On the Role of an Isaacs Equation in Importance Sampling for Stochastic Networks

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Main points of the talk:

- Importance sampling in large deviation context needs stronger theoretical foundation
- Stochastic networks provide compelling evidence–no general results for even simplest models
- Performance of optimal importance sampling is characterized in terms of value function of a differential game
- Key tool for the construction and analysis of schemes is related Isaacs equation, and especially subsolutions
- Ideas applicable in much broader setting than stochastic networks

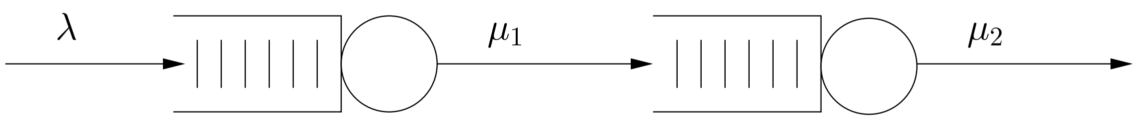
Outline

1. Problem formulation
2. Large deviation and PDE background
3. Some importance sampling generalities
4. The differential game and optimal performance
5. A standard heuristic for importance sampling, and its breakdown
6. Subsolutions for the Isaacs equation and IS schemes
7. Examples of subsolutions and remarks on implementation
8. A comment on the proof and concluding remarks

(1)

Problem formulation

Markov tandem queue, 2 station for illustration.



A tandem queue

Work with discrete time problem, normalization $\lambda + \mu_1 + \mu_2 = 1$.

A benchmark problem for IS: Let $Y(j) = (Y_1(j), Y_2(j))$ be the state. Estimate

$$P\{Y_1(j) + Y_2(j) \text{ exceeds } n \text{ before returning to } 0 \mid Y(0) = nx\},$$

when n is large. Can assume $\mu_1 \geq \mu_2$ without loss of generality, and $\lambda < \mu_2$ for stability.

Let

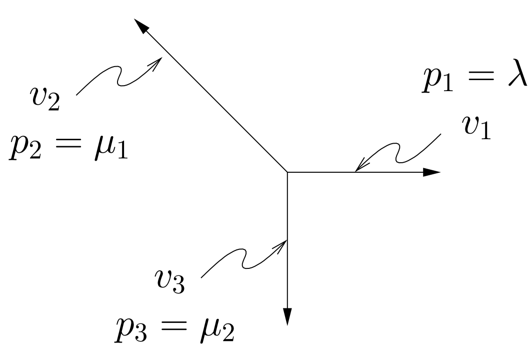
$$V^n(x) \doteq -\frac{1}{n} \log P\{Y_1(j) + Y_2(j) \geq n \text{ before hitting } 0 \mid Y(0) = nx\}.$$

Some references:

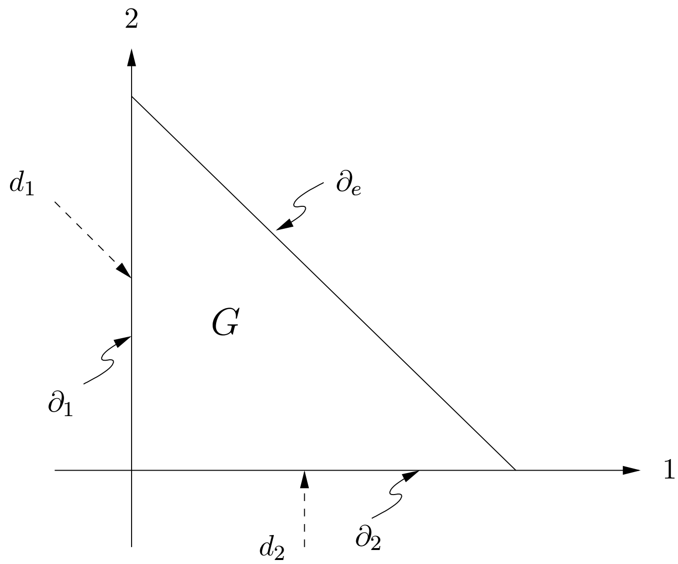
- V. Anantharam, P. Heidelberger, and P. Tsoucas. Analysis of rare events in continuous time Markov chains via time reversal and fluid approximation. *IBM Technical Reports*, RC16820, 1990.
- M. Frater and B. Anderson. Fast simulation of the statistics of excessive backlogs in tandem networks of queues. *Australian Telecommun. Res.*, 23:49–55, 1989.
- M. Frater, T. Lenon, and B. Anderson. Optimally efficient estimation of the statistics of rare events in queueing networks. *IEEE Trans. Autom. Control*, 36:1395–1405, 1991.
- P. Glasserman and S. Kou. Analysis of an importance sampling estimator for tandem queues. *ACM Trans. Modeling Comp. Simulation*, 4:22–42, 1995.
- D. Kroese and V. Nicola. Efficient simulation of a tandem Jackson network. *ACM Trans. Modeling Comp. Simulation*, 12:119–141, 2002.
- S. Parekh and J. Walrand. A quick simulation method for excessive backlogs in networks of queues. *IEEE. Trans. Autom. Control*, 34:54–66, 1989.
- S. Asmussen et. al. Large deviations and fast simulation in the presence of boundaries. *Stoch. Proc. and Appl.*, 102:1–23, 2002.

(2)
Large deviation and PDE background

Jump directions and probabilities for the network:



Relevant domain and constraint directions (Skorohod problem) for the scaled process:



Large deviation properties. We have

$$V^n(x) \to V(x),$$

where $V(x)$ is the solution to the following variational inequality:

$$V(x) = \inf \left\{ \int_0^\tau \sum_{i=1}^{3} \log\left(\frac{\bar{p}_i(t)}{p_i}\right) \bar{p}_i(t) dt \right.$$
$$\left. : \dot{\phi}(t) = \pi\left(\phi(t), \sum_{i=1}^{3} \bar{p}_i(t) v_i\right), \phi(0) = x, \phi \text{ hits } \partial_e \text{ before } 0 \right\}$$

and π is the projected velocity for the given Skorohod problem.

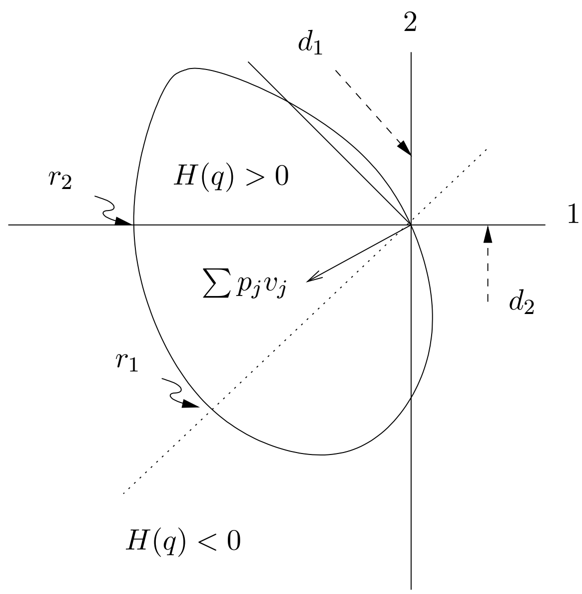
Hamilton-Jacobi equation. V can be characterized as a viscosity solution to the following PDE:

$$\begin{cases} H(DV(x)) = 0, & x \in G, \\ \langle DV(x), d_1 \rangle = 0, & x \in \partial_1, \\ \langle DV(x), d_2 \rangle = 0, & x \in \partial_2, \\ V(x) = 0, & x \in \partial_e, \\ V(0) = \infty \end{cases}$$

where

$$H(q) = \inf_{\bar{p}} \left[\left\langle q, \sum_{i=1}^{3} \bar{p}_i v_i \right\rangle + \sum_{i=1}^{3} \log \left(\frac{\bar{p}_i}{p_i} \right) \bar{p}_i \right].$$

Roots of the Hamiltonian:



with $r_1 = -2 \log \left(\frac{\mu_2}{\lambda} \right) (1, 1)$.

