Applications of stochastic recursive equations

and branching processes in

queueing networks with stationary ergodic driving sequences

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1 Introduction

•Consider the stochastic recursion

$$Y_{n+1} = A_n(Y_n) + B_n, \qquad n \geq n_0. \tag{1}$$

Y_n is a random variable defined on a subset $\mathcal{Y}$ of R^m for some integer m, equipped with some norm $\|\cdot\|$.

•We assume that $0 \in \mathcal{Y}$ where 0 is the zero vector. $\{A_n(\cdot)\}_{-\infty<n<\infty}$ are a sequence of random processes taking values in R^m.

•**A1:** For each sample path, the vectors $A_n(y)$ and B_n are nonnegative (componentwize) for all n and y. Moreover, $A_n(y)$ are monotone increasing in y for all n and for each sample path.

•In applications considered in the talk, we assume $\{A_n\}_n$ to be independent, and independent of B_n. B_n will be general stationary ergodic.

- **Example 1: TCP model. Wireless networks, random losses**
  - $Y(t)$ is the transmission rate.
  - It increases linearly in time.
  - When a packet is lost, Y decreases by a multiplicative constant ν. S_n is the interloss times.
  - Y_n denotes the rate just before loss n. Then

$$Y_{n+1} = \nu Y_n + \alpha S_n.$$

    Then $A_n(y) = \nu y, B_n = \alpha S_n$.
    One dimensional system!
  - (Ref: Altman, Avratchenkov, Barakat, Sigcomm 2000).

- **Example 2: TCP model, congestion losses.** [El-Azouzi, Ross, Tuffin, Vojvonic and EA]
  - N flows sharing a bottleneck link with capacity C.
  - T_n:- n-th congestion epoch (capacity reached), $\tau_{n+1} = T_{n+1} - T_n$.
  - η_i:= additive increase rate for session i,
  - $\beta^{(i)}$ its multiplicative decrease rate.
  - $X_n^{(i)}$ the throughput of session i *before* the n-th congestion epoch.
  - Denote $\eta = \sum_{j=1}^{N} \eta_j$ and $\tilde{\eta}_i = \eta_i/\eta$, $i = 1, 2, \ldots, N$.
  - let $Z_n^{(i)}$ be random variable with value 1 if session i experiences a loss at the n-th congestion epoch, and 0 otherwise.

We have

$$X_{n+1}^{(i)} = \gamma_n^{(i)} X_n^{(i)} + \tau_{n+1}\eta_i \tag{2}$$

where

$$\gamma_n^{(i)} = (1 - Z_n^{(i)}) + \beta^{(i)} Z_n^{(i)} = 1 - (1 - \beta_i) Z_n^{(i)}.$$

Assume that there is a loss as soon as

$$\sum_{i=1}^{N} \gamma_n^{(i)} X_n^{(i)} + \tau_{n+1} \sum_{i=1}^{N} \eta_i = C. \tag{3}$$

Using Equation (3), we get the time between the n-th and $(n+1)$-th congestion epochs

$$\tau_{n+1} = \frac{C - \sum_{i=1}^{N} \gamma_n^{(i)} X_n^{(i)}}{\eta} = \frac{1}{\eta} \sum_{i=1}^{N} (1 - \gamma_n^{(i)}) X_n^{(i)} = \frac{1}{\eta} \sum_{i=1}^{N} (1 - \beta_i) Z_n^{(i)} X_n^{(i)}. \quad (4)$$

Combining this with (2) we obtain the relation in vector form:

$$X_{n+1} = \mathsf{A}_n X_n + \mathsf{B} \quad (5)$$

where

$$[\mathsf{A}_n]_{ij} = \gamma_n^{(i)} 1\{i = j\} - \tilde{\eta}_i \gamma_n^{(j)} \qquad \text{and} \qquad \mathsf{B}_i = \tilde{\eta}_i C.$$

