Fluid Limit of a Network with Fair Bandwidth Sharing and General Document Size Distributions

Stochastic Networks Conference

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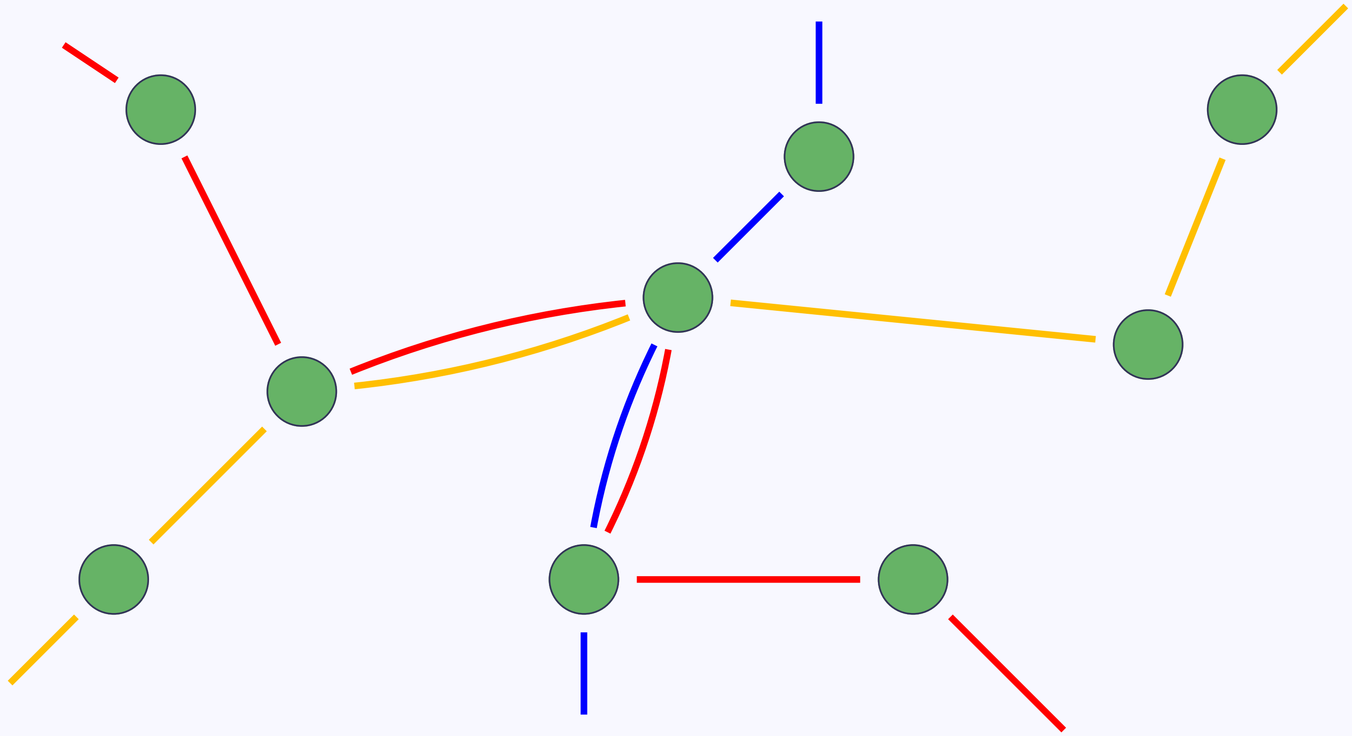
Joint work with Ruth Williams

Flow level network model (Roberts & Massoulié ’00)



Link resource or server

Route nonempty subset of links

Flow continuous transfer of document on a route

Simultaneous resource possession

Network structure

J links

I routes

$C_j > 0$ capacity of link j

$J \times I$ incidence matrix A

$$A_{ji} = \begin{cases} 1, & \text{if route } i \text{ uses link } j \\ 0, & \text{else} \end{cases}$$

Weighted α-fair bandwidth sharing (Mo & Walrand '00)

$\alpha \in (0, \infty)$, κ_i weight for route i

$\Lambda_i(Z)$ dynamic bandwidth allocation for route i

Z_i number of flows on route i

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$$\Lambda(Z) = \operatorname{argmax}\{G_Z(\Lambda) : A\Lambda \leq C, \Lambda \geq 0\}$$

where $$G_Z(\Lambda) = \begin{cases} \sum_i \kappa_i Z_i^{\alpha} \frac{{\Lambda_i}^{1-\alpha}}{1-\alpha}, & \text{for } \alpha \neq 1 \\ \sum_i \kappa_i Z_i \log(\Lambda_i), & \text{for } \alpha = 1 \end{cases}$$