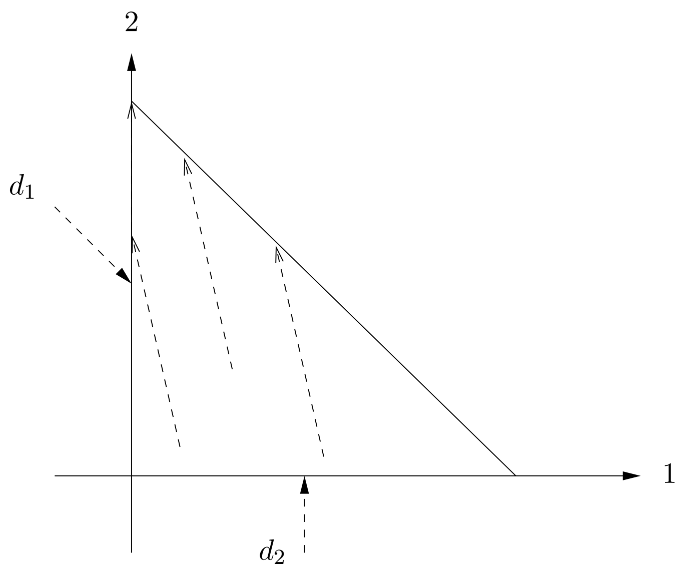
V can be characterized as the "fastest growing" viscosity solution (cf. McEneaney), and thus

$$V(x) = 2\log\left(\frac{\mu_2}{\lambda}\right)(1 - x_1 - x_2).$$

The minimizing $\bar{p}$ in the H-J equation is

$$(\bar{p}_1, \bar{p}_2, \bar{p}_3) = (p_3, p_2, p_1) = (\mu_2, \mu_1, \lambda).$$

The minimizing trajectories:



(3)
Some importance sampling generalities

Basic setup. Given a real-valued random variable of the form $Z = 1_A(Y)$, where Y has distribution θ, compute

$$EZ = P\{Y \in A\} = \theta(A).$$

Consider alternative sampling distribution ν, with $\theta \ll \nu$. Let $f(x) \doteq (d\theta/d\nu)(x)$. Let $\tilde{Y}_i$ be iid ν, and form estimator

$$\tilde{Q}_K \doteq \frac{1}{K}\left(1_A(\tilde{Y}_0)f(\tilde{Y}_0) + \cdots + 1_A(\tilde{Y}_{K-1})f(\tilde{Y}_{K-1})\right).$$

Since

$$E1_A(\tilde{Y}_0)f(\tilde{Y}_0) = \int_A f(y)\nu(dy) = \int_A \theta(dy) = EZ,$$

this is unbiased, with variance

$$\mathrm{var}\left[1_A(\tilde{Y}_0)f(\tilde{Y}_0)\right] = \int_A f(y)^2\nu(dy) - \theta(A)^2.$$

Goal is to minimize variance, so minimize 2nd moment

$$\int_A f(y)^2\nu(dy) = \int_A f(y)\theta(dy)$$

Unconstrained minimization not well defined. We focus on 2nd moment.

Issues related to rare events–guaranteed bounds:

Introduce a *large deviation* parameter n, and assume

$$\lim_{n\to\infty} -\frac{1}{n}\log P\left\{Y^n \in A\right\} = \gamma \in (0,\infty),$$

which we informally write as

$$P\left\{Y^n \in A\right\} \approx e^{-n\gamma}.$$

Particular difficulties here due to relative sizes. With standard Monte Carlo ($f = 1$)

$$\frac{\text{Stand. dev. of estimator}}{\text{Quantity to estimate}} = \frac{\sqrt{\frac{1}{K}\text{var}\left[1_A(\tilde{Y}_0^n)f(\tilde{Y}_0^n)\right]}}{P\left\{Y^n \in A\right\}} \approx \frac{1}{\sqrt{K}}\frac{e^{-\frac{n}{2}\gamma}}{e^{-n\gamma}}.$$