Sufficient condition for A1, relaxing linearity

A2: For each realization, the vectors $A_n(y)$ and B_n are nonnegative (componentwize) for all n and y. Moreover, $A_n(\cdot)$ is divisible, i.e. for each n and k, there exist $A_n^{(i)}(\cdot)$, $i = 0, ..., k$ such that for any $x_i \in \mathcal{Y}, i = 0, ..., k$ satisfying $\sum_{i=0}^k x_i \in \mathcal{Y}$,

$$A_n \left(\sum_{i=0}^{k} x_i \right) = \sum_{i=0}^{k} A_n^{(i)} (x_i) \tag{6}$$

where $\{A_n^{(i)}(\cdot)\}_{i=0,1,2,...,k}$ are i.i.d. with the same distribution as $A_n(\cdot)$. In particular, $A_n(0) = 0$.

• **Example:** any non-negative Lévy process.

• **Example:** X_n are integer valued and $A_n(x) = \sum_{i=1}^{x} \xi_n^{(i)}$ where $\xi_n^{(i)}$ are i.i.d.

• It follows from Assumption A2 that if $A_n(\cdot)$ are stationary, then there exists a matrix A such that for any y,

$$E[A_n(y)] = \mathsf{A} y. \tag{7}$$

Moreover, for A_n i.i.d. $\quad E[A_n A_{n-1} ... A_{m+1}(y)] = \mathsf{A}^{n-m} y, \qquad \forall n > m.$

Example 3: Queue with Vacations, Gated Regime

•$M/G/1/\infty$ queue,

•Arrival rate λ, i.i.d. service times $\{D_n\}$ with first and second moments d, $d^{(2)}$.

•Sequence of vacations: V_n. Will be assumed stationary ergodic, with first and second moments v, $v^{(2)}$.

•Gated regime: at the nth end of vacation, a gate is closed (nth polling instant). Then the server goes on serving the customers present at the queue at that polling instant: X_n.
Then the server leaves on vacation.

•We denote:

- B_n:= the number of arrivals during the nth vacation.
- $A_n(X_n)$:= the number of arrivals during the service time of X_n customers i.e. the number of arrivals during the busy period S_n of the nth "cycle" C_n.

•Then:

$$X_{n+1} = A_n(X_n) + B_n, \qquad n \geq n_0.$$

One dimensional system!

Example 4: Queue with Vacations, Gated Regime

•Define the time to serve N customers as:

$$\tau(N) := \sum_{i=1}^{N} D_i$$

•Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T.

•Denote by $\hat{\mathcal{A}}_n(C_n) = \tau(\mathcal{N}(C_n))$, i.e. the sum of service times of all the arrivals during C_n.

•We obtain

$$C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}. \quad (8)$$

One dimensional system!

Example 5: Discrete time infinite server queue

•Service times are considered to be i.i.d. and independent of the arrival process.

•We represent the service time as the discrete time analogous of a phase type distribution: there are N possible service phases.

•The initial phase k is chosen at random according to some probability $p(k)$.

•If at the beginning of slot n a customer is in a service phase i then it will move at the end of the slot to a service phase j with probability P_{ij}.

•With probability $1 - \sum_{j=1}^{N} P_{ij}$ it ends service and leaves the system at the end of the time slot.

•P is a sub-stochastic matrix (it has nonnegative elements and it's largest eigenvalue is strictly smaller than 1), which means that services ends in finite time w.p.1. and that $(I - P)$ is invertible.

•Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be i.i.d. random matrices of size $N \times N$. Each of its element can take values of 0 or 1, and the elements are all independent.

•The ijth element of $\xi^{(k)}(n)$ has the interpretation of the indicator that equals one if at time n, the kth customer among those present at service phase i moved to phase j.

•Obviously, $E[\xi_{ij}^{(k)}(n)] = P_{ij}$.

•Let $B_n = (B_n^1, ..., B_n^N)^T$ be a column vector for each integer n, where B_n^i is the number of arrivals at the nth time slot that start their service at phase i.

•B_n is a stationary ergodic sequence and has finite expectation.

