UNIVERSAL NEAR-OPTIMAL FEEDBACKS

F. H. Clarke
Institut Desargues
Université de Lyon I (bât. 101)
69622 Villeurbanne, France

S. Nobakhtian
Department of Mathematics and Statistics
McGill University
Montreal, Quebec H3A 2K6, Canada

R. J. Stern*
Department of Mathematics and Statistics
Concordia University
Montreal, Quebec H4B 1R6, Canada

April 15, 1999

Abstract. For a general fixed duration optimal control problem, the proximal aiming technique of nonsmooth analysis is employed in order to construct a discontinuous feedback law, all of whose Euler solutions are optimal to within a prescribed tolerance, universally for all intitial data in a prescribed bounded set. The technique is adapted in order to construct universal near-saddle points for two-player fixed duration differential games of Krasovskii-Subbotin type.

1. Introduction

Consider a control system

$$\dot{x} = f(t, x, u), \tag{1}$$

where $f : \Re \times \Re^n \times \Re^m \to \Re^n$ is continuous and locally Lipschitz in the state variable x. Controls are Lebesgue measurable functions $u : \Re \to U$, where the control restraint set $U \subseteq \Re^m$ is compact. Assume also that f satisfies a growth condition. Then for each control function $u(\cdot)$ and each initial phase (i.e. data pair) $(\tau, \alpha) \in \Re \times \Re^n$ there is a unique solution $x(t) = x(t; \tau, \alpha, u(\cdot))$ on $[\tau, \infty)$. A central issue in control theory is the existence of feedback control laws which achieve desired behavior of the control system (1) *universally*; that is, for all initial states or initial phases in a prescribed set.

In fixed duration endpoint cost optimal control problems, which will be the main focus of our attention in the present work (with differential games being viewed as an

Key words: Optimal control, differential games, Euler polygonal arc, Euler solution, nonsmooth analysis, proximal aiming, infinitesimal decrease, inwardness, discontinuous universal near-optimal feedback.

Mathematical Subject Classification: 49J24, 49J52, 49N55, 90D25

*Research supported by the Natural Sciences Engineering Research Council of Canada and Le Fonds pour la Formation de Chercheurs et l'Aide à la Recherche du Québec.

application of the techniques to be developed), one seeks to design a feedback law $k : (-\infty, T] \times \Re^n \to U$ such that for any initial phase (τ, α) in some specified subset of $(-\infty, T] \times \Re^n$, all trajectories of

$$\dot{x} = f(t, x, k(t, x)) \tag{2}$$

satisfying $x(\tau) = \alpha$ minimize a cost $\ell(x(T))$, where $\ell : \Re^n \to \Re$ is assumed to be continuous. We emphasize that the feedback k is *nonautonomous*; that is, it depends upon the current time t as well as the position $x(t)$ (as it must in a fixed-time problem).

Even in very simple cases, one cannot expect the existence of universal feedback laws k which are continuous, this being the minimal condition for the classical existence theory of ordinary differential equations to apply to (2). This inadequacy of continuous feedback can be illustrated via the following example due to Clarke, Ledyaev and Subbotin [10].

Example 1.1. Consider the optimal control problem with dynamics

$$\begin{aligned} \dot{x} = u, \quad & t \in [0, 1], \\ & x(0) = \alpha, \end{aligned}$$

where $x \in \Re$, $u(\cdot)$ is valued in $U = [-1, 1]$, and where it is desired to minimize the endpoint cost functional $-|x(1)|$. First consider any continuous feedback $k(t, x)$ which is Lipschitz in x and such that solutions of (2) are defined on $[0, 1]$ for every $\alpha \in [-1, 1]$. Then there is a unique solution $x(t) = x(t; \alpha)$ of $\dot{x} = k(t, x)$ on $[0, 1]$ for each such α. Now note that the function $Q : [-1, 1] \to [-1, 1]$ given by

$$Q(\alpha) := -\int_0^1 k(t, x(t; \alpha))dt$$

is continuous, and therefore has a fixed point $\hat{\alpha}$ by virtue of Brouwer's fixed point theorem. Furthermore, $x(1; \hat{\alpha}) = 0$. Now consider the discontinuous feedback law

$$k(t, x) := \begin{cases} \frac{x}{|x|} & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

Upon substituting into the dynamics, we see that for any $\alpha \in \overline{B}_2$, the resulting trajectory satsifies $-\|x(1)\| \leq -1$. This argument can be extended, by approximation, to the case of merely continuous k. Therefore, given any continuous feedback k, there is at least one initial phase $(0, \hat{\alpha})$ with $\hat{\alpha} \in \overline{B}_2$, such that the above discontinuous feedback produces a better outcome than the continuous one. Hence, there does not exist a continuous feedback which is optimal universally for initial phases in the set $\{0\} \times \overline{B}_2$. Of course, it is not always possible to obtain a classical solution to the differential equation

$$\dot{x} = f(t, x, k(t, x)) =: g(t, x)$$

when the feedback k is discontinuous, since the existence theory of ordinary differential equations can break down, but in the present example, this difficulty does not occur.

A reference which is relevant to the present work, is Clarke, Ledyaev, Sontag and Subbotin [7], which dealt with the construction of (discontinuous) stabilizing feedback and a *discretized* solution concept, wherin controls are iteratively reset and held constant on successive time intervals. The discretized solution concept utilized in this article, that of "Euler polygonal arcs" is somewhat akin to this. The departure point of the analysis in [7] is the key fact, due to Sontag [17] (see also and Sontag and Sussmann [18]) that asymptotic controllability (which is assumed) is equivalent to the existence of a continuous (but nonsmooth) control Lyapunov function, or CLF. The methods of nonsmooth analysis were then brought to bear; the stabilizing feedback is constructed by using the sublevel sets of the Moreau-Yosida infimal convolution of the CLF and exploiting a nonsmooth infinitesimal decrease property of this function. The methods of [7] were adapted to differential games of pursuit by Clarke, Ledyaev, and Subbotin [11]. In that work, instead of the sublevel sets of a CLF, the sublevel sets of the value function of the problem were utilized. (See also Remark 3.5 below for further discussion of the connections between [11] and other references with the present work.)

A major difference between the methods of [7] (or [11]) and the present work is that here the Moreau-Yosida infimal convolutions is not required, and the *proximal aiming* method introduced in Clarke, Ledyaev, Stern and Wolenski [8], [9] can be applied in a direct way. Proximal aiming is a geometric version of the "extremal aiming" method of Krasovskii and Subbotin [13] in differential game theory. One important feature that [7], [11] and the present work have in common is that feedbacks are constructed by utilizing a nonsmooth "infinitesimal decrease" property of either a CLF (in stabilizability problems) or a value function (in optimal control problems), with this property being expressed via a generalized Hamilton-Jacobi inequality.

Berkowitz [2] provided a method of (universal) feedback construction for optimal control, quite different from those mentioned above, but one which also relies upon a nonsmooth Hamilton-Jacobi approach. In the context of the present article, Berkowitz's approach can be described as follows. Since the value function $V = V(t,x)$ of the problem is known to satisfy the generalized Hamilton-Jacobi inequality

$$\min_{v \in f(t,x,U)} DV(t,x;1,v) = 0, \quad (t,x) \in (-\infty, T) \times \Re^n, \tag{3}$$

where $DV(t,x;1,v)$ denotes the lower Dini derivate of V at (t,x) in the direction $(1,v)$ (see (31) below), one approach (which is known to work when V is smooth) is to consider a set-valued "feedback map" $U(t,x)$ such that

$$f(t,x,U(t,x)) = \text{argmin}_{v \in f(t,x,U)} DV(t,x;1,v). \tag{4}$$

One is then led to consider the differential inclusion

$$\dot{x} \in f(t,x,U(t,x)). \tag{5}$$

It transpires that under the present hypotheses, any solution of this differential inclusion corresponds to an optimal trajectory of the optimal control problem. On the other hand, as is noted in [2], the multifunction $f(t,x,U(t,x))$ on the right-hand-side of (5) in general lacks sufficient regularity (most notably, convexity and upper semicontinuity) for existence of solutions to hold in general, or, for that matter, for discretized solution procedures to be applicable. A reference related to Berkowitz [2] is Rowland and Vinter [16]. There a modification of Berkowitz's method is given which overcomes this lack of regularity without imposing extra conditions. Rowland and Vinter provided a discretization procedure (but not a feedback law) which in the limit produces an optimal trajectory for any intitial phase. Another related approach to feedback construction was undertaken by Cannarsa and Frankowska in [3]; in that work, additional conditions on the cost functional and dynamics were given which provide the requisite regularity in Berkowitz's original procedure.

The plan of this article is as follows. In the next section, we provide the required preliminaries in nonsmooth analysis, differential inclusions and Euler solutions. Then in §3 we present a proximal aiming based method for constructing near-optimal feedbacks, universal with respect to all initial phases in a specified generalized rectangle of (t, x)-space, in fixed duration endpoint cost optimal control problems. The dynamics considered in §3 are taken to be in the form of a differential inclusion

$$\dot{x} = F(t, x), \tag{6}$$

which can serve to model a classical control system of the form (1) upon identifying $F(t, x) := f(t, x, U)$; the trajectory equivalence of (6) and (1) is well understood. It appears that the weakening of "optimal" to "near-optimal" (to arbitrary tolerance) in our constructions is the price to be paid for universality. Then in §4 our proximal aiming method is extended to differential games of Krasovskii-Subbotin type, where the goal is the construction of a universal ε-saddle point. Finally, §5 consists of concluding remarks concerning the relaxation of hypotheses on the dynamics in the results of §3.