A benchmark. By Jensen's $\leq$, for *any* change of measure

$$\begin{aligned}\limsup_{n\to\infty} -\frac{1}{n}\log E[(1_A(\tilde{Y}_0^n)f(\tilde{Y}_0^n))^2] &\leq \lim_{n\to\infty} -\frac{1}{n}\log\left(E1_A(\tilde{Y}_0^n)f(\tilde{Y}_0^n)\right)^2 \\ &= \lim_{n\to\infty} -\frac{1}{n}\log\left(P\left\{Y^n \in A\right\}\right)^2 \\ &= 2\gamma.\end{aligned}$$

Definition. Estimator $\tilde{Q}_K^n$ is *asymptotically optimal* if

$$\lim_{n\to\infty} -\frac{1}{n}\log E[(1_A(\tilde{Y}_0^n)f(\tilde{Y}_0^n))^2] = 2\gamma.$$

We are also interested in schemes that at *nearly* optimal. Drop K, focus on 2nd moment of single sample $1_A(\tilde{Y}_0^n)f(\tilde{Y}_0^n)$.

(4)
The differential game and optimal performance

With regard to network problem, optimal change of measure shown to correspond to jump probabilities

$$(\tilde{p}_1^n(y), \tilde{p}_2^n(y), \tilde{p}_3^n(y)) \,.$$

Radon-Nikodym derivative (essentially) can be written

$$\left[\prod_{i=0}^{T^n-1} \frac{p_{z^n(i)}}{\tilde{p}^n_{z^n(i)}(Y^n(i))}\right] 1_{S^n}(Y^n),$$

where $z^n(i)$ is the type of jump that occurred at time i, T^n is the first time $Y^n(i)_1 + Y^n(i)_2 \geq n$, and S^n are the trajectories where escape occurs before hitting zero. Let

$$W^n(x) = -\frac{1}{n} \log \min_{\tilde{p}^n} E_x \left[\prod_{i=0}^{T^n-1} \frac{p_{z^n(i)}}{\tilde{p}^n_{z^n(i)}(Y^n(i))}\right] 1_{S^n}(Y^n).$$

Owing to the log (large deviations) transform, $W^n(x)$ is the value for a stochastic game, with Isaacs equation

$$\begin{aligned} W^n(x) = \sup_{\tilde{p}} \inf_{\bar{p}} \Bigg[& \sum_{j=1}^{3} W^n\left(x + \frac{1}{n} v_j\right) \\ & + \frac{1}{n} \sum_{j=1}^{3} \left(\log\left(\frac{\bar{p}_j}{p_j}\right) + \log\left(\frac{\tilde{p}_j}{p_j}\right)\right) \bar{p}_j \Bigg] , \end{aligned}$$

plus boundary constraining behavior plus boundary conditions.

THEOREM.

$$W^n(x) \to W(x),$$

where W is the value of the dynamic game with dynamics

$$\dot{\phi}(t) = \pi\left(\phi(t), \sum_{i=1}^{3} \bar{p}_i(t) v_i\right), \phi(0) = x,$$

cost

$$\int_0^\tau \sum_{j=1}^{3} \left(\log\left(\frac{\bar{p}_j(t)}{p_j}\right) + \log\left(\frac{\tilde{p}_j(t)}{p_j}\right)\right) \bar{p}_j(t) dt,$$

and where the tilde player maximizes while the bar player minimizes over all controls that force escape prior to hitting zero. This value function is a viscosity solution to

$$\begin{cases} \bar{H}(DW(x)) = 0, & x \in G, \\ \langle DW(x), d_1 \rangle = 0, & x \in \partial_1, \\ \langle DW(x), d_2 \rangle = 0, & x \in \partial_2, \\ W(x) = 0, & x \in \partial_e, \\ W(0) = \infty \end{cases}$$

where

$$\bar{H}(q) = \sup_{\tilde{p}} \inf_{\bar{p}} \left[\left\langle q, \sum_{i=1}^{3} \bar{p}_i v_i \right\rangle + \sum_{j=1}^{3} \left(\log\left(\frac{\bar{p}_j}{p_j}\right) + \log\left(\frac{\tilde{p}_j}{p_j}\right)\right) \bar{p}_j \right]$$