•Y_n^i:= number of customers in phase i at time n. Satisfies

$$Y_{n+1} = A_n(Y_n) + B_n$$

where the ith element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{9}$$

Objectives:

•Obtain stability conditions

•Obtain explicitly the distribution of the discrete process Y_n, for the general setting.

•Obtain explicitly the expectation of related continuous time processes
Ex 1-2: the transmission rate at arbitrary moment,
Ex 3-4: and polling systems: number of customers in the system at arbitrary moment, waiting times.
Ex 5: expected and variance of number of customers

•The challenge: Obtain tools for studying systems with stationary ergodic driving sequences, without Markovian assumptions.

Solution of the stochastic recursion:

Iterating $Y_{n+1} = A_n(Y_n) + B_n$, we obtain from A2:

$$
\begin{aligned}
Y_2 &= A_1(Y_1) + B_1 \\
&= A_1(A_0(Y_0) + B_0) + B_1 \\
&= A_1^{(0)}(A_0(Y_0)) + A_1^{(1)}(B_0) + B_1 \\
&= A_1^{(0)} A_0^{(0)}(Y_0) + A_1^{(1)}(B_0) + B_1.
\end{aligned}
$$

$$
\begin{aligned}
Y_3 &= A_2(Y_2) + B_2 \\
&= A_2(A_1(Y_1) + B_1) + B_2 \\
&= A_2(A_1(A_0(Y_0) + B_0) + B_1) + B_2 \\
&= A_2^{(0)} A_1^{(0)} A_0^{(0)}(Y_0) + A_2^{(1)} A_1^{(1)}(B_0) + A_2^{(2)}(B_1) + B_2
\end{aligned}
$$

In general:

$$Y_n = \sum_{j=0}^{n-1} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}) + \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (Y_0), \quad n > 0 \qquad (10)$$

(we understand $\prod_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\prod_{i=n}^{k} A_i(x) = A_k A_{k-1} ... A_n$ whenever $k > n$).

•Assume A2, and that the sequence $\{(A_n(\cdot), B_n), -\infty < n < \infty\}$ is stationary ergodic, defined on some probability space $(\Omega, \mathcal{F}, P)$.

Under fairly general assumptions, $\lim_{n\to\infty} \left(\prod_{i=0}^{n-1} A_i^{(0)} \right) (y) = 0$, so Y_n has a limit as $n \to \infty$ distributed like

$$Y_n^* =_d \sum_{j=0}^{\infty} \left(\prod_{i=n-j}^{n-1} A_i^{(n-j)} \right) (B_{n-j-1}), \qquad n \in Z, \qquad (11)$$

where for each integer i, $\{A_i^{(j)}(\cdot)\}_j$ are independent of each other and have the same distribution as $A_i(\cdot)$. Moreover:

•The sum on the right side of (11) converges absolutely P-almost surely.

•For all initial conditions Y_0, $\|Y_n - Y_n^*\| \to 0$, P-almost surely on the same probability space.

•In particular, the distribution of Y_n converges to that of Y_0^* as $n \to \infty$.

Example of sufficient conditions:

•$A_n(\cdot)$ is an i.i.d. sequence, independent of the stationary ergodic sequence B_n;

•Further,

$$E[\|A_0(y)\|] \leq \alpha \|y\|, \qquad \text{and} \qquad E[\|B_0\|] < \infty, \tag{12}$$

for some $\alpha < 1$ and all y, and thus

$$\|\mathsf{A}\| < 1 \tag{13}$$

where A is given in (7).

(Recall that under A2, there exists Ay such that $E[A_n(y)] = \mathsf{A}y$.)

Queueing with gated vacations (Ex. 4)

Expected waiting time: (FIFO queue)

Consider an arbitrary customer. Upon arrival, it has to wait for

1. The residual cycle time C_{res},
2. The service time of all the customers that arrived during C_{past} which is the past cycle time: $d(\lambda E[C_{past}]) = \rho E[C_{past}]$

We have from [Baccelli & Brémaud, 1994]

$$E[C_{res}] = E[C_{past}] = \frac{E[C_0^2]}{2E[C_0]}.$$

Thus the expected waiting time of an arbitrary customer is given by

$$E[W_n] = (1+\rho)\frac{E[C_0^2]}{2E[C_0]},$$

The expected number of customers in queue in stationary regime (not including service) is obtained using Little's Theorem: $\lambda E[W_n]$.