2. Preliminaries

2.1. Nonsmooth analysis background. A general reference for this section is Clarke, Ledyaev, Stern and Wolenski [9]; see also [8], Clarke [5], [6] and Loewen [15]. First some basic notation is provided: The Euclidean norm is denoted $\|\cdot\|$, and $\langle\, , \rangle$ is the usual inner product. The open unit ball in $\Re^n$ is denoted B_n. For a set $Z \subseteq \Re^n$, we denote by $\mathrm{co}(Z)$, $\overline{Z}$, $\mathrm{bdry}(Z)$ and $\mathrm{int}(Z)$ the convex hull, closure, boundary and interior of Z, respectively. We denote the cone generated by Z as $\mathrm{cone}(Z)$; that is,

$$\mathrm{cone}(Z) := \{\alpha z : \alpha \geq 0, z \in Z\}.$$

Let S be a nonempty subset of $\Re^n$. The distance of a point u to S is given by

$$d_S(u) := \inf\{\|u - x\| : x \in S\}.$$

The *metric projection* of u on S is denoted

$$\mathrm{proj}_S(u) := \{x \in S : \|u - x\| = d_S(u)\}.$$

If $u \notin S$ and $x \in \mathrm{proj}_S(u)$, then the vector $u - x$ is called a *perpendicular* to S at x. The cone consisting of all nonnegative multiples of these perpendiculars is denoted $N_S^P(x)$, and is referred to as the *proximal normal cone* (or P-normal cone) to S at x. If $x \in \mathrm{int}(S)$ or no perpendiculars to S exist at x, then we set $N_S^P(x) = \{0\}$. Observe that the P-normal cone is a local construct, since as can readily be shown,

$$N_S^P(x) = N_{S\cap\{x+\delta B_n\}}^P(x) \quad \forall\, \delta \geq 0.$$

Let $f : U \to \Re$ be continuous, where $U \subseteq \Re^n$ is open. Denote the epigraph of f by

$$\text{epi}(f) := \{(x, y) \in U \times \Re : x \in U,\ y \geq f(x)\}.$$

A vector $\zeta \in \Re^n$ is said to be a *proximal subgradient* (or P-subgradient) of f at $x \in U$ provided that

$$(\zeta, -1) \in N_{\text{epi}(f)}^P(x, f(x)).$$

The set of all such vectors is called the P-*subdifferential* of f at x, denoted $\partial_P f(x)$. One can show that $\zeta \in \partial_P f(x)$ iff there exists $\sigma > 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x\rangle$$

for all $y \in U$ near x, and that $\partial_P f(x) \neq \phi$ for a dense set of $x \in U$.

The *limiting normal cone* (or L-normal cone) to S at $x \in S$ is defined to be the set

$$N_S^L(x) := \{\zeta : \zeta_i \to \zeta,\ \zeta_i \in N_S^P(x_i),\ x_i \to x\}.$$

In particular, $N_S^P(x) \subseteq N_S^L(x)$ and $N_S^L(x) = \{0\}$ if $x \in \text{int}(S)$. One can show that the multifunction $x \to N_S^L(x)$ is closed on S. Also, if $S \cap \{x + r\overline{B}_n\}$ is nonempty and closed and $x \in \text{bdry}(S)$, then $N_S^L(x) \neq \{0\}$. The L-normal cone leads to a corresponding *limiting subdifferential* (or L-subdifferential) set for f:

$$\partial_L f(x) := \{\zeta : (\zeta, -1) \in N_{\text{epi}(f)}^L(x, f(x)\},$$

the members of which are called *limiting subgradients* (or L-subgradients). One has

$$\partial_L f(x) = \{\zeta : \zeta_i \to \zeta,\ \zeta_i \in \partial_P f(x_i),\ x_i \to x,\ f(x_i) \to f(x)\}.$$

We now summarize some required facts from nonsmooth calculus:

(a) *Sum rule:* Suppose that g is C^2 near a point $x \in U$. Then

$$\partial_P(g + f)(x) \subseteq g'(x) + \partial_P f(x), \tag{7}$$

where g' denotes the Fréchet derivative.

(b) *Local Lipschitzness:* Assume U to be convex as well as open. Then f is Lipschitz of rank K on U iff

$$\|\zeta\| \leq K \quad \forall\, \zeta \in \partial_P f(x), \quad \forall\, x \in U.$$

Also, $\partial_L f(x) \neq \phi$ if f is Lipschitz near x, and we have f is Lipschitz of rank K on U iff

$$\|\zeta\| \leq K \quad \forall\, \zeta \in \partial_L f(x), \quad \forall\, x \in U.$$

(c) *L-normals to sublevel sets:* Let f be Lipschitz on U (open and convex) and let $a \in \Re$. Denote

$$S(a) := \{x \in U : f(x) \leq a\}.$$

Suppose that $\bar{x} \in U$ is such that $f(\bar{x}) = a$, and assume that

$$0 \notin \partial_L f(\bar{x}). \tag{8}$$

Then

$$N^L_{S(a)}(\bar{x}) \subseteq \text{cone}[\partial_L f(\bar{x})]. \tag{9}$$

2.2. Differential inclusions and Euler solutions. Consider the differential inclusion (or generalized control system)

$$\dot{x} \in F(t, x), \tag{10}$$

where by a *solution* or *trajectory* of (10) on an interval J we mean an absolutely continuous function $t \rightarrow x(t) \in \Re^n$ satisfying (10) a.e. on J. Here F is a multifunction which is assumed to satisfy the following standing hypotheses:

(SH) (a) For each point $(t, x) \in \Re \times \Re^n$, $F(t, x)$ is a nonempty compact convex subset of $\Re^n$.

(b) *Linear growth:* There exist $\gamma_1 > 0$ and γ_2 such that

$$\|v\| \leq \gamma_1 \|x\| + \gamma_2 \quad \forall\, v \in F(t, x), \quad \forall\, (t, x) \in \Re \times \Re^n.$$

(c) F is *locally Lipschitz* on $\Re \times \Re^n$; that is, to every bounded set $S \subseteq \Re \times \Re^n$ there corresponds $K > 0$ such that

$$F(t_1, x_1) \subset F(t_2, x_2) + K\|(t_1, x_1) - (t_2, x_2)\| B \quad \forall\, (t_i, x_i) \in S, \quad i = 1, 2.$$

Remark 2.1. The main results of this article can be extended to the case where the sets $F(t, x)$ are not assumed to be convex or closed, but only bounded. Furthermore, the Lipschitz hypothesis on F can be relaxed to upper semicontinuity. This is taken up in §5; until then, however, (SH) will be assumed to hold.

(SH) suffices for global existence of solutions of (10); that is, for any initial data $(\tau, \alpha) \in \Re \times \Re^n$, there exists a solution of (10) on $[\tau, \infty)$. Furthermore, we require the following well known fact, commonly referred to as "compactness of trajectories": Given a sequence of solutions $\{x_k(\cdot)\}_{k=1}^{\infty}$ of (10) on a compact interval $[\tau, T]$, each satisfying $x_k(\tau) = \alpha$, there exists a subsequence converging uniformly to a solution $x(\cdot)$ of (10) on that interval, also satisfying $x(\tau) = \alpha$. If convexity of the velocity sets $F(t, x)$ is omitted from the hypothesis, then a limit arc $x(\cdot)$ still exists, which is a solution of the differential inclusion $\dot{x} = \text{co}[F(t, x)]$ on $[\tau, T]$. (In addition to [9], see e.g. Aubin and Cellina [1], Deimling [12] or Castaing and Valadier [4] for this fact as well as the general existence theory of differential inclusions.)

A *feedback* f for F is simply a selection of F; that is, for every $(t, x) \in \Re \times \Re^n$ one has $f(t, x) \in F(t, x)$. In the classical control system case, where the dynamics are of the form $\dot{x} = f(t, x, u)$, upon defining $F(t, x) := f(t, x, U)$, such a feedback corresponds to a selection $k(t, x) \in U$, where U is the control restraint set.

The generalized solution concept with which we will be working will now be described. Given $\alpha \in \Re^n$ and a compact time interval $[\tau, T]$, then should it exist, an *Euler solution* of the initial value problem

$$\dot{x} = f(t, x), \quad x(\tau) = \alpha \tag{11}$$

is the uniform limit of piecewise linear functions as follows: Given a partition π of $[\tau, T]$,

$$\tau =: t_0 < t_1 < \ldots < t_{N_\pi} := T,$$

the *Euler polygonal arc* $x_\pi(\cdot)$ is generated on successive subintervals via the recursive formula

$$x_\pi(t) = x_\pi(t_i) + (t - t_i) f(t_i, x_\pi(t_i)), \quad t \in [t_i, t_{i+1}], \; i = 0, 1, \ldots, N_\pi - 1,$$

where $x_\pi(\tau) = \alpha$. The instants t_i are referred to as the *meshpoints* of the partition, and for convenience we denote $x_\pi(t_i) = x_i$. The *nodes* of the scheme are the points (t_i, x_i), $i = 0, 1, \ldots, N_\pi - 1$. An Euler solution on $[\tau, t]$ is then defined to be the uniform limit of a sequence $x_{\pi_j}(\cdot)$ such that $\mathrm{diam}(\pi_j) \to 0$; here

$$\mathrm{diam}(\pi_j) := \max\{|t_i - t_{i-1}| : 1 \le i \le N_{\pi_j}\}$$

is referred to as the *mesh diameter* of the partition π. Under the standing hypotheses (SH), if f is a feedback for F, then at least one Euler solution on $[\tau, T]$ exists, and any such Euler solution is necessarily also a trajectory for the underlying differential inclusion (10) on $[\tau, T]$; see [9].

3. Universal feedback construction in optimal control

Suppose that F satifies the standing hypotheses (SH) and that $\ell : \Re^n \to \Re$ is a continuous function. Given $T \in \Re$, we shall consider the following parametrized family of optimal control problems $\{P(\tau, \alpha)\}$, where $(\tau, \alpha) \in (-\infty, T] \times \Re^n$:

minimize $\ell(x(T))$ over all trajectories x of the differential inclusion $\dot{x}(t) \in F(t, x(t))$ satisfying $x(\tau) = \alpha$.

In view of compactness of trajectories, the minimum in each problem $P(\tau, \alpha)$ is attained; we denote this minimum by $V(\tau, \alpha)$, and call the function $V : (-\infty, T] \times \Re^n \to \Re$ the *value function*.

We shall require a characterization of V as the unique solution of a proximal Hamilton-Jacobi partial differential equation with an associated natural boundary condition. Prior to stating it, we need to introduce the *lower Hamiltonian* $h_F : \Re \times \Re^n \times \Re^n \to \Re$ and the *augmented lower Hamiltonian* $\bar{h}_F : \Re \times \Re^n \times \Re \times \Re^n \to \Re$. These functions are defined as

$$h_F(t, x, \zeta) := \min_{v \in F(t,x)} \langle v, \zeta \rangle$$

and

$$\bar{h}_F(t,x,\theta,\zeta) := \theta + h_F(t,x,\zeta) = \min_{v\in F(t,x)} \langle (1,v)\,,\,(\theta,\zeta)\rangle.$$

Theorem 3.1.