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Bandwidth Λ_i / Z_i provided to each flow

Route i is processor sharing queue for fixed Z

Literature

Roberts & Massoulié '98

Mo & Walrand '00

de Veciana, Lee & Konstantopoulos '01

Bonald & Massoulié '01

Ye, Ou & Yuan '02

Kelly & Williams '03

Lakshmikantha & Srikant '04

Key, Massoulié, Bain & Kelly '03

Goals

Study stochastic model with *general* interarrival & document size distributions

- Stability
- Heavy traffic behavior

Important tools: *fluid models*

First step

- Propose fluid model
- Justify approximation via limit theorem

Strategies

Generalize measure valued process techniques for GI/GI/1 processor sharing queue

Grishechkin '94

Jean-Marie & Robert '94

G, Puha & Williams '02

Puha & Williams '03

G '03

Puha, Stolyar & Williams '03

G & Kruk '04

Outline

Stochastic model

Fluid model

Limit theorem

Intuition

Stochastic model

Network

- J links
- I routes
- $J \times I$ incidence matrix A
- $C \in \mathbb{R}_+^J$ link capacities

Stochastic model

Primitives

For each route i

- Renewal arrival process $E_i(t)$ with rate ν_i
  kth document arrives at time τ_{ik}
- i.i.d. document sizes $\{v_{ik}\}_{k=1}^{\infty}$ with distribution ϑ_i, mean μ_i^{-1}, cumulative process $V_i(n) = \sum_{k=1}^{n} v_{ik}$

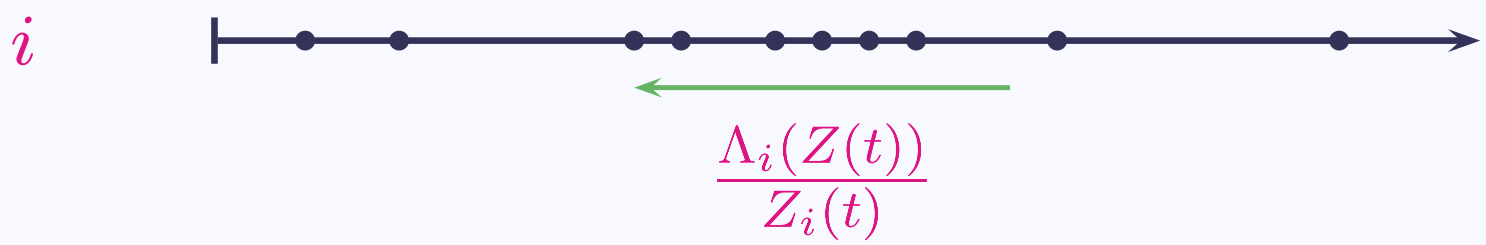
Stochastic model

Performance processes

$U_j(t)$ unused capacity for link j

For each route i

- $Z_i(t)$ queue length (number of documents)
- $W_i(t)$ workload
- $\mathcal{Z}_i(t)$ residual document size measure



Stochastic model

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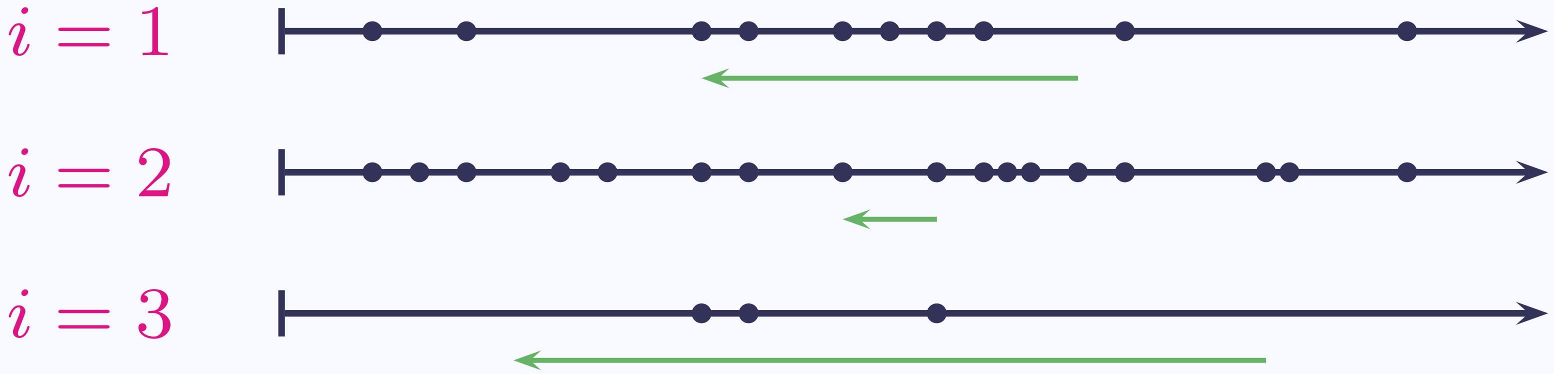

