Lagrangian time-discretization of the Korteweg-de Vries equation

Alexeï V. Penskoï*

CRM-2595

January 1999

*Centre de recherches mathématiques, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Québec, H3C 3J7, Canada. penskoi@crm.umontreal.ca

Abstract

A natural Lagrangian time-discretization of the Korteweg-de Vries equation with periodic boundary conditions is proposed. The corresponding discrete system is defined on the Virasoro group.

Résumé

Une discrétisation lagrangienne naturelle de l'équation de Korteweg-de Vries avec des conditions aux frontières périodiques est proposée. Le système discret correspondant est défini sur le groupe de Virasoro.

1 Introduction

This paper concerns investigations of discrete Lagrangian systems. Let M be a manifold, let L be a function on $M \times M$. A discrete Lagrangian system describes stationary points of a functional $S = S(X)$ defined on the space of sequences $X = (x_k), k \in \mathbb{Z}$ by a formal sum

$$S = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).$$

The function L is called a Lagrangian.

Such systems were treated in the papers of A. P. Veselov (see [5], [6], [7]), J. Moser and A. P. Veselov (see [2], [3]) and the author (see [4]). The works [2], [5], [6], [7] deals with finite-dimensional manifolds. The work [3] was the first attempt to investigate such discrete Lagrangian systems on an infinite-dimensional group $\mathrm{SDiff}(\mathbb{R}^2)$. The work [4] was the first attempt to investigate such systems on the Virasoro group.

The interest in discrete Lagrangian systems on the Virasoro group goes back to the observation of B. A. Khesin and V. Yu. Ovsienko [1] that the Korteweg-de Vries equation can be treated as an Euler equation on the Virasoro algebra. This observation suggested that it might be possible to find a “natural” Lagrange discretization of the Korteweg-de Vries equation as a discrete Lagrangian system on the Virasoro group. In this article we construct a discrete Lagrangian on the Virasoro group. Our Lagrangian is required to be symmetric and right-invariant. Such an additional natural requirement is very rigid and permits us to restrict the set of possible Lagrangians. A. P. Veselov [5] seems to be the first who realized that such a requirement often implies integrability.

The proposed Lagrangian is “natural” enough and has the KdV equation as a continious limit. Nevertheless the question of integrability of the obtained discrete Lagrangian systems is still open. Only two integrals are found.

2 The Virasoro group

Let $\mathrm{Diff}_+(S^1)$ be the group of diffeomorphisms of S^1 preserving the orientation. We shall represent an element of $\mathrm{Diff}_+(S^1)$ as a function $f : \mathbb{R} \to \mathbb{R}$ such that

1. $f \in C^\infty(\mathbb{R})$,
2. $f'(x) > 0$,
3. $f(x + 2\pi) = f(x) + 2\pi$.

Such a representation is not unique. Indeed, the functions f and $f + 2\pi$ represent one element of $\mathrm{Diff}_+(S^1)$.

There exists a nontrivial central extension of $\mathrm{Diff}_+(S^1)$ which is unique up to an isomorphism

This extention is called the Virasoro group and is denoted by Vir. Elements of Vir are pairs (F, f), where $F \in \mathbb{R}$, $f \in \mathrm{Diff}_+(S^1)$. The product of two elements is defined as

$$(F, f) \circ (G, g) = (F + G + \int_0^{2\pi} \log(f \circ g)' \, d \log g', f \circ g).$$

The unit element of Vir is $(0, id)$. The inverse element of (F, f) is $(-F, f^{-1})$.

3 Discrete Lagrangian systems

Let M be a manifold, let L be a function on $M \times M$. A discrete Lagrangian system describes stationary points of a functional $S = S(X)$ defined on the space of sequences $X = (x_k), k \in \mathbb{Z}$ by a formal sum

$$S(X) = \sum_{k \in \mathbb{Z}} L(x_k, x_{k+1}).$$

The function L is called the Lagrangian.