Conclusion: we need to compute $E[C_0]$ and $E[C_0^2]$!

Computing $E[C_0]$ and $E[C_0^2]$

•Recall: $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$.

•Due to the Poisson arrivals, A2 holds.
Assume throughout $\rho < 1$. Then

$$E[|\hat{\mathcal{A}}_n(y)|] = \rho y \text{ and thus } \|\mathsf{A}\| < 1;$$

we can use our expressions for the steady state distributions!

•Introduce the correlation function: $r(n) = E[V_0 V_n]$.

•The first and second moments of C_n in stationary regime are given by

$$E[C_n] = \frac{v}{1-\rho},$$

$$E[C_n^2] = \frac{1}{(1-\rho^2)} \left(\frac{\lambda v d^{(2)}}{1-\rho} + r(0) + 2 \sum_{j=1}^{\infty} \rho^j r(j) \right). \tag{14}$$

Proof of expressions for $E[C_0^2]$

Useful relations: 2nd moment of workload arriving during T

•If N is a random variable independent of the sequence D_n, and $\tau(N) := \sum_{i=1}^{N} D_i$ then

$$E[\tau(N)^2] = E[N^2]d^2 + E[N](d^{(2)} - d^2). \tag{15}$$

•Let $\mathcal{N}(T)$ denote the number of arrivals during a random duration T, where the arrival process is Poisson with rate λ, and is independent of T. Then

$$E[\mathcal{N}(T)^2] = \lambda^2 E[T^2] + \lambda E[T]. \tag{16}$$

•If we take an arbitrary T and choose $N = \mathcal{N}(T)$, then we get from (15)-(16)

$$\begin{aligned} E[(\hat{\mathcal{A}}(T))^2] &= E[\tau(\mathcal{N}(T))^2] \\ &= d^2(\lambda^2 E[T^2] + \lambda E[T]) + \lambda E[T](d^{(2)} - d^2) \\ &= d^2 \lambda^2 E[T^2] + \lambda E[T] d^{(2)}. \end{aligned} \tag{17}$$

•Also, if we take $T = \tau(N)$, then

$$E[\mathcal{N}(\tau(N))]^2 = \lambda^2 \Big[E[N^2]d^2 + E[N](d^{(2)} - d^2)\Big] + \lambda d E[N]. \tag{18}$$

•From $C_{n+1} = \hat{\mathcal{A}}_n(C_n) + V_{n+1}$ we have

$$\begin{aligned} E[C_{n+1}^2] &= E[\hat{\mathcal{A}}_n(C_n)^2] + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] \\ &= \left(\rho^2 E[C_n^2] + \lambda E[C_n] d^{(2)}\right) + v^{(2)} + 2E[\hat{\mathcal{A}}_n(C_n)V_{n+1}]. \end{aligned}$$

•To compute the last term, we now use the explicit form of C_0:

$$C_0 = \sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}).$$

•We use the fact that the processes $\{\hat{\mathcal{A}}_i^{(j)}\}$ are independent of $\{V_n\}$. We get:

$$\begin{aligned} E[\hat{\mathcal{A}}_n(C_n)V_{n+1}] &= E[\hat{\mathcal{A}}_0(C_0)V_1] = E\left[\hat{\mathcal{A}}_0 \left(\sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} \hat{\mathcal{A}}_i^{(-j)} \right) (V_{-j}) \right) V_1 \right] \\ &= \rho \sum_{j=0}^{\infty} \rho^j E[V_{-j}V_1] = \sum_{j=1}^{\infty} \rho^j r(j). \end{aligned}$$

Substituting this, we obtain the second moment.

Number of customers at beginning of busy periods.