Stochastic model

Performance processes

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For each route i

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$\mathcal{Z}(\cdot)$ takes values in $\mathbf{D}([0,\infty),\mathbf{M}^I)$
$\mathbf{M}$ finite nonnegative Borel measures on $\mathbb{R}_+$

Stochastic model

Performance processes

$U_j(t)$ unused capacity for link j

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Weak topology on $\mathbf{M}$

$\zeta_n \xrightarrow{\mathrm{w}} \zeta$ iff $\langle g, \zeta_n \rangle \to \langle g, \zeta \rangle$ for all $g \in \mathbf{C}_b$

Stochastic model

Performance processes

$U_j(t)$ unused capacity for link j

For each route i

- $Z_i(t)$ queue length (number of documents)
- $W_i(t)$ workload
- $\mathcal{Z}_i(t)$ residual document size measure

$$Z_i(t) = \langle 1, \mathcal{Z}_i(t) \rangle \qquad W_i(t) = \langle x, \mathcal{Z}_i(t) \rangle$$

Stochastic model

Rich description

Other functionals. . .

$\langle 1_{[x,\infty)}, \mathcal{Z}(t) \rangle$

$\langle \log(x), \mathcal{Z}(t) \rangle$

Stochastic model

Dynamic equations

For the vector valued processes

$$W(t) = W(0) + V(E(t)) - T(t)$$
$$U(t) = Ct - AT(t)$$

where $T_i(t) = \int_0^t \Lambda_i(Z(u))du$

Stochastic model

Dynamic equations

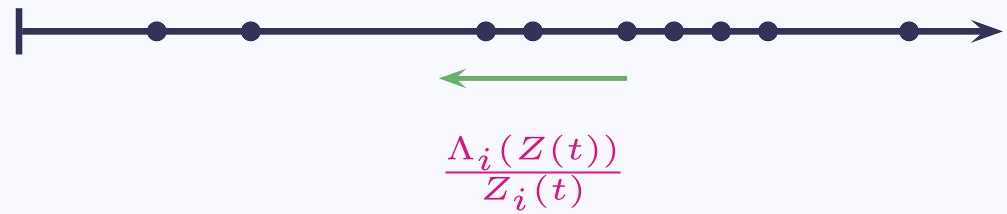
For the measure valued process

Projections $\langle g, \mathcal{Z}_i(\cdot) \rangle$ for all $g \in \mathcal{C}$

$$\mathcal{C} = \{g \in \mathbf{C}_b^1 : g(0) = g'(0) = 0\}$$

Stochastic model

Dynamic equations



Route i

$$\langle g, \mathcal{Z}_i(t) \rangle = \langle g(\cdot - S_i(t)), \mathcal{Z}_i(0) \rangle + \sum_{k=1}^{E_i(t)} g\left(v_{ik} - S_i(t) + S_i(\tau_{ik}) \right)$$

where $\quad S_i(t) = \int_0^t \frac{\Lambda_i(Z(u))}{Z_i(u)} du$

Next. . .