Let us assume that M is a Lie group G. In this case it is reasonable to require that the function L has the following properties:

1. $L(x, y) = L(y, x)$ (symmetry),

2. $\forall g \in G \quad L(xg, yg) = L(x, y)$ (right-invariance).

Let $H(x) = L(x, e)$, where e is the unit of G. Hence $H(x^{-1}) = H(x)$; indeed,

$$H(x^{-1}) = L(x^{-1}, e) = L(e, x^{-1}) = L(e \cdot x, x^{-1} \cdot x) = L(x, e) = H(x).$$

We say that H is inverse-invariant if $H(x) = H(x^{-1})$.

Equivalently, suppose $H : G \to \mathbb{R}$ is an inverse-invariant function. Let

$$L(x, y) = H(x \cdot y^{-1}). \tag{1}$$

Thus this function $L : G \times G \to \mathbb{R}$ is symmetric and right-invariant.

This yields that the construction of right-invariant symmetric Lagrangians is equivalent to the construction of inverse-invariant functions on G.

4 Construction of Lagrangian on the Virasoro group

Consider a function $H : Vir \to \mathbb{R}$ such that

$$H((F, f)) = F^2 + \int_0^{2\pi} (\cos(f(x) - x) - 1)\sqrt{f'(x)}\, dx.$$

The function H is well-defined. Indeed, let $(F, f + 2\pi k), k \in \mathbb{Z}$ be another representative of the same element of $\mathrm{Diff}_+(S^1)$. We have

$$H((F, f + 2\pi k)) = F^2 + \int_0^{2\pi} (\cos(f(x) + 2\pi k - x) - 1)\sqrt{(f(x) + 2\pi k)'}\, dx =$$

$$= F^2 + \int_0^{2\pi} (\cos(f(x) - x) - 1)\sqrt{f'(x)}\, dx = H((F, f)).$$

Lemma 1. The function H is an inverse-invariant function on Vir.
Proof. We have

$$H((-F, f^{-1})) = F^2 + \int_0^{2\pi} (\cos(f^{-1}(x) - x) - 1)\sqrt{(f^{-1}(x))'}\, dx.$$

The map $f:[f^{-1}(0),f^{-1}(2\pi)]\to[0,2\pi]$ is a diffeomorphism. Hence we can do a change of variables $x=f(y)$. This implies that

$$H((-F,f^{-1}))=F^2+\int\limits_{f^{-1}(0)}^{f^{-1}(2\pi)}(\cos(f^{-1}(f(y))-f(y))-1)\sqrt{(f^{-1})'(f(y))}f'(y)\,dy=$$

$$=F^2+\int\limits_{f^{-1}(0)}^{f^{-1}(2\pi)}(\cos(y-f(y))-1)\sqrt{\frac{1}{f'(y)}}f'(y)\,dy=$$

$$=F^2+\int\limits_{f^{-1}(0)}^{f^{-1}(2\pi)}(\cos(f(y)-y)-1)\sqrt{f'(y)}\,dy.$$

By the above we have $f^{-1}(2\pi)=f^{-1}(0)+2\pi$. Since the function under the integral is periodic, it follows that $H((-F,f^{-1}))=H((F,f))$. This completes the proof. □

By (1) H defines a symmetric and right-invariant Lagrangian L :

$$L((F,f),(G,g))=H((f,f)\circ(G,g)^{-1})=(F-G+\int\limits_0^{2\pi}\log(f\circ g^{-1})'\,d\log(g^{-1})')^2+$$

$$+\int\limits_0^{2\pi}(\cos(f\circ g^{-1}(x)-x)-1)\sqrt{(f\circ g^{-1})'(x)}\,dx.$$