•Recall that we had $X_{n+1} = A_n(X_n) + B_n$ where

- B_n:= the number of arrivals during the nth vacation.
- $A_n(X_n)$:= the number of arrivals during the service time of X_n customers i.e. the number of arrivals during the busy period S_n of the nth "cycle" C_n.

•We get from (18)

$$\begin{aligned} E[X_{n+1}^2] &= E[A_n(X_n)^2] + E[B_n^2] + 2E[A_n(X_n)B_n] \\ &= \lambda^2 \Big(E[X_n^2]d^2 + E[X_n](d^{(2)} - d^2) \Big) + \lambda d E[X_n] \\ &\quad +(\lambda^2 v^{(2)} + \lambda v) + 2\lambda d E[X_n B_n] \end{aligned}$$

In stationary regime, we have $E[X_n^2] = E[X_{n+1}^2]$. Hence:

$$(1-\rho^2)E[X_0^2] = E[X_0](\lambda^2(d^{(2)} - d^2) + \lambda d) + (\lambda^2 v^{(2)} + \lambda v) + 2\lambda d E[X_0 B_0] \quad (19)$$

We need to compute the last term!

•We use the explicit expression for X_0:

$$X_0 = \sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} A_i^{(-j)} \right) (B_{-j-1})$$

and obtain

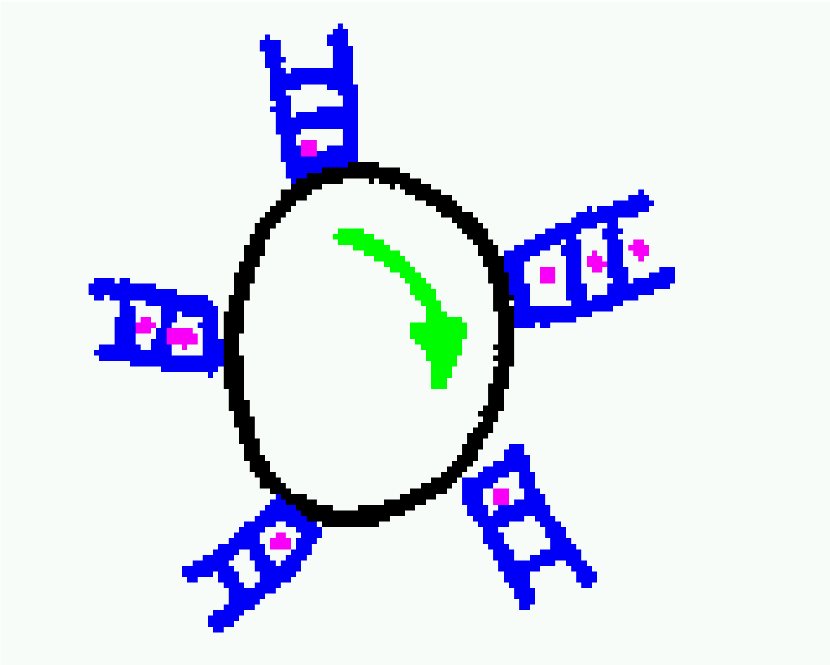
$$\begin{aligned} E[X_0 B_0] &= E \left(\sum_{j=0}^{\infty} \left(\prod_{i=-j}^{-1} A_i^{(-j)} \right) (B_{-j-1}) B_0 \right) \\ &= \sum_{j=0}^{\infty} \rho^j E[B_{-j-1} B_0] = \lambda^2 \sum_{j=0}^{\infty} \rho^j r(j+1). \end{aligned}$$

•Substituting, we obtain the second moment of X_n:

$$E[X_n^2] = \frac{\lambda^3 v d^{(2)}}{(1-\rho)^2(1+\rho)} + \frac{\lambda v}{1-\rho} + \frac{\lambda^2 (r(0) + 2\sum_{j=1}^{\infty} \rho^j r(j))}{1-\rho^2}$$

Other results: Polling systems with N queues

•The server moves cyclically (fixed order) between the queues $1, ..., M$.
It requires walking times (vacations) for moving from one queue to another.