(a) *The value function V is the unique continuous function $\varphi : (-\infty, T] \times \Re^n \to \Re$ satisfying*

$$\bar{h}_F(t,x,\partial_P\varphi(t,x)) = 0 \quad \forall\,(t,x) \in (-\infty,T)\times\Re^n, \tag{12}$$

$$\varphi(T,x) = \ell(x) \quad \forall\, x \in \Re^n. \tag{13}$$

(b) *Furthermore, if ℓ is assumed to be locally Lipschitz on $\Re^n$, then V is locally Lipschitz on $(-\infty,T]\times\Re^n$.*

In order to clarify the meaning of the equation (12), let us note that it is rephrasable as

$$\bar{h}_F(t,x,\theta,\zeta) = \theta + h_F(t,x,\zeta) = 0 \quad \forall\,(\theta,\zeta)\in\partial_P\varphi(t,x), \quad \forall\,(t,x)\in(-\infty,T)\times\Re^n.$$

For the proof of Theorem 3.1, see [9], §§4.3, 4.7 as well as [8]. The proximal characterization that the theorem provides can be rephrased in terms of viscosity, or alternatively, minimax solutions to the nonsmooth Hamilton-Jacobi equation; see [9], [8] for definitions, proofs of these equivalences, and for historical references regarding these generalized solution concepts. We wish to point out that the proximal viewpoint taken in the present work capitalizes upon the intrinisic (and quite natural) geometry of the problem in order to construct near-optimal feedbacks, a route which appears problematic in the context of viscosity or minimax solutions.

Our main result is the following.

Theorem 3.2. *Suppose that the multifunction F satisfies (SH) and that the cost functional ℓ is continuous. Let $M > 0$ and $t_0 \in (-\infty,T)$ be specified. Then for any given $\varepsilon > 0$, there exists a feedback f_ε for F and a scalar $\tilde{\mu} > 0$ such that the following holds: Given any initial data*

$$(\tau,\alpha) \in [t_0,T]\times M\overline{B}_n \tag{14}$$

and any partition π of $[\tau,T]$ with $\mathrm{diam}(\pi) \le \tilde{\mu}$, every Euler polygonal arc x_π of the initial value problem

$$\dot{x}(t) = f_\varepsilon(t,x(t)), \quad x(\tau) = \alpha \tag{15}$$

satisfies

$$\ell(x_\pi(T)) \le V(\tau,\alpha) + \varepsilon. \tag{16}$$

An immediate consequence of the theorem holds for Euler (limit) solutions:

Corollary 3.3. *Under the hypotheses of the theorem, for any intial data satisfying* (14) *and any given* $\varepsilon > 0$, *every Euler solution* x *of* (15) *on* $[\tau, T]$ *satisfies*

$$\ell(x(T)) \leq V(\tau, \alpha) + \varepsilon. \tag{17}$$

The statement of the above corollary can be rephrased as follows:

All Euler solutions generated by the feedback f_ε *are* ε*-optimal trajectories of* $P(\tau, \alpha)$ *for any given initial data* (τ, α) *in the* $(n+1)$*-dimensional rectangle* $[t_0, T] \times M\overline{B}_n$.

The proof of Theorem 3.2 also yields the following corollary, which is useful in case V is not known, but a continuous solution (also called a "semisolution") to a proximal Hamilton-Jacobi inequality is available; the other hypotheses of the theorem are unchanged.

Corollary 3.4. *Let* $\varphi : (-\infty, T] \times \Re^n \to \Re$ *be a continuous function satisfying*

$$\bar{h}_F(t, x, \partial_P \varphi(t, x)) \leq 0 \quad \forall\, (t, x) \in (-\infty, T) \times \Re^n, \tag{18}$$

$$\varphi(T, x) \geq \ell(x) \quad \forall\, x \in \Re^n. \tag{19}$$

Then given $\varepsilon > 0$, *there exists a feedback* f_ε *for* F *and a scalar* $\tilde{\mu} > 0$ *such that for any initial data* (14) *and any partition* π *of* $[\tau, T]$ *with* $\operatorname{diam}(\pi) \leq \tilde{\mu}$, *every Euler polygonal arc* x_π *of the initial value problem* (15) *satisfies*

$$\ell(x_\pi(T)) \leq \varphi(\tau, \alpha) + \varepsilon. \tag{20}$$

Furthermore, every Euler solution x *of* (15) *on* $[\tau, T]$ *satisfies*

$$\ell(x(T)) \leq \varphi(\tau, \alpha) + \varepsilon. \tag{21}$$

Remark 3.5. We pause to discuss the relationship of Theorem 3.2 and its corollaries to certain known results.

- If V is known, then a special case of Theorem 4.8.1 of [9] (which first appeared as Theorem 10.1 in [8]) provides a proximal aiming method for constructing a feedback (using the proximal characterization of V given by (12)-(13)), such that all its Euler solutions are optimal (that is, $\varepsilon = 0$), for a *given* initial data pair (τ, α). Actually, the invariance-based proof shows that a somewhat better result holds: The feedback produces optimal Euler solutions for any initial data in the set

$$S := \{(\tau', \alpha') \in (-\infty, T] \times \Re^n : V(\tau', \alpha') \leq V(\tau, \alpha)\}.$$

  In contrast to this scenario, the feedback f_ε that we will build is operative for *all* initial data in a prescribed generalized rectangle. This "universal" property of the feedback f_ε is an important distinction, and in a sense, the weakening of "optimal" to "ε-optimal for any given $\varepsilon > 0$" in our results can be viewed as the price paid for universality in Corollary 3.3, albeit a small one in any practical sense. Whether this price is truly unavoidable is an open question, since we do not at present have a counterexample to the $\varepsilon = 0$ case of Corollary 3.3. On the other hand, Krasovskii and Subbotin [13] have provided an example of a fixed duration differential game which does not possess a universal saddle point, under hypotheses which imply the existence of a saddle point for each *individual* startpoint; see also the discussion in §4.1 below. In §4.2 we will construct a universal ε-saddle point in proximal aiming feedback controls. We refer the reader to Krasovskii [14] (see also [13]) for a different construction of a universal ε-saddle point, one which is less in the spirit of our geometric and dynamic programming approach, in that it does directly utilize a generalized Hamilton-Jacobi system.

- In Theorem 10.2 of [8], a sufficient condition is given for the existence of a universal ε-optimal feedback, in the classical ordinary differential equations (as opposed to Euler) solution sense. This condition requires finding a Lipschitz semisolution to a strict version of (18) (along with the boundary condition (19)), but with the proximal subdifferential $\partial_P V$ replaced by the generalized subdifferential $\partial_C V$ of Clarke, which is in general a larger object than the P-subdifferential. Because of this, the value function in general does not satisfy this condition, so there is the difficulty of finding an appropriate semisolution if one seeks to apply this result.

- In Clarke, Ledyaev and Subbotin [11], a proximal analytic method is given for producing universal ε-optimal feedback controls in differential games of pursuit, in the Krasovskii-Subbotin framework. (The pursuit strategy constructed there happens to be autonomous, because the payoff in the game is the minimum-time function to a target.) The feedbacks in [11] are constructed with the aid of Moreau-Yosida infimal convolutions of a not necessarily continuous proximal semisolution to a Hamilton-Jacobi inequality; this lack of continuity is a natural feature of the value function in time-optimal control, and more generally, in differential games of pursuit. In the present work, we will employ a technique dubbed *proximal aiming* in [8], which has as an antecedent the "extremal aiming" method employed by Krasovskii and Subbotin [13] in their theory of differential games. The proximal aiming method we shall employ involves metric projection *directly* onto the sublevel sets of the value function (which in many cases is determined first heuristically, and then verified via Theorem 3.1); the infimal convolution operation is not required.

For easier exposition, we will without loss of generality assume that $0 < t_0 < T$. Also, let us denote

$$C := [0, T] \times M\overline{B}_n. \tag{22}$$

Recalling the discussion of Euler polygonal arcs in §2.2, it is a straightforward exercise to show that there exists $M_1 > 0$ such that the following holds: For any feedback f of F, any initial data $(\tau, \alpha) \in C$, and any partition π of $[\tau, T]$, the Euler polygonal arc x_π generated by f on $[\tau, T]$ satisfies

$$||x_\pi(t)|| \leq M_1 \quad \forall\, t \in [\tau, T]. \tag{23}$$

We denote

$$C_1 := [0,T] \times 2M_1\overline{B}_n. \tag{24}$$

Then for any $(\tau,\alpha) \in C$, we clearly have

$$(t, x_\pi(t)) \in [0,T] \times M_1\overline{B}_n \subseteq C_1 \quad \forall\, t \in [\tau, T]. \tag{25}$$

For $\theta > 0$, consider the mollifier of ℓ given by

$$\ell_\theta(x) = \int_{\Re^n} \ell(x+\theta y)\omega(y)dy,$$

where $\omega : \Re^n \to [0,\infty)$ is a function in $C^\infty(\Re^n)$ with support in $\overline{B}_n$ such that

$$\int_{\Re^n} \omega(y)dy = 1.$$

Then standard arguments concerning regularization of continuous functions yield that $\ell_\theta \in C^\infty(\Re^n)$, and what is more, for any given $\chi > 0$, θ may be chosen small enough to ensure that

$$|\ell_\theta(x) - \ell(x)| \le \chi \quad \forall\, x \in M_1\overline{B}.$$

Upon denoting by V_θ the value function obtained by replacing ℓ with ℓ_θ, it follows that

$$|V_\theta(\tau,\alpha) - V(\tau,\alpha)| \le \chi \quad \forall\,(\tau,\alpha) \in C.$$

The next lemma follows readily from the preceding discussion. It will prove to be quite useful, since for technical reasons that will become plain as we proceed, we shall require (in the proof of Lemma 3.13, specifically) Lipschitz behavior of the value function, as opposed to mere continuity.

Lemma 3.6. *In Theorem 3.2, it suffices to consider only the case where ℓ is C^∞ on $\Re^n$.*

From this point on, whether explicitly stated or not, the hypotheses of Theorem 3.2 will be in force, and we will assume that ℓ is C^∞ on $\Re^n$. Then in view of part (b) of Theorem 3.1, V is locally Lipschitz on $(-\infty, T] \times \Re^n$.

The proof of Theorem 3.2 is quite long and somewhat technical, and its component parts will be dealt with in the two ensuing subsections.

3.1. Artificial inwardness and sublevel sets. Let us fix $\beta > 0$, a parameter which will later be taken sufficiently small. Denote by $\{P^\beta(\tau,\alpha)\}$ the version of the family of problems $\{P(\tau,\alpha)\}$ one obtains by adding a "running cost" $\beta(T-\tau)$ to the original cost $\ell(x(T))$. Thus the problem $P^\beta(\tau,\alpha)$ is

minimize $\ell(x(T)) + \beta(T-\tau)$ over all trajectories x of the differential inclusion $\dot{x}(t) \in F(t,x(t))$ satisfying $x(\tau) = \alpha$.