Stochastic model

Fluid model

Limit theorem

Intuition

Law of large numbers approximation

Stochastic model

$$\mathscr{X} = (\mathscr{Z}, Z, W, U)$$

↓ scaling

Fluid model

$$\chi = (\zeta, z, w, u)$$

Law of large numbers approximation

Stochastic model

$$\mathscr{X} = (\mathcal{Z}, Z, W, U)$$

stability?
state space collapse?

scaling

Fluid model

$$\chi = (\zeta, z, w, u)$$

Fluid model

Network (J, I, A, C) as before

Primitive parameters (ν, ϑ)

- $\nu > 0$
- ϑ probability measure on $\mathbb{R}_+$ with mean μ^{-1}

Performance functions

$\chi = (\zeta, z, w, u)$

Fluid model

A *fluid model solution* $\chi = (\zeta, z, w, u)$ is a continuous function $\chi : [0, \infty) \to \mathbf{M}^I \times \mathbb{R}_+^I \times \mathbb{R}_+^I \times \mathbb{R}_+^J$ such that

(i) $z_i = \langle 1, \zeta_i \rangle$ for all $i \leq I$

(ii) For each $i \leq I$

$$w_i(t) = w_i(0) + \int_0^t \left(\rho_i - \Lambda_i(z(u)\right) 1_{(0,\infty)}(z_i(u))du$$

(iii) For each $j \leq J$

$$u_j(t) = C_j t - \sum_i A_{ji} \int_0^t \Lambda_i(z(u))du$$

is nondecreasing

Fluid model

A *fluid model solution* $\chi = (\zeta, z, w, u)$ is a continuous function $\chi : [0, \infty) \to \mathbf{M}^I \times \mathbb{R}_+^I \times \mathbb{R}_+^I \times \mathbb{R}_+^J$ such that

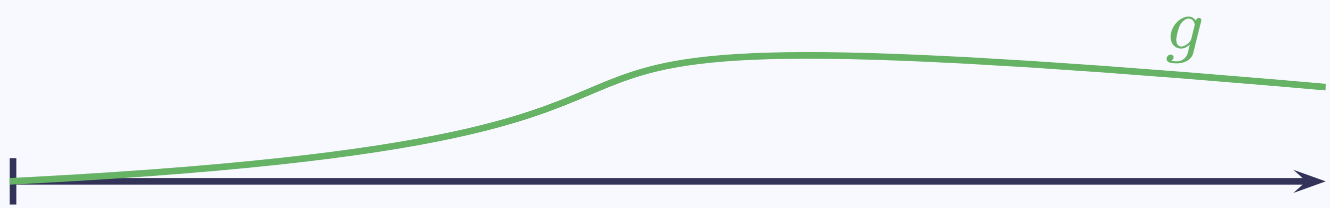
(iv) $\langle 1_{\{0\}}, \zeta_i \rangle = 0$, for all $i \le I$

(v) For each $g \in \mathcal{C}$, $i \le I$, $t \ge 0$

$$\langle g, \zeta_i(t) \rangle = \langle g, \zeta_i(0) \rangle - \int_0^t \langle g', \zeta_i(u) \rangle \frac{\Lambda_i(z(u))}{z_i(u)} du + \nu_i \langle g, \vartheta_i \rangle \int_0^t 1_{(0,\infty)}(z_i(u)) du$$

Fluid model

Dynamic equation for ζ



$$\langle g, \zeta_i(t) \rangle = \langle g, \zeta_i(0) \rangle - \int_0^t \langle g', \zeta_i(u) \rangle \frac{\Lambda_i(z(u))}{z_i(u)} du$$

$$+ \nu_i \langle g, \vartheta_i \rangle \int_0^t 1_{(0,\infty)}(z_i(u)) du$$

Next. . .

Stochastic model

Fluid model

Limit theorem

Intuition

Sequence of systems

$\mathscr{X}^r = (\mathscr{Z}^r, Z^r, W^r, U^r)$

Fluid scaling $\bar{\mathscr{X}}^r(t) = \frac{1}{r}\mathscr{X}^r(rt)$

e.g. $\bar{\mathscr{Z}}_i^r(t) = \frac{1}{r}\mathscr{Z}_i^r(rt)$

Assymptotic assumptions

Arrivals

$\nu^r \to \nu$

$\bar{E}^r(t) \to \nu t$ u.o.c.