•Upon arrival at a queue, some customers are served. The number to be served is determined by the "polling regime":

•**Globally Gated (GG) regime** (Boxma, Levy, Yechiali 1992):
A cycle starts whenever the servers arrives at queue 1. A gate is closed simultaneously at all queues at the beginning of a cycle.
Customers to be served at a cycle are those that were present at the beginning of the cycle.

•**Gated regime** [see e.g. book by Takagi 1986]:
Upon arrival at each queue, a gate is closed at that queue, and customers present there at the polling instant are served.

•**Exhaustive regime:** [see e.g. book by Takagi 1986]:
At each queue, the server stays serving as long as there are customers.

All regimes can be modeled within our framework:

•A one dimensional recursion can be used for the cycle time for the GG polling system.
The first two moments of the cycle time allows to obtain the expected waiting times at all queues.

•The Exhaustive and Gated regime require M-dimensional recursive equations.
Walking time distributions depend on the stations, so they are not stationary ergodic. We cannot directly use our framework!
If we write the evolution of the system at beginning of cycles then the stationarity and ergodicity hold. This requires more complex recursions.

Possible extensions:

The following extensions preserve our framework of stochastic recursive equations:

•Batch arrivals,

•Polling: correlation between arrivals to different queues.

•We may allow the service times and arrival intensities to vary according to a stationary process Z_n, where the index n corresponds the state of the process at the beginning of the nth cycle.

Back to Example 1: TCP model.

- $Y(t)$ is the transmission rate. It increases linearly in time.
- When a packet is lost, Y decreases by ν. S_n is the interloss times.
- Y_n denotes the rate just before loss n. Then $Y_{n+1} = \nu Y_n + \alpha S_n$.
- Taking expectations, we obtain in the stationary regime:

$$E[Y_0] = \frac{\alpha E[S]}{1-\nu}.$$

- $E[Y^2]$ is obtained from:

$$\begin{aligned} EY_{n+1}^2 &= E(\nu Y_n + \alpha S_n)^2 \\ &= \nu^2 EY_n^2 + \alpha^2 ES_n^2 + 2\nu\alpha E[Y_n S_n] \end{aligned}$$

$$E[Y_n S_n] = \alpha E\left[\left(\sum_{k=0}^{\infty} \nu^k S_{n-1-k}\right) S_n\right]$$

•Define $R(k) = E[S_n S_{n+k}]$. We get

$$E[(Y_n^*)^2] = \frac{\alpha^2}{1-\nu^2}[R(0) + 2\sum_{k=1}^{\infty} \nu^k R(k)]$$

• The average transmission rate: $\bar{Y} = \lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)dt$ is given by

$$\begin{aligned}
\bar{Y} &= \frac{1}{E[S]} E_0\left[\int_0^{S_1} Y(t)dt\right] = \frac{1}{E[S]} E_0\left[\int_0^{S_1} (\nu Y_0 + \alpha t)dt\right] \\
&= \frac{1}{E[S]}(\nu E[Y_0 S_1] + \alpha E[S^2]/2) \\
&= \frac{1}{E[S]} E_0[\alpha\nu \sum_{k=0}^{\infty} \nu^k S_{-1-k} S_0] + \frac{\alpha}{2} E[S_0^2] \\
&= \frac{1}{E[S]} \alpha \sum_{k=0}^{\infty} \nu^{k+1} R(k+1) + \frac{\alpha}{2E[S]} R(0) \\
&= \frac{1}{E[S]} \alpha \left(\sum_{k=1}^{\infty} \nu^k R(k) + \frac{1}{2} R(0)\right)
\end{aligned}$$

Note: in this example the "A_n" are completely linear, and the theory for linear stochastic recursive equations can be used, see e.g.