By compactness of trajectories, the minimum is attained in $P^\beta(\tau,\alpha)$; we denote the associated value function by V^β.

Straightforward arguments yield the following:

Lemma 3.7. *For every* $(\tau, \alpha) \in C$, *one has*

$$V^{\beta}(\tau, \alpha) = V(\tau, \alpha) + \beta(T - \tau), \tag{26}$$

and in particular,

$$V^{\beta}(\tau, \alpha) \leq V(\tau, \alpha) + \beta T. \tag{27}$$

Since one the functions in the sum defining V^{β} is linear, the sum rule (7) implies

$$\partial_P V^{\beta}(\tau, \alpha) = \partial_P V(\tau, \alpha) - (\beta, 0).$$

Then Theorem 3.1 together with (26) readily yields the following.

Lemma 3.8. *The value function* V^{β} *is locally Lipschitz on* $(-\infty, T] \times \Re^n$ *and satisfies*

$$\bar{h}_F(t, x, \partial_P V^{\beta}(t, x)) = -\beta \quad \forall\, (t, x) \in (-\infty, T) \times \Re^n, \tag{28}$$

as well as

$$V^{\beta}(T, x) = \ell(x) \quad \forall\, x \in \Re^n. \tag{29}$$

The addition of the running cost term $\beta(T-\tau)$ has the effect of "artificially" producing a type of decrease property, at the rate β, of the value function, via the Hamiltonian inequality (28). This is most naturally observed by invoking the fact that (28) is equivalent to the Dini version

$$\min_{v \in F(t,x)} DV^{\beta}((t, x); (1, v)) = -\beta \quad \forall\, (t, x) \in (-\infty, T) \times \Re^n, \tag{30}$$

where for a locally Lipschitz function g, the *lower Dini derivate* of g at y in the direction w is

$$Dg(y; w) := \liminf_{\lambda \downarrow 0} \frac{g(y + \lambda w) - g(y)}{\lambda}. \tag{31}$$

The equivalence of (28) and (30) rests upon a deep result of Subbotin linking proximal and directional calculus; in this regard we refer the reader to [9], §§3.4, 4.7. On the other hand, to our knowledge, no way is known to directly use the Dini inequality to construct optimal or suboptimal feedbacks in problems of stabilizability or control. As will be seen below (in Lemma 3.13), a geometric version of the decrease property can meaningfully be termed "artificial strict inwardness" of the velocity sets F with respect to a certain family of sets constructed from the sublevel sets of V^{β}. This property will be vital in the design of the feedback f_{ε} in Theorem 3.2.

The following simple example, which is part of Exercise 4.7.11 in [9], illustrates a related effect of adding the running cost to the problem; namely, the creation of a new family of level sets for the value function which can be suitably "packed" in a way needed for the implementation of the proximal aiming methodology.

Figure 1: Altered level sets

Example 3.9. Let $n = 1$, $F(x) = [-1, 1]$, $\ell(x) = |x|$, and $T = 1$. It is not difficult to see that

$$V(t, x) = \max\{|x| + t - 1, 0\}.$$

Figure 1(a) shows some sublevel sets of the original V as well as a region where $V = 0$ when the cost functional is $|x(1)|$. Figure 1(b) shows sublevel sets $\{(t, x) : V^\beta(t, x) \leq a\}$ (where $0 < a \leq \beta$) obtained when the cost is $|x(1)| + \beta(1 - \tau)$ for a fixed parameter value $\beta > 0$. These sublevel sets form a nested family of triangles, reducing to $\{(1, 0)\}$ when $a = 0$. The reader is invited to (heuristically at least) verify the picture.

Let

$$a_m := \min\{V^\beta(t, x) : (t, x) \in C_1\},$$

$$a_M := \max\{V^\beta(t, x) : (t, x) \in C_1\},$$

and for $a \in [a_m, a_M]$, let us denote the a-sublevel sets of V^β (relative to C_1) by

$$S(a) := \{(t, x) \in C_1 : V^\beta(t, x) \leq a\}.$$

Obviously one has the nesting property

$$a_m \leq a < a' \leq a_M \Longrightarrow S(a) \subseteq S(a'),$$

and clearly $S(a_M) = C_1$. Also, one can use the infinitesimal decrease property (30) in order to show that $S(a_m) \cap \operatorname{int}(C_1) = \phi$.

The next lemma provides basic properties of the multifunction S.

Lemma 3.10. *For each $a \in [a_m, a_M]$, the following hold:*

(a) *The set $S(a)$ is nonempty and compact.*

(b) *The multifunction S is Hausdorff continuous on $[a_m, a_M]$.*

Proof: The verification of (a) is easy and left to the reader. As for part (b), let $a \in [a_m, a_M]$. Given $\varepsilon > 0$, we must verify the existence of $\delta > 0$ such that

$$\left.\begin{array}{c} a' \in [a_m, a_M] \\ |a' - a| < \delta \end{array}\right\} \Longrightarrow \left\{\begin{array}{ll} S(a') \subseteq S(a) + \varepsilon B_{n+1} & \text{(i)} \\ S(a) \subseteq S(a') + \varepsilon B_{n+1} & \text{(ii)} \end{array}\right. \tag{32}$$

Case 1. $a' \geq a$.

In this case, (ii) is obvious. Suppose (i) was false. Then there would exist sequences $a_i \downarrow a$, $a_i \in [a_m, a_M]$, and $(t_i, x_i) \in S(a_i)$ such that $d_{S(a)}(t_i, x_i) \geq \varepsilon$. Since (t_i, x_i) is clearly a bounded sequence, we can assume convergence to a point $(t, x) \in C_1$. Then continuity of the distance function implies $d_{S(a)}(t, x) \geq \varepsilon$. But continuity of V^β implies $V^\beta(t, x) \leq a$, providing a contradiction.

Case 2. $a > a'$.

In this case, (i) is obvious. Again arguing by way of contradiction, suppose that (ii) was false. Then there would exist $\varepsilon > 0$ such that for sequences $a_i \uparrow a$, $a_i \in [a_m, a_M]$, and $(t_i, x_i) \in S(a)$, one had

$$d_{S(a_i)}(t_i, x_i) > \varepsilon \quad \forall\, i \geq i_0$$

for some index i_0. By compactness, we can assume $(t_i, x_i) \to (t, x) \in S(a)$. Then

$$d_{S(a_i)}(t, x) > \frac{\varepsilon}{2} \quad \forall\, i \geq i_0. \tag{33}$$

Now, since C_1 is compact, convex and has nonempty interior, it is the closure of its interior. Hence without loss of generality, we may assume that the point (t, x) in (33) is in $\mathrm{int}(C_1)$, and in particular, $t < T$. Then, by the infinitesimal decrease property of V^β provided by (30), there exist points $(t', x') \in C_1$ such that

$$\|(t', x') - (t, x)\| \leq \frac{\varepsilon}{4} \tag{34}$$

and

$$V^\beta(t', x') < V^\beta(t, x). \tag{35}$$

Then (35) implies $V^\beta(t', x') < a$, and therefore there exists i_0' such that

$$(t', x') \in S(a_i) \quad \forall\, i \geq i_0'. \tag{36}$$

But (33) and (34) together imply

$$d_{S(a_i)}(t', x') > \frac{\varepsilon}{4} \quad \forall\, i \geq i_0,$$

which contradicts (36). □

The next lemma's proof follows readily from part (b) of the preceding one, along with a routine uniform continuity argument; we therefore omit the proof.

Lemma 3.11. *Let $\gamma > 0$ be given. Then there exists $k_0 > 0$ such that for all integers $k \geq k_0$, the partition $\{b_j\}_{j=0}^k$ of the interval $[a_m, a_M]$ given by*

$$b_0 := a_m, \quad b_k := a_M, \quad b_{j+1} := b_j + \frac{a_M - a_m}{k}, \quad j = 0, 1, \ldots, k-1. \tag{37}$$

satisfies

$$S(b_{j+1}) \subseteq S(b_j) + \gamma B_{n+1}, \quad j = 0, 1, \ldots, k-1. \tag{38}$$

Let us denote

$$\nu := \sup\{\|v\| : v \in F(t,x), \, (t,x) \in C_1\}.$$

Lemma 3.12. *There exists $\delta > 0$ such that one has*

$$\|(\theta, \zeta)\| \geq \delta \quad \forall\, (\theta, \zeta) \in \partial_P V^\beta(t,x), \quad \forall\, (t,x) \in \operatorname{int}(C_1). \tag{39}$$

Furthermore,

$$0 \not\in \partial_L V^\beta(t,x) \quad \forall\, (t,x) \in \operatorname{int}(C_1). \tag{40}$$

Proof: The infinitesimal decrease property (28) implies

$$\bar{h}_F(t, x, \partial_P V^\beta(t,x)) = -\beta.$$

Let $(\theta, \zeta) \in \partial_P V^\beta(t,x)$. Then for some $\tilde{v} \in F(t,x)$ one has

$$\theta + \langle \tilde{v}, \zeta \rangle = -\beta,$$

or equivalently,

$$\langle (1, \tilde{v}) \, , \, (\theta, \zeta) \rangle = -\beta.$$

Then by the Cauchy-Schwartz inequality, we obtain

$$\|(1, \tilde{v})\| \, \|(\theta, \zeta)\| \geq \beta,$$

whence

$$\|(\theta, \zeta)\| \geq \delta = \frac{\beta}{\nu + 1},$$

which verifies (39). Now (40) follows from the definition of the L-subdifferential. □

The following lemma provides a key property, which amounts to a type of "uniform strict inwardness" of F with respect to the family of sets $\{S(a) \cap \operatorname{int}(C_1)\}$ as a varies in

$[a_m, a_M]$; note that boundary points of C_1 are excluded in this family. The proof of the lemma requires locally Lipschitz behavior of the value function V^β. We denote by κ' a Lipschitz constant for V on C_1. Then in view of (26), $\kappa := \kappa' + 1$ is a Lipschitz constant for V^β on C_1 provided that $\beta \leq 1$, which will henceforth be tacitly assumed. (This upper bound on β will subsequently be decreased.) Also, we denote by $\tilde{\kappa}$ a Lipschitz constant for the multifunction F on C_1.

Lemma 3.13. *For any given $a \in [a_m, a_M]$, one has*

$$\bar{h}_F(t,x,\eta) \leq -\frac{\beta}{\kappa}\|\eta\| \quad \forall\,(t,x) \in S(a) \cap \mathrm{int}(C_1), \quad \forall\,\eta \in N^P_{S(a)}(t,x). \tag{41}$$

Proof: Note that (41) is vacuously true if $S(a) \cap \mathrm{int}(C_1) = \phi$.

