $\{\alpha, -\alpha\}$ IS DEFINABLE OVER $\mathbb{Q}$.

PROOF $\mathbb{Z}$ AND KER ARE. $\square$

LEMMA THE SET $\{\alpha/2i, -\alpha/2i\}$ IS DEFINABLE OVER $\mathbb{Q}$.

PROOF EXERCISE. □

LEMMA THERE IS JUST ONE ELEMENT x IN $\{\alpha/2i, -\alpha/2i\}$ FOR WHICH $\text{SINE}(x/2) = 1$

FOR THE OTHER, $-x$, $\text{SINE}\left(\frac{-x}{2}\right) = -1$.

PROOF EXERCISE. □

DEFINE π TO BE THE x OF THE PREVIOUS LEMMA. THIS IS A CORRECT DEFINITION (OVER $\mathbb{Q}$) IN $\mathbb{C}$, AND WE NOW TAKE IT IN GENERAL.

NOTE THE STARK CONTRAST TO REAL CASE, WHERE SC $\Rightarrow$ π NOT DEFINABLE.

LET US NOW ASSUME K IS ALGEBRAICALLY CLOSED (e.g. $\mathbb{C}$ OR $\mathbb{B}$).

LET U BE THE SET OF ROOTS OF UNITY IN K.

<u>LEMMA</u> U IS DEFINABLE OVER $\mathbb{Q}$.

<u>PROOF</u> $U = \{ E(2\pi i/n) : n \in \mathbb{Z}, n \neq 0 \}$ AND $\mathbb{Z}$ DEFINABLE. □.

$\mathbb{Q}^{ab}$ IS $\mathbb{Q}(U) = \mathbb{Q}[U]$, THE MAXIMAL ABELIAN EXTENSION OF $\mathbb{Q}$. IT IS KNOWN THAT $\mathbb{Q}^{ab}$ HAS A UNIQUE MAXIMAL FORMALLY REAL SUBFIELD $\mathbb{Q}^{ab,\mathrm{Real}}$ (WITH $\mathbb{Q}^{ab} = \mathbb{Q}^{ab,\mathrm{Real}}(i)$) ALSO, $\mathbb{Q}^{ab,\mathrm{Real}}$ IS GOT FROM $\mathbb{Q}$ BY ADJOINING ALL $\cos(r \cdot \pi)$, $r \in \mathbb{Q}$.

DEFINABLE ALGEBRAIC NUMBERS

IN $\mathbb{C}$, COMPLEX CONJUGATION SHOWS THAT ANY NUMBER DEFINABLE OVER $\mathbb{Q}$ MUST BE REAL.

BUT IN GENERAL K THIS IS MEANINGLESS.

STILL:

THEOREM (K AS ABOVE) (KIRBY-MACINTYRE-ONSHUUS) ALL ELEMENTS OF $\mathbb{Q}^{ab, real}$ ARE DEFINABLE.

HOWEVER,

THEOREM IN $\mathfrak{B}$, THE ONLY DEFINABLE ALGEBRAIC NUMBERS ARE THOSE IN $\mathbb{Q}^{ab, real}$.

PROOF USES SOME DELICATE ALGEBRA BY ZILBER. □

COR: i NOT DEFINABLE.

COR: NO $\sqrt[3]{2}$ DEFINABLE

[$\sqrt{2}$ IS DEFINABLE AS $1/\text{COSINE}(\pi/4)$]

GALOIS THEORY AND EXTENDING PARTIAL EXPONENTIALS

IN THIS SUBJECT WE OFTEN MEET THE FOLLOWING SITUATION :

$K = D \oplus \Delta$, A $\mathbb{Q}$-SPACE DECOMPOSITION, E DEFINED ONLY ON D. WE WANT TO EXTEND E TO A $\mathbb{Q}$-SUBSPACE OF Δ, WITH SPANNING SET

$\{\alpha_1, \cdots \alpha_t\}$.

WE MAKE SOME CHOICE OF

$E(\alpha_1), \cdots E(\alpha_t)$.

HOW TO EXTEND TO THE $\mathbb{Q}$-SPACE ?

FIX n. WHAT SHOULD

$E(\alpha_1/n), \cdots E(\alpha_t/n)$ BE ?

ENTER THE n^{th} ROOTS OF THE $E(\alpha_j)$ (AND THE n^{th} ROOTS OF 1).

GIVEN ONE CHOICE $E(\alpha_1/n), \cdots E(\alpha_r/n)$

THE OTHERS ARE

$\theta_{n,1} E(\alpha_1/n), \cdots \theta_{n,r} E(\alpha_r/n)$

FIX SOME FIELD L.

CONSIDER FIRST V_1, THE VARIETY OF

$\alpha_1, \cdots \alpha_r, E(\alpha_1), \cdots E(\alpha_r)$

OVER L.

NOW LET n VARY AND CONSIDER ALL $V_{n,\bar{\theta}}$,

THE VARIETY OF

$(\alpha_1/n, \cdots \alpha_r/n, \theta_{n,1} E(\alpha_1/n), \cdots, \theta_{n,r} E(\alpha_r/n))$

THESE FORM A PROJECTIVE SYSTEM, UNDER THE MORPHISMS $\gamma_{m,n}$ (FOR $n \mid m$) GIVEN BY

$$(x_1, \cdots x_r, y_1, \cdots y_r) \longmapsto (n/m \cdot x_1, \cdots n/m \cdot x_j, y_1^{n/m}, \cdots y_j^{n/m})$$

SOLUTIONS TO EXTENSION PROBLEM (AS FAR AS NORMALIZED RELATIONS OVER L ARE CONCERNED) CORRESPOND TO BRANCHES THROUGH THIS PROJECTIVE SYSTEM.

ONE CAN EXPECT "UNSTABLE" MODEL THEORY IF THERE IS INFINITE BRANCHING.

BUT KUMMER THEORY YIELDS A BASIC STABILITY RESULT.

FORGET THAT THE y_j ARE TO BE $E(x_j)$ AND JUST WORK WITH V, THE VARIOUS $V_{n,\bar{\theta}}$ AND THE CONNECTING MORPHISMS (A FINITE-BRANCHING TREE)

WE MAKE <u>ONE ASSUMPTION</u> ON V, NAMELY THAT THERE IS NO j-TUPLE $(s_1, \cdots s_j)$ FROM $\mathbb{Z}$ $\bar{s} \neq \bar{0}$, SO THAT GENERICALLY, $y_1^{s_1} \cdots y_j^{s_j} \in L(\bar{x})$.

ASSUME, TOO, L ALG. CLOSED (MORE FOR CONVENIENCE)

THEN ONE HAS, FOR FIXED V,

<u>THEOREM</u> $\exists n$ SUCH THAT THERE IS NO BRANCHING BELOW THE LEVEL n.

(PROVED BY ZILBER USING KUMMER THEORY)

THIS IS A KIND OF NOETHERIANITY, GIVING A KIND OF MODEL-THEORETIC STABILITY.