Moreover,

$$W(x) = 2V(x).$$

Thus asymptotic optimality can be achieved. The result is natural, since interchanging the max/min shows

$$\bar{H}(q) = \inf_{\bar{p}} \left[\left\langle q, \sum_{i=1}^{3} \bar{p}_i v_i \right\rangle + 2 \sum_{i=1}^{3} \log\left(\frac{\bar{p}_i}{p_i}\right) \bar{p}_i \right] = 2H(q/2).$$

(5)
A STANDARD HEURISTIC FOR IMPORTANCE SAMPLING AND ITS BREAKDOWN

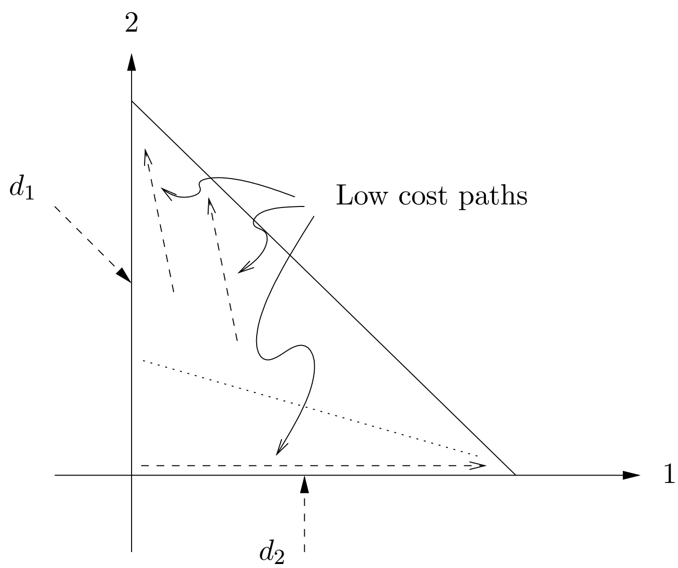
Based on a min/max calculation that is sometimes valid in the setting of Cramér's Theorem, it has been traditional to use the following guess for a nearly optimal IS change of measure:

Let $\tilde{p}_j^n(y)$ *be the same as the asymptotic optimizer in the large deviations analysis.* Thus

$$(\tilde{p}_1^n(y), \tilde{p}_2^n(y), \tilde{p}_3^n(y)) = (\mu_2, \mu_1, \lambda).$$

Unfortunately, this is often far from an optimal policy in the game. E.g., when $\mu_1 = \mu_2 = \mu$, it is always true that the large deviation player can exploit an advantage.

The minimizing trajectories:



EXAMPLE. Simulation results for $(\lambda, \mu_1, \mu_2) = (0.1, 0.45, 0.45)$ and buffer size $n = 25$. The theoretical value is $p_n = 4.04 \times 10^{-15}$. The sample size is $K = 20000$.

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ ($\times 10^{-15}$)	2.58	2.47	5.63	13.65
Standard Error ($\times 10^{-15}$)	0.22	0.24	2.50	10.27
95% C.I. ($\times 10^{-15}$)	[2.14, 3.02]	[1.99, 2.95]	[0.63, 10.63]	[-6.89, 34.19]

Table 1. IS based on standard heuristic for overflow probability of a tandem network