As far as proving the inequality (41) is concerned, let us first note that it is equivalent to its limiting form

$$\bar{h}_F(t,x,\eta) \leq -\frac{\beta}{\kappa}\|\eta\| \quad \forall\,(t,x) \in S(a) \cap \mathrm{int}(C_1), \quad \forall\,\eta \in N^L_{S(a)}(t,x), \tag{42}$$

as follows directly from the definition of the L-normal cone and the continuity of h_F in η.

Given $a \in [a_m, a_M]$, let $(t,x) \in S(a) \cap \mathrm{int}(C_1)$ and $\eta \in N^L_{S(a)}(t,x)$. Since the lower Hamiltonian $h_F(t,x,\eta)$ is positively homogeneous in η, it suffices to verify (42) for $\|\eta\| = 1$, which will be assumed from this point on in the proof. Then $(t,x) \in \mathrm{bdry}(S(a))$, since $\eta = 0$ if (t,x) in the interior of $S(a)$, and therefore $V^\beta(t,x) = a$. We are to prove that

$$\bar{h}_F(t,x,\eta) \leq -\frac{\beta}{\kappa}. \tag{43}$$

To this end, recall (9), which we are in position to apply by virtue of (40). Then

$$\eta = \lambda(\theta, \zeta)$$

for some $\lambda > 0$, where $(\theta,\zeta) \in \partial_L V^\beta(t,x)$. Since V^β is rank κ Lipschitz on C_1, we have $\|(\theta,\zeta)\| \leq \kappa$. Therefore, since $\|\eta\| = 1$, we have

$$\lambda \geq \frac{1}{\kappa}. \tag{44}$$

In view of (28),

$$\bar{h}_F(t,x,\theta,\zeta) = -\beta,$$

and so

$$\bar{h}_F(t,x,\eta) = -\lambda\beta.$$

Upon combining this with (44), we arrive at (43). □

3.2. Completing the proof of Theorem 3.2. From this point on, we shall take

$$\gamma := \frac{\beta}{2\kappa\tilde{\kappa}}, \tag{45}$$

where β will subsequently be adjusted.

Consider a partition $\{b_j\}_{j=0}^k$ of $[a_m, a_M]$ as in Lemma 3.11; that is, such that (37) and (38) hold, for this value of γ. For $j = 0, 1, \ldots, k-1$, we denote the open γ-neighborhood of $S(b_j)$ by Z_j; that is,

$$Z_j := S(b_j) + \gamma B_{n+1}, \quad j = 0, 1, \ldots, k-1.$$

We will refer to Z_j as the *absorption zone* of the set $S(b_j)$, for reasons that will subsequently become clear; the value γ given in (45) is called the *absorption radius.* Then (38) says

$$S(b_{j+1}) \subseteq Z_j, \quad j = 0, 1, \ldots, k-1. \tag{46}$$

Given $(t, x) \in S(b_k) = S(a_M)$, denote

$$j(t, x) = \min\{j : (t, x) \in Z_j,\ j = 0, 1, \ldots, k-1\}.$$

In other words, $S(b_{j(t,x)})$ is the "most inner " (or smallest) sublevel set $S(b_j)$ (i.e. the sublevel set with the lowest possible j) whose absorption zone Z_j contains the point (t, x). Note that if $(t, x) \in S(a_m) = S(b_0)$, then $j(t, x) = 0$.

We shall now define a certain feedback f of F.

- For $(t, x) \in S(b_0)$, $f(t, x)$ is taken to be an arbitrary element of $F(t, x)$.
- Likewise, for $(t, x) \in \Re^n \backslash \{S(b_k\}$, $f(t, x)$ is an arbitrary element of $F(t, x)$.
- For $(t, x) \in S(b_k) \backslash \{S(b_0)\}$, let

$$s = s(t, x) \in \mathrm{proj}_{S(b_{j(t,x)})}(t, x), \tag{47}$$

  and denote $\eta := (t, x) - s$. Note that $\eta \in N^P_{S(b_{j(t,x)})}(s)$, and that

$$0 < \|s - (t, x)\| = \|\eta\| = d_{S(b_{j(t,x)})}(t, x) < \gamma. \tag{48}$$

  Let $v = v(t, x) \in F(s)$ be any point such that

$$\bar{h}_F(s, \eta) = \langle (1, v)\,,\, \eta \rangle.$$

  We take $f(t, x)$ to be the (unique) closest point in $F(t, x)$ to v.

It is important to observe that in view of Lemma 3.13, if $(t, x) \in S(a) \cap \mathrm{int}(C_1)$, then we have

$$\langle (1, v)\,,\, \eta \rangle = \langle (1, v), ((t, x) - s) \rangle \leq -\frac{\beta}{\kappa} \|(t, x) - s\|. \tag{49}$$

Therefore for such points (t, x), since

$$\langle (1, f(t,x)), (t,x) - s \rangle = \langle (1, v), (t,x) - s \rangle + \langle (1, f(t,x)) - (1, v), (t,x) - s \rangle$$

and F is $\tilde{\kappa}$-Lipschitz on C_1, one has

$$\langle (1, f(t,x)), (t,x) - s \rangle \leq \tilde{\kappa} d^2_{S(b_{j(t,x)})}(t,x) - \frac{\beta}{\kappa} d_{S(b_{j(t,x)})}(t,x). \tag{50}$$

The next lemma asserts that if the running cost parameter β and the mesh diameter of the partition π are small enough, then for any initial data in C, the Euler polygonal arc x_π generated by f exhibits a type of nonincrease property with respect to V^β. After the lemma is proven, the proof of Theorem 3.2 will be completed by adjusting the value of β (depending upon the given ε). We first need to specify certain constants.

- We set
$$c_1 = \frac{\beta - \kappa\tilde{\kappa}\gamma}{2\kappa}.$$
- It is clear from (46) that there exists $c_2 > 0$ such that
$$d_{S(b_{j+1})}(t,x) \leq c_2 \Longrightarrow (t,x) \in Z_j, \quad j = 0, 1, \ldots, k-1. \tag{51}$$

Lemma 3.14. *Let $0 < \hat{\varepsilon} < \frac{T}{2}$, and assume that*

$$\beta \leq \min\{\kappa\tilde{\kappa}M_1, \kappa\tilde{\kappa}\hat{\varepsilon}\}. \tag{52}$$

Let $(\tau, \alpha) \in [\hat{\varepsilon}, T - \hat{\varepsilon}] \times M\overline{B}_n \subseteq C$, and let π be a partition of $[\tau, T]$ such that its mesh diameter satisfies

$$\operatorname{diam}(\pi) \leq \hat{\mu} := \frac{c_1 c_2}{\nu^2}, \tag{53}$$

where γ is given by (45). Set

$$i* := \max\{i : t_i \leq T - \hat{\varepsilon}\}.$$

Then node $i*$ of the Euler polygonal arc x_π on $[\tau, T]$ generated by the feedback f in the initial value problem

$$\dot{x}(t) = f(t, x(t)), \quad x(\tau) = \alpha \tag{54}$$

satisfies

$$(t_{i*}, x_{i*}) \in Z_{j(\tau,\alpha)}. \tag{55}$$

Proof: In view of (45) and the bound (52), one has

$$\gamma < \min\{M_1, \hat{\varepsilon}\}. \tag{56}$$

Then

$$(t,x) \in [\hat{\varepsilon}, T - \hat{\varepsilon}] \times M_1\overline{B}_n \Longrightarrow (t,x) + \gamma\overline{B}_{n+1} \in \operatorname{int}(C_1), \tag{57}$$

and therefore, in view of (48),

$$(t,x) \in [\hat{\varepsilon}, T-\hat{\varepsilon}] \times M_1\overline{B}_n \Longrightarrow s = s(t,x) \in \mathrm{int}(C_1). \tag{58}$$

Since obviously

$$(t, x_\pi(t)) \in [\hat{\varepsilon}, T-\hat{\varepsilon}] \times M_1\overline{B}_n \quad \forall\, t \in [0, t_{i*}],$$

(58) implies that the nodes of x_π satisfy

$$s(t_i, x_i) \in \mathrm{int}(C_1) \quad i = 0, 1, \ldots, i*, \tag{59}$$

a fact which enables us to employ Lemma 3.13 in the ensuing arguments.

We will show that for $i = 0, 1, \ldots, i*$, the nodes of x_π, beginning with the intitial point $(t_0, x_0) = (\tau, \alpha)$, continually enter absorption zones with nonincreasing indices $j \le j(\tau, \alpha)$. This will yield (55).

We introduce the function

$$g(t,x) = 2\left(\tilde{\kappa} d_{S(b_{j(\tau,\alpha)})}(t,x) - \frac{\beta}{\kappa}\right).$$

We claim that

$$g(t,x) < -2c_1 \quad \forall\,(t,x) \in Z_{j(\tau,\alpha)} = S(b_{j(\tau,\alpha)}) + \gamma B_{n+1}. \tag{60}$$

To see this, first note that (60) will hold provided that

$$\tilde{\kappa} d_{S(b_{j(\tau,\alpha)})}(t,x) - \frac{\beta}{\kappa} < \frac{\kappa\tilde{\kappa}\gamma - \beta}{2\kappa}.$$

After some arithmetic, this last inequality is seen to be equivalent to

$$\gamma < \frac{\beta}{\kappa\tilde{\kappa}},$$

which holds due to (45), and (60) follows.

In view of (50), for every $(t,x) \in Z_{j(\tau,\alpha)}$, one has

$$\langle (1, f(t,x))\,,\,(t,x) - s\rangle \le \frac{g(t,x)}{2} d_{S(b_{j(\tau,\alpha)})}(x) \quad \forall\, s \in \mathrm{proj}_{S(b_{j(\tau,\alpha)})}(t,x). \tag{61}$$

Then

$$\begin{aligned}
d^2_{S(b_{j(\tau,\alpha)})}(t_1,x_1) &\le ||(t_1,x_1) - s(\tau,\alpha)||^2 \\
&= ||(t_1,x_1) - (\tau,\alpha)||^2 + ||(\tau,\alpha) - s(\tau,\alpha)||^2 \\
&\qquad + 2\langle (t_1,x_1) - (\tau,\alpha)\,,\,(\tau,\alpha) - s(\tau,\alpha)\rangle \\
&\le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) + \nu^2(t_1-\tau)^2 + 2\int_\tau^{t_1} \langle (1, f(\tau,\alpha))\,,\,(\tau,\alpha) - s(\tau,\alpha)\rangle dt \\
&\le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) + \nu^2(t_1-\tau)^2 + g(\tau,\alpha) d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha)(t_1-\tau) \\
&\le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) + [\nu^2 \mathrm{diam}(\pi) + g(\tau,\alpha) d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha)](t_1-\tau).
\end{aligned}$$

We now claim that

$$\nu^2 \mathrm{diam}(\pi) + g(\tau,\alpha) d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) < -c_1 c_2. \tag{62}$$

To show this, first note that (60) implies

$$g(\tau,\alpha) d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) < -2c_1 d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha).$$

Therefore, in view of (53), the inequality (62) will hold if

$$d_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) > c_2.$$

But this follows from the definition of c_2. Hence (62) holds.