5 The equation of motion

Let $(F_k,f_k),k\in\mathbb{Z}$ be a sequence of elements of $\mathrm{Diff}_+(S^1)$, let

$$S=\sum_{k\in\mathbb{Z}}L((F_k,f_k),(F_{k+1},f_{k+1})).$$

Let $(\Omega_k,\omega_k)=(F_{k-1},f_{k-1})\circ(F_k,f_k)^{-1}$; this is a discrete analog of an angular velocity.
THEOREM 1. The "discrete Euler-Lagrange equations" $\delta S=0$ are

$$\Omega_k=\Omega_{k+1},\tag{2}$$

and

$$\begin{aligned}&-4\Omega_k(\log\omega_k'(x))''+\sin(\omega_k(x)-x)\omega_k'(x)\sqrt{\omega_k'(x)}+\\&+\sin(\omega_k(x)-x)\sqrt{\omega_k'(x)}-\tfrac{1}{2}\cos(\omega_k(x)-x)\tfrac{\omega_k(x)''}{\sqrt{\omega_k'(x)}}+\tfrac{1}{2}\tfrac{\omega_k''(x)}{\sqrt{\omega_k'(x)}}=\\&=-4\Omega_{k+1}(\log(\omega_{k+1}^{-1}(x))')''-\sin(\omega_{k+1}^{-1}(x)-x)(\omega_{k+1}^{-1})'(x)\sqrt{(\omega_{k+1}^{-1})'(x)}-\\&-\sin(\omega_{k+1}^{-1}(x)-x)\sqrt{(\omega_{k+1}^{-1})'(x)}+\tfrac{1}{2}\cos(\omega_{k+1}^{-1}(x)-x)\tfrac{(\omega_{k+1}^{-1}(x))''}{\sqrt{(\omega_{k+1}^{-1})'(x)}}-\tfrac{1}{2}\tfrac{(\omega_{k+1}^{-1})''(x)}{\sqrt{(\omega_{k+1}^{-1})'(x)}}\end{aligned}\tag{3}$$

PROOF. The proof is by direct calculation. □

6 Integrals of motion

The question of the integrability of this discrete Lagrangian system is still open. From (2) we have one trivial integral of motion:

$$\Omega_k = \Omega_{k+1}.$$

Now we can find only one non-trivial integral for this system.
THEOREM 2. The function

$$I_k = \int_0^{2\pi} \sin(\omega_k(x) - x)\sqrt{\omega_k'(x)}dx$$

is an integral of motion, i.e. $I_k = I_{k+1}$.
PROOF. Integrating the equation (3) with respect to x, we get

$$\int_0^{2\pi} (\sin(\omega_k(x) - x)\omega_k'(x)\sqrt{\omega_k'(x)} + \sin(\omega_k(x) - x)\sqrt{\omega_k'(x)} -$$

$$-\frac{1}{2}\cos(\omega_k(x) - x)\frac{\omega_k(x)''}{\sqrt{\omega_k'(x)}} + \frac{1}{2}\frac{\omega_k''(x)}{\sqrt{\omega_k'(x)}})\, dx =$$

$$= \int_0^{2\pi} (-\sin(\omega_{k+1}^{-1}(x) - x)(\omega_{k+1}^{-1})'(x)\sqrt{(\omega_{k+1}^{-1})'(x)} -$$

$$-\sin(\omega_{k+1}^{-1}(x) - x)\sqrt{(\omega_{k+1}^{-1})'(x)} -$$

$$-\frac{1}{2}\cos(\omega_{k+1}^{-1}(x) - x)\frac{(\omega_{k+1}^{-1}(x))''}{\sqrt{(\omega_{k+1}^{-1})'(x)}} + \frac{1}{2}\frac{(\omega_{k+1}^{-1})''(x)}{\sqrt{(\omega_{k+1}^{-1})'(x)}})\, dx.$$

Let us remark that

$$0 = \int_0^{2\pi} \left(\frac{(\cos(\omega_k(x) - x) - 1)\omega_k'(x)}{\sqrt{\omega_k'(x)}} \right)' =$$

$$-\sin(\omega_k(x) - x)\omega_k'(x)\sqrt{\omega_k'(x)} + \frac{(\cos(\omega_k(x) - x) - 1)\omega_k''(x)}{2\sqrt{\omega_k'(x)}}.$$