Document sizes

$\vartheta^r \xrightarrow{\text{w}} \vartheta$

$\limsup \langle x^{1+\epsilon}, \vartheta^r \rangle < \infty$

$(\Rightarrow \mu^r \to \mu)$

Initial condition

$(\bar{\mathcal{Z}}^r(0), \bar{W}^r(0)) \xrightarrow{\text{w}} (\mathcal{Z}^0, \langle x, \mathcal{Z}^0 \rangle) \in \mathbf{M}^I \times \mathbb{R}_+^I$

$\langle 1_{\{x\}}, \mathcal{Z}^0 \rangle = 0$ for all $x \in \mathbb{R}_+$

Limit theorem

Theorem (G-Williams) The sequence $\{\bar{\mathscr{X}}^r\}$ is $\mathbf{C}$-tight; each weak limit point $\bar{\mathscr{X}} = (\bar{\mathcal{Z}}, \bar{Z}, \bar{W}, \bar{U})$ is a.s. a fluid model solution

$\Rightarrow$ processes $\bar{\mathscr{X}}$ approximate the dynamics of $\bar{\mathscr{X}}^r$

Measure valued approximation

- detailed description of residual document sizes
- facilitates proof
- extensible, e.g., add information tracking timing requirements or sojourn times (cf. G-Kruk ’04)

Other work on queueing systems using m.v.p.

Krichagina & Puhalskii ’97

Doytchinov, Lehoczky & Shreve ’01

Kruk, Lehoczky, Shreve & Yeung ’02 ’03

Limic ’00 ’01

Next. . .

Stochastic model

Fluid model

Limit theorem

Intuition

Overview

$\mathbf{C}$-tightness of $\{\bar{\mathscr{X}}^r\}$

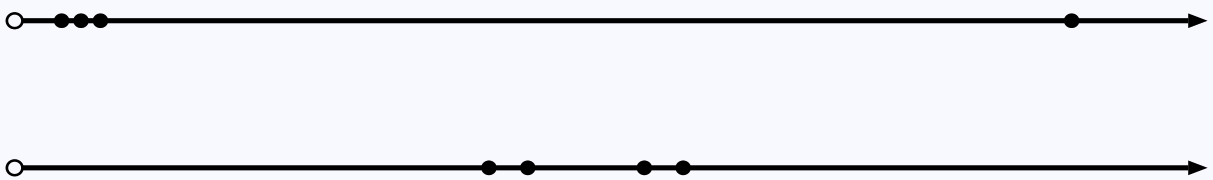
- compact containment
- oscillation control

Characterization of limit points

Controlling oscillations

For each $g \in \mathbf{C}_b$ and $\epsilon > 0$

$$\lim_{h\to 0} \liminf_{r\to\infty} \mathbf{P}\left(\left\| \langle g, \bar{\mathcal{Z}}^r(\cdot + h)\rangle - \langle g, \bar{\mathcal{Z}}^r(\cdot)\rangle \right\|_T \le \epsilon \right) = 1$$



For $i = 1$,2, $\langle 1, \bar{\mathcal{Z}}_i^r(t)\rangle = a$ and $\langle x, \bar{\mathcal{Z}}_i^r(t)\rangle = b$

Must rule out large drop in $\langle g, \bar{\mathcal{Z}}_1^r(\cdot)\rangle$ during $[t, t+h]$

Characterizing the limit ζ $\quad (\bar{\mathcal{Z}}_i^r \xrightarrow{\text{w}} \zeta_i$ u.o.c.)

Suppose $\zeta_i(u) > 0$ on $[s, t]$

Dynamic equation for $\bar{\mathcal{Z}}_i^r$

$$\langle g, \bar{\mathcal{Z}}_i^r(t)\rangle = \langle g(\cdot - \bar{S}_i^r(t) + \bar{S}_i^r(s)), \bar{\mathcal{Z}}_i^r(s)\rangle$$

$$+ \frac{1}{r} \sum_{k=r\bar{E}^r(s)+1}^{r\bar{E}_i^r(t)} g\left(v_{ik} - \bar{S}_i^r(t) + \bar{S}_i^r(r^{-1}\tau_{ik})\right)$$

Dynamic equation for ζ_i

$$\langle g, \zeta_i(t)\rangle = \langle g, \zeta_i(s)\rangle - \int_s^t \langle g', \zeta_i(u)\rangle \frac{\Lambda_i(z(u))}{z_i(u)} du$$

$$+ \nu_i \langle g, \vartheta_i\rangle \int_s^t 1_{(0,\infty)}(z_i(u)) du$$

Characterizing the limit ζ ($\bar{\mathcal{Z}}_i^r \xrightarrow{\text{w}} \zeta_i$ u.o.c.)