We conclude that

$$d^2_{S(b_{j(\tau,\alpha)})}(t_1,x_1) \le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) - c_1 c_2 (t_1 - \tau). \tag{63}$$

Possibly (t_1, x_1) is in Z_j for some $i < j(\tau,\alpha)$. If so, restart the procedure just described, with the node (t_1, x_1) taking over the role played by $(\tau,\alpha) = (t_0, x_0)$ above. If not, then since x_1 is in $Z_{j(\tau,\alpha)}$, we can repeat the argument and obtain

$$d^2_{S(b_{j(\tau,\alpha)})}(t_2,x_2) \le d^2_{S(b_{j(\tau,\alpha)})}(t_1,x_1) - c_1 c_2 (t_2 - t_1).$$

Then

$$d^2_{S(b_{j(\tau,\alpha)})}(t_2,x_2) \le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) - c_1 c_2 (t_2 - \tau).$$

Continuing in this way, if $(\tau,\alpha), (t_1,x_1), \ldots, (t_{i-1}, x_{i-1})$ are not in some Z_j with $j < j(\tau,\alpha)$, then since all these points are in $Z_{j(\tau,\alpha)}$, we have

$$d^2_{S(b_{j(\tau,\alpha)})}(t_i,x_i) \le d^2_{S(b_{j(\tau,\alpha)})}(\tau,\alpha) - c_1 c_2 (t_i - \tau). \tag{64}$$

The above argument shows that for $i = 0, 1, \ldots, i*$, the nodes of the Euler polygonal arc remain in $Z_{j(\tau,\alpha)}$, possibly entering more and more inner absorption zones. In particular, (55) holds, which completes the proof. □

We now are in position to complete the proof of the theorem.

Proof of Theorem 3.2: Let $\hat{\varepsilon}$ and (τ,α) be as in the statement of the preceding lemma, and let us denote

$$j' = j'(\tau,\alpha) = \min\{j : V^\beta(\tau,\alpha) \le b_j\}.$$

Noting that $(\tau,\alpha) \not\in S(b_0)$, we have $j(\tau,\alpha) \le j' - 1$, and therefore the Euler polygonal arc produced in the lemma by the feedback f satisfies

$$(t_{i*}, x_\pi(t_{i*})) \in Z_{j'-1} = S(b_{j'-1}) + \gamma B_{n+1}.$$

Hence there exists (t', x') such that $V^\beta(t', x') \le b_{j'-1}$ (that is, $(t', x') \in S(b_{j'-1})$) and

$$\|(t', x') - (t_{i*}, x_\pi(t_{i*}))\| \le \gamma. \tag{65}$$

We have

$$\begin{aligned}\ell(x_\pi(t_{i*}) &= V^\beta(T, x_\pi(t_{i*}))\\ &\le V^\beta(t_{i*}), x_\pi(t*)) + \kappa(T - t_{i*})\\ &\le V^\beta(t', x') + \kappa(T - t_{i*}) + \kappa\gamma\\ &\le b_{j'-1} + \kappa(T - t_{i*}) + \kappa\gamma\\ &\le V^\beta(\tau, \alpha) + \kappa(T - t_{i*}) + \kappa\gamma\\ &\le V^\beta(\tau, \alpha) + \kappa(\mathrm{diam}(\pi) + \hat{\varepsilon}) + \kappa\gamma.\end{aligned}$$

Let K denote a Lipschitz constant for the cost functional ℓ on C_1. From (27) we then obtain, for any $(\tau, \alpha) \in [\hat{\varepsilon}, T - \hat{\varepsilon}] \times M\overline{B}_n$, the inequality

$$\ell(x_\pi(T) \le V(\tau, \alpha) + \kappa(\mathrm{diam}(\pi) + \hat{\varepsilon}) + \kappa\gamma + K\nu(\mathrm{diam}(\pi) + \hat{\varepsilon}) + \beta T. \tag{66}$$

Then for for any $(\tau, \alpha) \in [\hat{\varepsilon}, T] \times M\overline{B}_n$, one has

$$\ell(x_\pi(T) \le V(\tau, \alpha) + \kappa(\mathrm{diam}(\pi) + \hat{\varepsilon}) + \kappa\gamma + K\nu(\mathrm{diam}(\pi) + \hat{\varepsilon}) + \beta T + K\nu\hat{\varepsilon}. \tag{67}$$

Now, bearing in mind that $t_0 > 0$, observe that (16) holds with the feedback f_ε taken to be f as described above, when all the bounds demanded on $\hat{\varepsilon}$, the mesh diameter $\mathrm{diam}(\pi)$ and running cost coefficient β are small enough to ensure that the right-hand-side of (67) is less that ε. This completes the proof of the theorem. □

4. Application to differential games

4.1. Krasovskii-Subbotin differential games. In this subsection we will describe a variant of the Krasovskii-Subbotin framework for differential games (Krasovskii and Subbotin [13], Subbotin [19]). The primary difference between the model presented here and theirs is that in the discretized Krasovskii-Subbotin model, Euler polygonal arcs are not utilized, but an alternate concept, that of "step-by-step motions" is employed. We will not dwell upon this distinction here.

The dynamics of the differential game are furnished by the two-controller system

$$\dot{x}(t) = g(t, x(t), u(t), w(t)), \quad t \in (-\infty, T], \tag{68}$$

where $g : R \times \Re^n \times \Re^p \times \Re^q \to \Re^n$. Admissible controls are Lebesgue measurable functions

$$u : (-\infty, T] \to P, \quad w : (-\infty, T] \to Q,$$

where the control restraint sets P and Q are compact subsets of $\Re^p$ and $\Re^q$, respectively.

The following standing hypotheses on the dynamics will be in effect throughout:

(a) The function g is continuous.

(b) For each bounded set $G \subseteq (-\infty, T] \times \Re^p$ there exists a number $\lambda(G) > 0$ such that

$|g(t_1, x_1, u, v) - g(t_2, x_2, u, v)| \leq \lambda(G)\|(t_1, x_1) - (t_2, x_2)\| \quad \forall\, (t_i, x_i, u, v) \in G \times P \times Q,$

$i = 1, 2.$

(c) There exist positive constants γ_1 and γ_2 such that

$$\|g(t, x, u, v)\| \leq \gamma_1 \|x\| + \gamma_2, \quad \forall\, (t, x, u, v) \in (-\infty, T] \times \Re^n \times P \times Q.$$

(d) The set $g(t, x, p, Q)$ is convex for all $(t, x, u) \in (-\infty, T] \times \Re^n \times P$, and the set $g(t, x, P, v)$ is convex for all $(t, x, v) \in (-\infty, T] \times \Re^n \times Q$.

(e) The *Isaacs condition* holds; that is, for all $(t, x) \in (-\infty, T] \times \Re^n$ one has

$$\min_{u \in P} \max_{w \in Q} \langle g(t, x, u, w), \zeta \rangle = \max_{w \in Q} \min_{u \in P} \langle g(t, x, u, w), \zeta \rangle \quad \forall\, \zeta \in \Re^n. \tag{69}$$

Remark 4.1. It is well known that if the Isaacs condition is omitted from the hypotheses, then it nevertheless holds in the differential game wherein controls are replaced by relaxed controls; see the discussion of mixed strategies in Chapter 10 of [13].

The *payoff functional* of the differential game to be described here is of endpoint type, and is given by $\ell(x(T))$, where $\ell : \Re^n \to \Re$ is assumed to be locally Lipschitz. The u-player is the minimizer, and the w player is the maximizer.

Feedback controls for the two players are simply any functions $U : (-\infty, T] \times \Re^n \to P$ and $W : (-\infty, T] \times \Re^n \to Q$, with no continuity or measurability assumptions made. Note that a pair (U, W) of feedback controls generates a feedback f for the multifunction

$$F(t, x) := g(t, x, P, Q),$$

namely

$$f(t, x) = g(t, x, U(t, x), W(t, x)).$$

Let $(\tau, \alpha) \in (-\infty, T] \times \Re^n$ be given initial data. We now describe our "Euler" differential game model, denoted $G(\tau, \alpha)$. Consider the partition π of $[\tau, T]$ given by

$$\pi = \{\tau =: t_0, t_1, t_2, \ldots, t_{N_\pi} := T\}.$$

Then the Euler polygonal arc $x(\cdot) = x(\cdot\,; \tau, \alpha, U, W, \pi)$ generated[1] by a pair of feedback controls (U, W) is described by

$$\begin{aligned} x(t) &= x(t_i) + \int_{t_i}^{t} g(t_i, x(t_i), U(t_i, x(t_i)), W(t_i, x(t_i)))ds, \\ x(\tau) &= \alpha, \quad s \in [t_i, t_{i+1}], \quad i = 0, 1, \ldots, N_\pi - 1. \end{aligned}$$

Consider a sequence of such Euler polygonal arcs

$$x_k(\cdot) = x_k(\cdot\,; \tau, \alpha, U, W, \pi_k), \quad k = 1, 2, \ldots,$$

where

$$\pi_k = \{t_0^k = \tau, t_1^k, t_2^k, \ldots, t_{N_\pi}^k = T\}$$

and the mesh diameters satisfy $\operatorname{diam}(\pi_k) \to 0$ as $k \to \infty$. A uniform limit of such Euler polygonal arcs on $[\tau, T]$ is called a *motion* generated by the pair of feedback controls (U, W). The discussion of Euler solutions in §2 implies that this set of motions, denoted $X(\tau, \alpha, U, W)$, is nonempty because, as pointed out above, each $x_k(\cdot)$ is an Euler polygonal arc generated by a certain feedback for the multifunction $F(t, x) = g(t, x, P, Q)$, and each such motion is a solution to the differential inclusion $\dot{x} \in \operatorname{co}[g(t, x, P, Q)]$.