•Vervaat, 1979,

•Brandt, Franken and Lisek, 1992,

•Glasserman and Yao, 1995,

General form of first and second moment:

Let:

$$Y_{n+1} = A_n(Y_n) + B_n$$

where the i the element of the column vector $A_n(Y_n)$ is given by

$$[A_n(Y_n)]_i = \sum_{j=1}^{N} \sum_{k=1}^{Y_n^j} \xi_{ji}^{(k)}(n) \tag{20}$$

$\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ are i.i.d. random matrices of size $N \times N$. Each of its element is a nonnegative integer.

•Denote $E[\xi_{ij}^{(k)}(n)] = \mathsf{A}_{ji}$.

•The N-dimensional vector B_n is a stationary ergodic stochastic whose entries $B_n^i, i = 1, ..., N$ are nonnegative integers.

•Denote by y_i and $y_i^{(2)}$ the first and second moment of the ith element of Y_n^*.

•Denote $cov(Y)_{ij} = E[(Y_0^*)_i(Y_0^*)_j] - y_i y_j$.

•Let b_i and $b_i^{(2)}$ denote the two first moments of B_n^i.

•Denote $cov(\xi)_{jk}^i = E(\xi_{ij}^{(0)} \xi_{ik}^{(0)}) - \mathsf{A}_{ji}\mathsf{A}_{ki}$.

•Define the following $N \times N$ matrices:
$\mathcal{B}(k)$ is the matrix whose ijth entry equals $E[B_0^i B_k^j]$, where k is an integer.
$\hat{B}$ is the matrix whose ijth entry equals $b_i b_j$,
$cov(B)$ is the matrix whose ijth entry equals $E[B_0^i B_0^j] - b_i b_j$.
Define $\hat{\mathcal{B}}(k) := \mathcal{B}(k) - \hat{B}$.

•(i) The first moment of Y_n^* is given by

$$E[Y_0^*] = (I - \mathsf{A})^{-1} b, \tag{21}$$

•(ii) Assume that the first and second moments b_i and $b_i^{(2)}$'s are finite. Define Q to be the matrix whose (ij)th entry is

$$Q_{ij} \quad = \quad \sum_{k=1}^{N} y_k (cov(\xi)_{ij}^k).$$

•Then the matrix $cov(Y^*)$ is the unique solution of the set of linear equations:

$$\begin{aligned} cov(Y) \quad &= \quad cov(B) + \sum_{r=1}^{\infty} \left(\mathsf{A}^r \hat{\mathcal{B}}(r) + \left[\mathsf{A}^r \hat{\mathcal{B}}(r) \right]^T \right) \\ &\quad + \mathsf{A} cov(Y) \mathsf{A}^T + Q. \end{aligned} \tag{22}$$

•The second moment matrix $E[YY^T]$ in steady state is the unique solution of the set of linear equations:

$$E[YY^T] \quad = \quad E[B_0 B_0^T] + \sum_{r=1}^{\infty} \left(\mathsf{A}^r \mathcal{B}(r) + \left[\mathsf{A}^r \mathcal{B}(r) \right]^T \right) + \mathsf{A} E[YY^T] \mathsf{A} + Q_{ij}.$$

•We show uniqueness. Let Z_1 and Z_2 be two solutions of (22) and define $Z = Z_1 - Z_2$. Then Z satisfies $Z = \mathsf{A}^T Z \mathsf{A}$. Iterating that we obtain that

$$Z = \lim_{n \to \infty} \mathsf{A}^n Z (\mathsf{A}^T)^n = 0$$

where the last equality follows since $\|\mathsf{A}\| < 1$. This implies the uniqueness of the solution for (22). The uniqueness of the solution of (23) is obtained similarly.

•An explicit simple formula is obtained if the correlation is created by a Markov chain.

Example 5: Discrete time infinite server queue

•Service times are geometrically distributed,

•The SRE becomes one dimensional. Y_n denotes the number of customers in the system.

•$\xi_n^{(k)}$ is the indicator that the kth customer present at the beginning of time-slot n will still be there at the end of the time-slot.

•The probability that a customer in the system finishes its service within a time slot is precisely $p = 1 - \mathsf{A} = 1 - E[\xi_n]$.