(6)
Subsolutions for the Isaacs equation and IS schemes

Definition. $\bar{W}$ a *classical subsolution* with fixed maximizing control $\tilde{p}(y)$ if $\bar{W}$ is continuously differentiable on an open neighborhood of G, and if

$$\begin{cases} \hat{H}(x, D\bar{W}(x)) \geq 0, & x \in G, \\ \langle D\bar{W}(x), d_1 \rangle \geq 0, & x \in \partial_1, \\ \langle D\bar{W}(x), d_2 \rangle \geq 0, & x \in \partial_2, \\ \bar{W}(x) \leq 0, & x \in \partial_e, \\ \bar{W}(0) \leq \infty \end{cases}$$

where

$$\hat{H}(x, q) = \inf_{\bar{p}} \left[\left\langle q, \sum_{i=1}^{3} \bar{p}_i v_i \right\rangle + \sum_{j=1}^{3} \left(\log\left(\frac{\bar{p}_j}{p_j}\right) + \log\left(\frac{\tilde{p}_j(x)}{p_j}\right)\right) \bar{p}_j \right]$$

Elementary fact: If $\bar{W}$ is a classical subsolution to the Isaacs equation, and if $\tilde{p}(y)$ is defined as a saddle point maximizer, then $\bar{W}$ is also a classical subsolution with fixed maximizing control $\tilde{p}(y)$.

<u>THEOREM.</u> Let $\bar{W}$ be a classical subsolution corresponding to $\tilde{p}(y)$. Consider the importance sampling scheme defined by $\tilde{p}^n(x) = \tilde{p}(x)$. Let

$$\bar{W}^n(x) = -\frac{1}{n}\log[\text{2nd moment under this IS scheme}].$$

Then

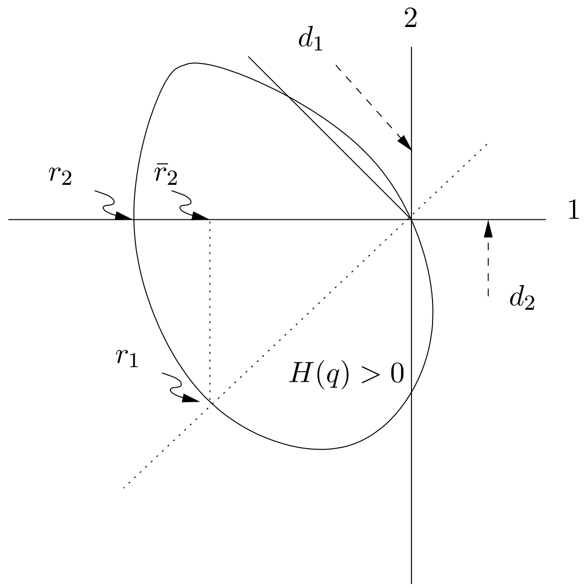
$$\liminf_{n\to\infty} \bar{W}^n(x) \geq \bar{W}(x).$$

Design problem: find a pair $(\bar{W}, \tilde{p})$ such that

- $\bar{W}$ is a subsolution,
- $\bar{W}(0) \geq 2V(0)$ [or at least $\bar{W}(0) \geq 2V(0) - \varepsilon$, given $\varepsilon > 0$].

(7)
Examples of subsolutions and remarks on implementation

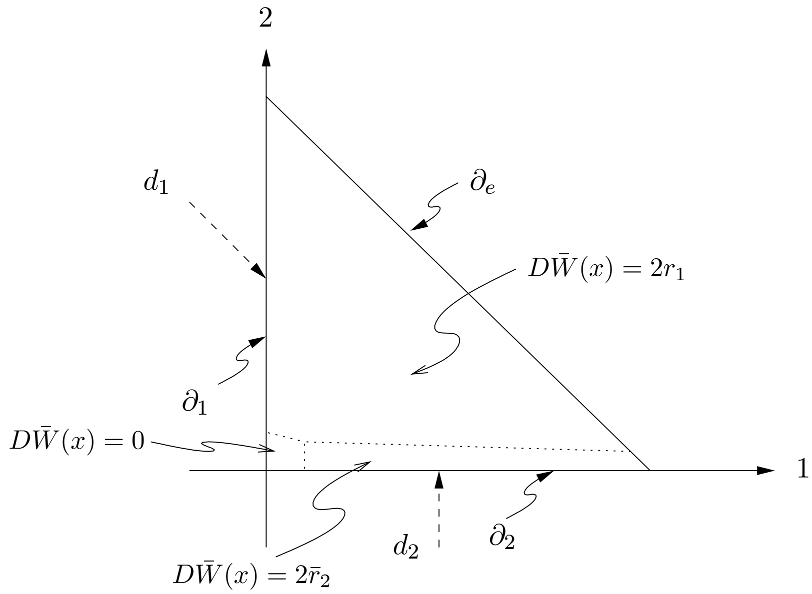
Problem with $\bar{W}(x) = 2V(x)$? It fails the boundary condition along ∂_2.