We denote

$$\begin{aligned} \Gamma_1(\tau, \alpha, U) &= \sup_W \{\ell(x(T) : x(\cdot) \in X(\tau, \alpha, U, W)\}, \\ \Gamma_2(\tau, \alpha, W) &= \inf_U \{\ell(x(T) : x(\cdot) \in X(\tau, \alpha, U, W)\}. \end{aligned}$$

Then for any pair of feedback controls (U, W), one has

$$\Gamma_2(\tau, \alpha, W) \leq \ell(x(T)) \leq \Gamma_1(\tau, \alpha, U) \tag{70}$$

for every $x(\cdot) \in X(\tau, \alpha, U, W)$.

Let $\varepsilon > 0$ be given. If there exists a pair of feedback controls $(U_\varepsilon, W_\varepsilon)$ such that

$$\Gamma_1(\tau, \alpha, U_\varepsilon) - \varepsilon \leq \Gamma_2(\tau, \alpha, W_\varepsilon) + \varepsilon, \tag{71}$$

then $(U_\varepsilon, W_\varepsilon)$ is said to be an *ε-saddle point* of the game $G(\tau, \alpha)$. Observe that then

$$\inf_U \Gamma_1(\tau, \alpha, U) - \varepsilon \leq \Gamma_1(\tau, \alpha, U_\varepsilon) - \varepsilon \leq \Gamma_2(\tau, \alpha, W_\varepsilon) + \varepsilon \leq \sup_W \Gamma_2(\tau, \alpha, W) + \varepsilon. \tag{72}$$

Suppose that an ε-saddle point exists for all small $\varepsilon > 0$. Since by (70) one always has

$$\inf_U \Gamma_1(\tau, \alpha, U) \geq \sup_W \Gamma_2(\tau, \alpha, W), \tag{73}$$

[1] In our discretized model, the two players make decisions at a common set of meshpoints, and it is at these times that the Euler polygonal arc is redirected. A different framework is also possible, truer to the original Krasovskii-Subbotin model, wherein the two players independently select sequences of partitions in generating Euler polygonal arcs. For notational ease, however, we have opted to use the simplified model.

(72) implies

$$\inf_U \Gamma_1(\tau,\alpha,U) = \sup_W \Gamma_2(\tau,\alpha,W), \tag{74}$$

in which case the differential game $G(\tau,\alpha)$ is said to have *value* equal to the expression in (74).

For each *individual* initial data point $(\tau,\alpha) \in (-\infty,T] \times \Re^n$, Krasovskii and Subbotin obtained, for their model, the existence of value $V(\tau,\alpha)$ and a saddle point. The value function of Krasovskii-Subbotin, which we will denote V, has a characterization in terms of generalized Hamiltonian inequalities which is central to their theory of differential games. One of its uses is in constructing (for specified initial data) a saddle point via a technique known as "extremal aiming". The characterization of V is usually given in terms of Dini directional derivates, but we state it here in a proximal form needed for our purposes, which can be seen to be a generalization of Theorem 3.1 (upon adopting the point of view that optimal control problems are differential games with one "invisible" player, by taking the control restraint set Q to be a singleton). The equivalence between the original and proximal forms is a straightforward exercise using Subbotin's theorem, mentioned earlier in the comment following (30) and (31).

Some further notation needs to be introduced. The *Hamiltonian* $H : (-\infty,T] \times \Re^n \times \Re \times \Re^n \to \Re$ of the game is given by

$$\begin{aligned} H(t,x,\theta,\zeta) &= \theta + \min_{p\in P}\max_{q\in Q} \langle g(t,x,p,q),\zeta\rangle \\ &= \theta + \max_{q\in Q}\min_{p\in P} \langle g(t,x,p,q),\zeta\rangle \quad \text{(by the Isaacs conditon).} \end{aligned}$$

The required characterization is as follows:

The value function V in the Krasovskii-Subbotin model is the unique locally Lipschitz function $\varphi : (-\infty,T] \to \Re^n$ such that the following hold:

$$H(t,x,\partial_P\varphi(t,x)) \le 0 \quad \forall\,(t,x) \in (-\infty,T) \times \Re^n, \tag{75}$$

$$H(t,x,\partial_P(-\varphi)(t,x)) \le 0 \quad \forall\,(t,x) \in (-\infty,T) \times \Re^n, \tag{76}$$

$$\varphi(T,x) = \ell(x) \quad \forall\, x \in \Re^n. \tag{77}$$

4.2. Construction of a universal ε-saddle point via proximal aiming. Our goal now is the construction of an ε-saddle point $(U_\varepsilon, W_\varepsilon)$ in the differential game model described above, universal with respect to initial data in a prescribed rectangle, for arbitrarily small ε. The significance of universality here is that one player more fully exploits any "bad play" of the other player, than in the ordinary (non-universal) case. In this regard, it is important to note that an example satisfying our hypotheses is provided in §8.5 of [13] wherein a universal "exact" (that is $\varepsilon = 0$) saddle point does *not* exist. We will extend the proximal aiming method of §3 for control problems to the differential game setting, by exploiting the information provided by the relations (75)-(77). An immediate consequence of the technique (Corollary 4.5 below) is that each differential game $G(\tau,\alpha)$

has value, and this value agrees with $V(\tau, \alpha)$.

Adaptations of the proof of Theorem 3.2 yield the following lemma, where the viewpoint of the minimizer is taken.

Lemma 4.2. *Let $M > 0$ and $t_0 < T$ be given. Then for any given $\varepsilon > 0$, there exists a feedback control U_ε for the minimizer such that the following holds: There exists a scalar $\tilde{\mu} > 0$ such that for any feedback control W for the maximizer, any initial data as in (14) and any partition π of $[\tau, T]$ with mesh diameter $\mathrm{diam}(\pi) \leq \tilde{\mu}$, every Euler polygonal arc x_π of the initial value problem*

$$\dot{x}(t) = g(t, x, U_\varepsilon(t, x), W(t, x)) \tag{78}$$

satisfies

$$\ell(x_\pi(T)) \leq V(\tau, \alpha) + \varepsilon. \tag{79}$$

Proof: As in the proof of Theorem 3.2, it suffices to consider $0 < t_0 < T$. With respect to the dynamics (68), which can be viewed as

$$\dot{x}(t) \in F(t, x) = g(t, x, P, Q), \tag{80}$$

we let C, M_1 and C_1 be as in (22)—(24).

Analogously to Lemma 3.7, given $\beta > 0$,

$$V^\beta(\tau, \alpha) = V(\tau, \alpha) + \beta(T - \tau)$$

is the Krasovskii-Subbotin value of the differential game where the payoff has been changed to $\ell(x(T)) + \beta(T - \tau)$. Then V^β is locally Lipschitz on $(-\infty, T] \times \Re^n$ and, because of (75), (77), analogously to (28), (29), V^β satisfies

$$H(t, x, \partial_P V^\beta(t, x)) \leq -\beta \quad \forall\, (t, x) \in (-\infty, T) \times \Re^n, \tag{81}$$

as well as the boundary condition

$$V^\beta(T, x) = \ell(x) \quad \forall\, x \in \Re^n. \tag{82}$$

By employing the notation and arguments of §3, one then arrives at the following analog of (41): For any $a \in [a_m, a_M]$, one has

$$H(t, x, \eta) \leq -\frac{\beta}{\kappa}\|\eta\| \quad \forall\, (t, x) \in S(a) \cap \mathrm{int}(C_1), \quad \forall\, \eta \in N^P_{S(a)}(t, x). \tag{83}$$

We now define a feedback control $\widehat{U}$ as follows:

- For $(t, x) \in S(b_0)$, $\widehat{U}(t, x))$ is an arbitrary element of P.
- For $(t, x) \in \Re^n \backslash \{S(b_k\}$, $\widehat{U}(t, x)$ is again an arbitrary element of P.

- For $(t,x) \in S(b_k)\backslash\{S(b_0)\}$, let

$$s = s(t,x) \in \mathrm{proj}_{S(b_{j(t,x)})}(t,x), \tag{84}$$

and denote $\eta = (t,x) - s$. Decompose η as $\eta = (\eta_1, \eta_2) \in \Re \times \Re^n$, and let

$$\widehat{U}(t,x) \in \arg\min_{u \in P}\{\max_{v \in Q}\langle g(s(t,x),u,v),\eta_2\rangle\}. \tag{85}$$

Now let W be an arbitrary feedback control for the maximizer, and let

$$v := v(t,x) := g(s(t,x), \widehat{U}(t,x), W(t,x)).$$

Then (49) holds for this v. Since $g(t,x,u,v)$ is by hypothesis locally Lipschitz in (t,x), we then have that (50) holds for the feedback for F given by

$$f(t,x) = g(t,x,\widehat{U}(t,x), W(t,x)). \tag{86}$$

The proof of the lemma is then completed by directly following the proofs of Lemma 3.13 and Theorem 3.2, now with f as in (86). In accordance with this, one takes $U_\varepsilon = \widehat{U}$ as constructed above. □

One also has the following version of Lemma 4.2, where the point of view of the maximizer is taken. It follows directly from Lemma 4.2, upon letting $-V$ take the role played by V in the preceding arguments, and utilizing the proximal Hamiltonian inequality (76) in place of (75).

Lemma 4.3. *Let $M > 0$ and $t_0 < T$ be given. Then for any given $\varepsilon > 0$, there exists a feedback control W_ε for the maximizer such that the following holds: There exists a scalar $\tilde{\mu} > 0$ such that for any feedback control U for the minimizer, any initial data as in* (14) *and any partition π of $[\tau, T]$ with mesh diameter $\mathrm{diam}(\pi) \leq \tilde{\mu}$, every Euler polygonal arc x_π of the initial value problem*

$$\dot{x}(t) = g(t,x,U(t,x), W_\varepsilon(t,x)) \tag{87}$$

satisfies

$$\ell(x_\pi(T)) \geq V(\tau,\alpha) - \varepsilon. \tag{88}$$

The two preceding lemmas yield the following result.