$t_n = n(t-s)/N$ and $t^n = t_{n+1}$

$$\langle g, \bar{\mathcal{Z}}_i^r(t)\rangle - \langle g, \bar{\mathcal{Z}}_i^r(s)\rangle = \sum_{n=0}^{N-1} \left(\langle g, \bar{\mathcal{Z}}_i^r(t^n)\rangle - \langle g, \bar{\mathcal{Z}}_i^r(t_n)\rangle\right)$$

Add and subtract

$$= \sum_{n=0}^{N-1} \Big(\langle g, \bar{\mathcal{Z}}_i^r(t^n)\rangle - \langle g(\cdot - \bar{S}_i^r(t^n) + \bar{S}_i^r(t_n)), \bar{\mathcal{Z}}_i^r(t_n)\rangle\Big)$$

$$+ \sum_{n=0}^{N-1} \Big(\langle g(\cdot - \bar{S}_i^r(t^n) + \bar{S}_i^r(t_n)), \bar{\mathcal{Z}}_i^r(t_n)\rangle - \langle g, \bar{\mathcal{Z}}_i^r(t_n)\rangle\Big)$$

Characterizing the limit ζ $\quad (\bar{\mathcal{Z}}_i^r \xrightarrow{\text{w}} \zeta_i$ u.o.c.)

$$= \sum_{n=0}^{N-1} \frac{1}{r} \sum_{k=r\bar{E}_i^r(t_n)+1}^{r\bar{E}_i^r(t^n)} g\left(v_{ik}^r - \bar{S}_i^r(t^n) + \bar{S}_i^r(r^{-1}\tau_{ik}^r)\right)$$

$$+ \sum_{n=0}^{N-1} \langle g(\cdot - \bar{S}_i^r(t^n) + \bar{S}_i^r(t_n)) - g(\cdot), \bar{\mathcal{Z}}_i^r(t_n)\rangle$$

Taylor's formula

$$\approx \sum_{n=0}^{N-1} \frac{1}{r} \sum_{k=r\bar{E}_i^r(t_n)+1}^{r\bar{E}_i^r(t^n)} g\left(v_{ik}^r\right)$$

$$+ \sum_{n=0}^{N-1} \left\langle -g'(\cdot)\Lambda_i(\bar{Z}^r(t_n))\bar{Z}_i^r(t_n)^{-1}N^{-1}, \bar{\mathcal{Z}}_i^r(t_n)\right\rangle$$

Characterizing the limit ζ $\quad (\bar{\mathcal{Z}}_i^r \xrightarrow{\text{w}} \zeta_i$ u.o.c.)

$$= \frac{1}{r} \sum_{k=\bar{E}_i^r(s)+1}^{r\bar{E}_i^r(t)} g\left(v_{ik}^r\right)$$

$$+ \sum_{n=0}^{N-1} \left\langle -g'(\cdot)\Lambda_i(\bar{Z}^r(t_n))\bar{Z}_i^r(t_n)^{-1}N^{-1}, \bar{\mathcal{Z}}_i^r(t_n) \right\rangle$$

Law of large numbers, bounded convergence

$$\to \nu_i\langle g, \vartheta_i\rangle(t-s) - \int_s^t \langle g', \zeta_i(u)\rangle \frac{\Lambda_i(z(u))}{z_i(u)} du$$

Characterizing the limit ζ $\quad (\bar{\mathcal{Z}}_i^r \xrightarrow{\text{w}} \zeta_i$ u.o.c.)

Can show absolute continuity of $\langle g, \zeta_i(\cdot)\rangle$

At a regular time t

$$\frac{d}{dt}\langle g, \zeta_i(t)\rangle = \begin{cases} -\langle g', \zeta_i(t)\rangle \frac{\Lambda_i(z(t))}{z_i(t)} + \nu_i \langle g, \vartheta_i\rangle, & \text{if } z_i(t) \neq 0 \\ 0, & \text{if } z_i(t) = 0 \end{cases}$$

Integrate to get

$$\langle g, \zeta_i(t)\rangle - \langle g, \zeta_i(0)\rangle = -\int_0^t \langle g', \zeta_i(u)\rangle \frac{\Lambda_i(z(u))}{z_i(u)} du + \nu_i \langle g, \vartheta_i\rangle \int_0^t 1_{(0,\infty)}(z_i(u)) du$$

Next steps

- Steady state behavior
- Stability
- State space collapse
- Diffusion approximation