•We consider a Markov chain with two states $\{\gamma, \delta\}$ with transition probabilities given by

$$\mathcal{P} = \begin{pmatrix} 1 - \epsilon p & \epsilon p \\ \epsilon q & 1 - \epsilon q \end{pmatrix}$$

•As an example, consider the following parameters: $p = q = 1$, at a given state there is at most one arrival with prob. $p_\gamma = 1, p_\delta = 0.5$. This gives:

$$var[Y^*] = \frac{1}{(1 - \mathsf{A}^2)} \left(\frac{3}{16} + \frac{2\mathsf{A}}{1 - \mathsf{A} + 2\epsilon \mathsf{A}} + \frac{3}{4}\mathsf{A} \right).$$

In Fig. 1 we plot the variance of the steady state number of customers, $var[Y^*]$, while varying ϵ and A.

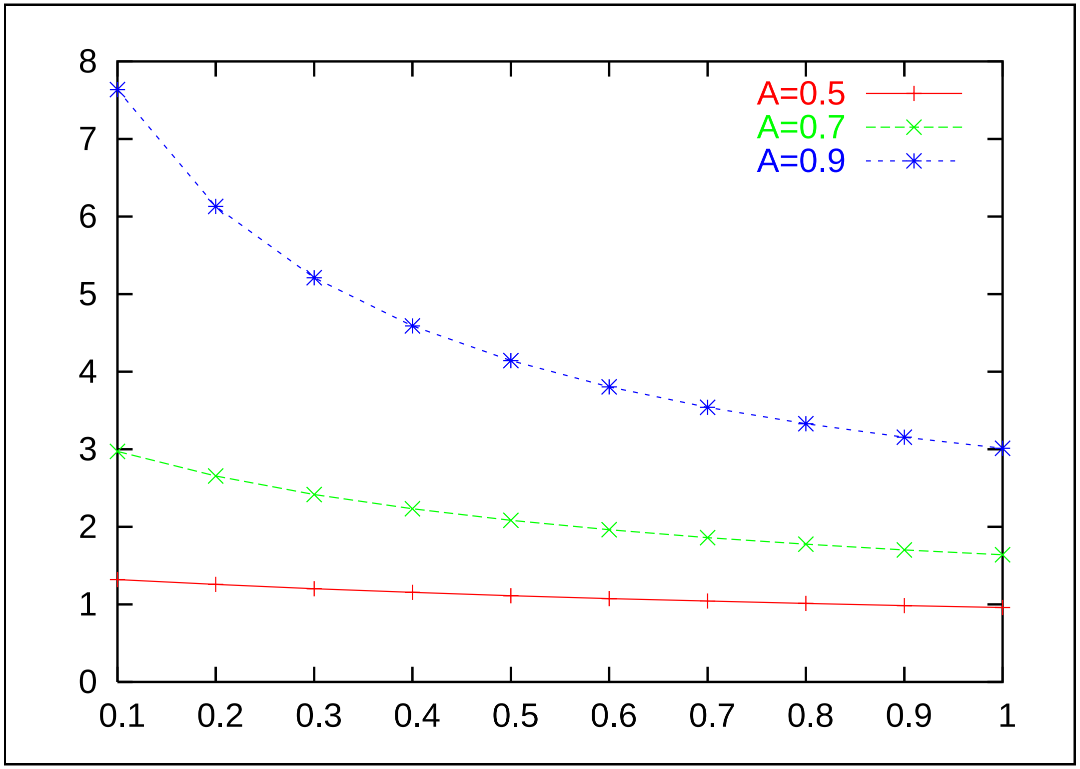


Figure 1: var$[Y*]$ as a function of ϵ and of A

Other issues:

•**No migration $Y_{n+1} = A_n(Y_n)$:** We can show using Kingman's subadditive ergodic theory that the following limit exists P-a.s.:

$$\lim_{n \to \infty} \frac{\log \|Y_n\|}{n} = \Lambda$$

•**The non contracting case:** In example 2 we have $\|A_n\| = 1$ so that $\|\mathrm{A}\| = 1$. Still the results hold.