Theorem 4.4. *Let $M > 0$ and $t_0 < T$ be given. Then the ε-saddle point condition* (71) *holds for the pair of feedback controls $(U_\varepsilon, W_\varepsilon)$, for any $(\tau,\alpha) \in [t_0,T] \times M\overline{B}_n$.*

Since every motion $x(\cdot) \in X(\tau,\alpha,U_\varepsilon,W_\varepsilon)$ satisfies

$$V(\tau,\alpha) - \varepsilon \leq \ell(x(T)) \leq V(x(T)) + \varepsilon \quad \forall\,(\tau,\alpha) \in C,$$

we have the following corollary (in which universality is not relevant):

Corollary 4.5. *For every $(\tau,\alpha) \in (-\infty,T] \times \Re^n$, the differential game $G(\tau,\alpha)$ has value equal to the Krasovskii-Subbotin value $V(\tau,\alpha)$.*

5. Relaxing hypotheses on the dynamics in §3

It is our intention here to justify our earlier Remark 2.1. We will show that Theorem 3.2 generalizes in a meaningful way if the standing hypotheses (SH) in that result are relaxed to

(SH*) (a) For each point $(t,x) \in \Re \times \Re^n$, $F(t,x)$ is a bounded subset of $\Re^n$.

(b) *Linear growth:* There exist $\gamma_1 > 0$ and γ_2 such that

$$\|v\| \leq \gamma_1 \|x\| + \gamma_2 \quad \forall\, v \in F(t,x), \quad \forall\, (t,x) \in \Re \times \Re^n.$$

(c) F is *upper semicontinuous* on $\Re \times \Re^n$; that is, given $(t,x) \in \Re \times \Re^n$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(t,x) - (t',x')\| < \delta \Longrightarrow F(t',x') \subseteq F(t,x) + \varepsilon B_{n+1}.$$

We now define a new multifunction

$$\widehat{F}(t,x) := \overline{\text{co}}[F(t,x)].$$

One can use Carathéodory's theorem in order to show that $\widehat{F}$, which is obviously compact convex valued, is also upper semicontinuous. Consider the parametrized family of optimal control problems $\{\widehat{P}(\tau,\alpha)\}$, where $(\tau,\alpha) \in (-\infty,T] \times \Re^n$, involving the minimization of $\ell(x(T))$ over all trajectories x of the differential inclusion $\dot{x}(t) \in \widehat{F}(t,x(t))$ satisfying $x(\tau) = \alpha$. Since compactness of trajectories holds for these dynamics, the minimum in each problem $\widehat{P}(\tau,\alpha)$ is attained, and we denote the associated value function by $\widehat{V}(\tau,\alpha)$. Also, if $\hat{f}$ is any feedback (i.e. selection) of $\widehat{F}$, then for any compact time interval $[\tau,T]$ and any $\alpha \in \Re^n$, there exists at least one Euler solution of the initial value problem $\dot{x} = \hat{f}(x)$ satisfying $x(\tau) = \alpha$, and it is necessarily a solution of the differential inclusion $\dot{x} \in \widehat{F}(x)$. (These facts can be found e.g. in [9].)

The generalization of Theorem 3.2 that we wish to prove is the following.

Theorem 5.1. *Suppose that the multifunction F satisfies (SH*) and that the cost functional ℓ is continuous. Let $M > 0$ and $t_0 \in (-\infty,T)$ be given. Then given $\varepsilon > 0$, there exists a feedback f_ε for F and a scalar $\tilde{\mu} > 0$ such that the following holds: Given any initial data*

$$(\tau,\alpha) \in [t_0,T] \times M\overline{B}_n \tag{89}$$

and any partition π of $[\tau,T]$ with $\text{diam}(\pi) \leq \tilde{\mu}$, every Euler polygonal arc x_π of the initial value problem

$$\dot{x}(t) = f_\varepsilon(t,x(t)), \quad x(\tau) = \alpha \tag{90}$$

satisfies

$$\ell(x_\pi(T)) \leq \widehat{V}(\tau,\alpha) + \varepsilon. \tag{91}$$

Proof: As in the proof of Theorem 3.2, we again assume $0 < t_0 < T$, and we work with a rectangle $C := [0,T] \times \Re^n$. As earlier, there exists $M_1 > 0$ such that for any feedback of $\widehat{F}$ (and therefore also of F), any initial data $(\tau,\alpha) \in C$, and any partition π of $[\tau, T]$, the Euler polygonal arc x_π generated by on $[\tau,T]$ satisfies (23). We again denote $C_1 := [0,T] \times 2M_1\overline{B}_n$ as in (24). Then for any $(\tau,\alpha) \in C$, (25) holds.

We recall a result on multifunction approximation (see Deimling [12]): Given $\delta > 0$, there exists a compact convex valued multifunction $\widehat{F}^\delta$, Lipschitz on C_1, such that

$$\widehat{F}(t,x) \subseteq \widehat{F}^\delta(t,x) \subseteq \widehat{F}(t,x) + \delta B_{n+1} \quad \forall\,(t,x) \in C_1. \tag{92}$$

Hence $\widehat{F}^\delta$ satisfies the original standing hypotheses (SH), and is an upper approximation of $\widehat{F} = \overline{\text{co}}F$ on S. We denote by $\widehat{V}^\delta$ the value function obtained when the dynamics are given by $\dot{x} \in \widehat{F}^\delta(x)$. In view of the first containment in (92), it is clear that for any $\delta > 0$ and $\widehat{F}^\delta$ as in (92), one has

$$\widehat{V}^\delta(\tau,\alpha) \le \widehat{V}(\tau,\alpha) \quad \forall\,(\tau,\alpha) \in C. \tag{93}$$

Hence, it suffices to prove a version of the theorem in which the inequality (91) is replaced by

$$\ell(x_\pi(T)) \le \widehat{V}^\delta(\tau,\alpha) + \varepsilon \tag{94}$$

for some $\delta > 0$. Observe that because $\widehat{V}^\delta$ is associated with dynamics satisfying (SH), Lemma 3.13 holds true, with the notational change that the extended lower Hamiltonian $\bar{h}_F$ be replaced by $\bar{h}_{\widehat{F}^\delta}$, and where the sublevel sets $S(a)$ are replaced by $\widehat{S}^\delta(a)$, which are those of $(\widehat{V}^\delta)^\beta$, where

$$(\widehat{V}^\delta)^\beta(\tau,\alpha) := \widehat{V}^\delta(\tau,\alpha) + \beta(T-\tau).$$

This version of the result will follow from the easily checked fact that given $\varepsilon' > 0$, one can choose $\delta > 0$ sufficiently small that for any $a \in [a_m, a_M]$, one has

$$\bar{h}_F(t,x,\eta) \le \bar{h}_{\widehat{F}^\delta}(t,x,\eta) + \varepsilon' \quad \forall\,(t,x) \in \widehat{S}^\delta(a) \cap \text{int}(C_1), \quad \forall\,\eta \in N^P_{\widehat{S}^\delta(a)}(t,x) \cap \overline{B}_n. \tag{95}$$

This leads to

$$\bar{h}_F(t,x,\eta) \le \left(-\frac{\beta}{\kappa} + \varepsilon'\right)\|\eta\| \quad \forall\,(t,x) \in \widehat{S}^\delta(a) \cap \text{int}(C_1), \quad \forall\,\eta \in N^P_{S(a)}(t,x), \tag{96}$$

which in the remainder of the proof can be used as (41) was in proving Theorem 3.2, since we can take ε' small enough that

$$-\frac{\beta}{\kappa} + \varepsilon' < 0.$$

Full details are left to the reader. □

Observe that, in analogy with Corollary 3.3, all Euler solutions on $[\tau,T]$ generated by the feedback f_ε of the original dynamics F (and such solutions exist since F satsifies linear growth) are ε-optimal trajectories of $\widehat{P}(\tau,\alpha)$ for any given initial data (τ,α) in the rectangle $[t_0,T] \times M\overline{B}_n$.

References

[1] J. P. Aubin and A. Cellina. *Differential Inclusions.* Springer–Verlag, Berlin, 1984.

[2] L. D. Berkowitz. Optimal feedback controls. *S.I.A.M. J. Control Optim.*, 27:991–1006, 1989.

[3] P. Cannarsa and H. Frankowska. Some characterizations of optimal trajectories in control theory. *S.I.A.M. J. Control Optim.*, 29:1322–1347, 1991.

[4] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions.* Lecture Notes in Mathematics, No. 580, Springer, Berlin, 1977.

[5] F. H. Clarke. *Optimization and Nonsmooth Analysis.* Wiley-Interscience, New York, 1983. Republished as Vol. 5 of Classics in Applied Mathematics, S.I.A.M., Philadelphia, 1990.

[6] F. H. Clarke. *Methods of Dynamic and Nonsmooth Optimization*, volume 57 of *CBMS-NSF Regional Conference Series in Applied Mathematics.* S.I.A.M., Philadelphia, 1989.

[7] F. H. Clarke, Yu. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *I.E.E.E. Trans. Automatic Control*, 42:1394–1407, 1997.

[8] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolenski. Qualitative properties of trajectories of control systems: a survey. *J. Dyn. Control Sys.*, 1:1–48, 1995.

[9] F.H. Clarke, Yu. S. Ledyaev, R.J. Stern, and P.R. Wolenski. *Nonsmooth Analysis and Control Theory*, Vol. 178. Springer-Verlag, Graduate Texts in Mathematics, New York, 1998.

[10] F. H. Clarke, Yu. S. Ledyaev, and A. I. Subbotin. Universal feedback control via proximal aiming in problems of control under disturbance and differential games. C.R.M. Preprint No. 2386, 1994.

[11] F. H. Clarke, Yu. S. Ledyaev, and A. I. Subbotin. Universal feedback strategies for differential games of pursuit. *S.I.A.M. J. Control Optim.*, 35:552–561, 1997.

[12] K. Deimling. *Multivalued Differential Equations.* de Gruyter, Berlin, 1992.

[13] N. N. Krasovskii and A. I. Subottin. *Game-Theoretical Control Problems.* Springer-Verlag, New York, 1988.

[14] N.N Krasovskii. Differential games, approximation and formal models. *Math. U.S.S.R. Sbornik*, 35:795–822, 1979.

[15] P. Loewen. *Optimal Control via Nonsmooth Analysis.* CRM Proceedings and Lecture Notes. Amer. Math. Soc., 1993.

[16] J. D. L. Rowland and R. B. Vinter. Construction of optimal feedback controls. *Systems Cont. Letters*, 16:357–357, 1991.

[17] E. D. Sontag. A Lyapunov-like characterization of asymptotic controllability. *S.I.A.M. J. Control Optim.*, 21:462–471, 1983.

[18] E. D. Sontag and H. J. Sussman. Nonsmooth control-Lyapunov functions. in *Proc. I.E.E.E. Conf. Decision and Control, New Orleans, Dec. 1995.*

[19] A. I. Subbotin. Generalization of the main equation of differential game theory. *J. Optim. Theor. Optim.*, 43:103–133, 1984.