Using this formula and analogous one for ω_{k+1}^{-1}, we get

$$\int_0^{2\pi} \sin(\omega_k(x) - x)\sqrt{\omega_k'(x)}dx = -\int_0^{2\pi} \sin((\omega_{k+1}^{-1})(x) - x)\sqrt{(\omega_{k+1}^{-1})'(x)}dx.$$

Using a change of variables $x = \omega_{k+1}(y)$ in the second integral, we get

$$\int_0^{2\pi} \sin(\omega_k(x) - x)\sqrt{\omega_k'(x)}dx = \int_0^{2\pi} \sin((\omega_{k+1})(x) - x)\sqrt{(\omega_{k+1})'(x)}dx.$$

This completes the proof. □

7 Continuous limit

The most important feature of our discrete Lagrangian system is the fact that a continuous limit of this system is the Korteweg-de Vries equation.

Let $\Omega_k = \epsilon\Omega(x,t)$, $\Omega_{k+1} = \epsilon\Omega(x,t+\epsilon)$. From (2), we get

$$\frac{d}{dt}\Omega(x,t) = 0,$$

hence, $\Omega = const$.

Let $\omega_k = id + \epsilon\xi(x,t)$, $\omega_{k+1} = id + \epsilon\xi(x,t+\epsilon)$. From (3) we get

$$-4\Omega\epsilon^2\Omega\xi'''(x,t) + 2\epsilon\xi(x,t) + 2\epsilon^2\xi(x,t)\xi'(x,t) + o(\epsilon^2) =$$

$$4\epsilon^2\Omega\xi'''(x,t+\epsilon) + 2\epsilon\xi(x,t+\epsilon) - 4\epsilon^2\xi(x,t+\epsilon)\xi'(x,t+\epsilon) + o(\epsilon^2).$$

Hence

$$2\frac{d}{dt}\xi(x,t) + 8\Omega\xi'''(x,t) - 6\xi(x,t)\xi'(x,t) = 0.$$

A simple change of variables converts this equation into the standard KdV equation form.

8 Conclusions

We have constructed a discrete equation (3) with the following properties:

1. Its continuous limit is the Korteweg-de Vries equation.
2. It is a Lagrangian equation with a Lagrangian that is symmetric and right invariant under the Virasoro group.
3. It allows (at least) two integrals of motion.

Acknowledgements

I am indebted to Professor Alexander P. Veselov for suggesting this problem. This work was completed while the author was a student at the Moscow State University. This work was supported by grant INTAS 96-0770.

References

1. Ovsienko, V. Yu., Khesin, B. A.: The super Korteweg-de Vries equation as an Euler equation. *Funkts. Anal. Pril.* **21** (1987), 81-82 (Russian).

2. Moser, J., Veselov, A. P.: Discrete versions of some classical integrable systems and factorization of matrix polynomials. *Comm. Math. Phys.* **139** (1991), 217-243.

3. Moser, J., Veselov, A. P.: *Two-dimensional "discrete hydrodynamics" and Monge-Ampère equation.* Preprint of ETH, Zürich, 1993.

4. Penskoï, A. V.: Discrete Lagrangian systems on the Virasoro group. *Vestnik. Mos. Univ. Ser. I Math. Mekh.* (1996), no 4, 99-102 (Russian).

5. Veselov, A. P.: Integrable Lagrangian relations and factorization of matrix polynomials. *Funkts. Anal. Pril.* **25** (1991), 38-49 (Russian). Translation in *Funct. Anal. Appl.* **25** (1991), 112-122.

6. Veselov, A.P.: Integrable mappings. *Uspekhi Mat. Nauk* **46** (1991), 3-45 (Russian). Translation in *Russian Math. Surveys* **46** (1991), 1-51.

7. Veselov, A. P.: Integrable systems with discrete time and difference operators. *Funkts. Anal. Pril.* **22** (1988), 1-13 (Russian). Translation in *Funct. Anal. Appl.* **22** (1988), 83-93.