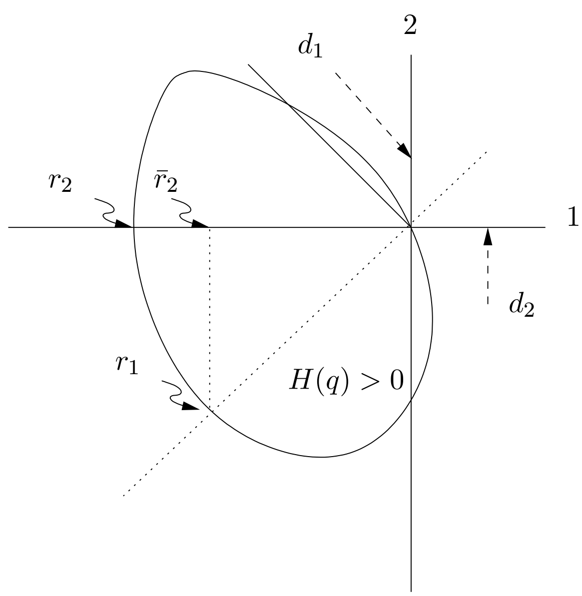
Can nearly optimal subsolutions be constructed in general? The main steps:

- Construct a nonsmooth subsolution as the minimum of a finite set of linear functions.
- Smooth to obtain a C^1 function.
- Define $\tilde{p}(y)$ as a saddle point maximizer.
- Alternatively, smooth (average) both the jump probabilities and gradients of subsolution.

The Hamiltonian for the Isaacs equation, $\bar{H}(q) = 2H(q/2)$, is concave. By Jensen's $\leq$ both smoothings preserve subsolution property.



Construction quite feasible, owing to *homogeneous Hamiltonian* and *flat Neumann and Dirichlet boundaries*.



EXAMPLE. Simulation results for $(\lambda, \mu_1, \mu_2) = (0.1, 0.45, 0.45)$ and buffer size $n = 25$. The theoretical value is $p_n = 4.04 \times 10^{-15}$. The sample size is $K = 20000$.

	No. 1	No. 2	No. 3	No. 4
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95% C.I. $(\times 10^{-15})$	[2.14, 3.02]	[1.99, 2.95]	[0.63, 10.63]	[-6.89, 34.19]

Table 1. IS based on standard heuristic for overflow probability of a tandem network

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ $(\times 10^{-15})$	3.98	4.00	4.10	3.96
Standard Error $(\times 10^{-15})$	0.09	0.08	0.09	0.07
95% C.I. $(\times 10^{-15})$	[3.80, 4.16]	[3.84, 4.16]	[3.92, 4.28]	[3.82, 4.10]

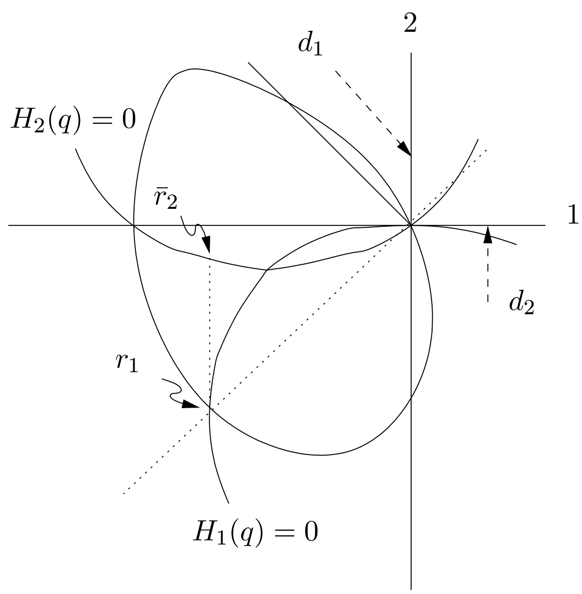
Table 2. IS based on smooth subsolution, probabilities inferred

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ $(\times 10^{-15})$	4.02	4.10	4.07	4.03
Standard Error $(\times 10^{-15})$	0.07	0.09	0.08	0.07
95% C.I $(\times 10^{-15})$	[3.88, 4.16]	[3.92, 4.28]	[3.91, 4.23]	[3.89, 4.17]

Table 3. IS based on direct smoothing of probabilities

An alternative construction based on a refined characterization of boundary dynamics. One can also characterize the boundary behavior via *boundary Hamiltonians* rather than Neumann boundary condition, prove analogous result.

$$\begin{cases} H(DV(x)) \geq 0, & x \in G, \\ H_1(DV(x)) \geq 0, & x \in \partial_1, \\ H_2(DV(x)) \geq 0, & x \in \partial_2, \\ V(x) \leq 0, & x \in \partial_e, \\ V(0) \leq \infty \end{cases}$$



In this example, we take $(\lambda, \mu_1, \mu_2) = (0.05, 0.5, 0.45)$ and buffer size $n = 20$. The theoretical value is $p_n = 5.77 \times 10^{-18}$. The sample size is $K = 20000$

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ ($\times 10^{-18}$)	6.37	6.65	5.39	16.02
Standard Error ($\times 10^{-18}$)	1.24	1.30	0.47	10.67
95 % C.I. ($\times 10^{-18}$)	[3.89, 8.85]	[4.06, 9.25]	[4.46, 6.33]	[-5.31, 37.36]

Table 4. IS based on standard heuristic for overflow probability of a tandem network

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ ($\times 10^{-18}$)	5.77	5.76	5.70	5.78
Standard Error ($\times 10^{-18}$)	0.05	0.06	0.06	0.05
95% C.I. ($\times 10^{-18}$)	[5.67, 5.87]	[5.64, 5.88]	[5.58, 5.82]	[5.68, 5.88]

Table 5. IS based on smoothing the probabilities, Neumann boundary condition

In the following table, we use the change of measure implied by boundary Hamiltonian. Thus the subsolution we obtained is not in classical sense for Neumann boundary data.

	No. 1	No. 2	No. 3	No. 4
Estimate $\hat{p}_n$ ($\times 10^{-18}$)	5.76	5.78	5.76	5.75
Standard Error ($\times 10^{-18}$)	0.02	0.02	0.02	0.02
95% C.I. ($\times 10^{-18}$)	[5.72, 5.80]	[5.74, 5.82]	[5.72, 5.80]	[5.71, 5.79]

Table 6. IS based on smoothing the probability, boundary Hamiltonian

(8)
A comment on the proof and concluding remarks

Idea of proofs. The proofs use a verification argument, with difficulties due to unbounded time interval. A key (non-trivial) estimate that is needed is the following. Given $M < \infty$, there is $T < \infty$ such that

$$P\left\{ Y \text{ first hits } 0 \text{ after time } nT \middle| \, Y(0) = nx \right\} \leq e^{-nM},$$

uniformly for $x \in G$. One expects this from the stability, but does not follow directly from basic large deviation properties of process.

Further work.

- Better understanding of implementation issues
- Networks with feedback (must use boundary Hamiltonians), other events, more general dynamics

References for other LD problems.

- P. Dupuis and H. Wang. Importance sampling, large deviations, and differential games. To appear in *Stoch. and Stoch. Reports.*
- P. Dupuis and H. Wang. Dynamic importance sampling for uniformly recurrent Markov chains. To appear in *Ann. Appl. Probab.*