Integrable and superintegrable Hamiltonian systems in magnetic fields

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Abstract

This article is devoted to the construction of integrable and superintegrable two-dimensional Hamiltonian systems with scalar and vector potentials. All integrable systems with a quadratic polar coordinate type integral of motion are found. Classical trajectories are calculated in integrable cases and compared with those for a system that is not integrable.

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Résumé

Cet article se consacre à la construction de systèmes hamiltoniens bidimensionnels intégrables et super-intégrables à potentiels scalaire et vectoriel. On y détermine tous les systèmes intégrables comportant une intégrale de mouvement de type polaire. On calcule aussi certaines trajectoires classiques dans les cas intégrables et on les compare avec les trajectoires obtenues pour un système non intégrable.

I Introduction

A classical result, due to Bertrand [1], states that the only central potentials in which all finite trajectories are closed are the Kepler potential $V = \frac{\alpha}{r}$ and the harmonic oscillator $V = \alpha r^2$. These two physical systems have many other interesting properties valid also in n-dimensional space for any finite n. The corresponding Hamiltonian systems are not only completely integrable, in the Liouville sense [2], they are both "maximally superintegrable". That means that they possess not only n integrals of motion in involution, but rather $2n-1$ integrals, amongst which it is possible to choose different subsets of n in involution. The corresponding quantum systems have degenerate energy levels. The degeneracy has been called "accidental", in that it goes beyond that explained by rotational invariance in any central potential. The integrals of motion in classical mechanics form Lie algebras: $o(n+1)$ in the case of the Kepler potential [3, 4] (or hydrogen atom) and $su(n)$ in the case of the harmonic oscillator [5]. In both cases, there is an $o(n)$ subalgebra, realized by first-order operators corresponding to the "geometric" rotational symmetry. The other operators, completing the algebra to $o(n+1)$, or $su(n)$, respectively are second order ones. This implies that the corresponding symmetry transformations are not point ones: "higher" symmetries are involved.

Both of the above systems are not only superintegrable, they are also separable in at least two coordinate systems in configuration space. That is, the corresponding Hamilton-Jacobi and Schrödinger equations can be separated in more than one coordinate system.

A systematic search for the Hamiltonian systems with higher symmetries was initiated some time ago [6, 7, 8, 9, 10] for space dimensions $n = 2$ and $n = 3$. The restriction to spherically symmetric systems was dropped, the integrals of motion were assumed to be second order polynomials in the momenta, or second order linear operators in the quantum case.

The results of the study can be summed up as follows. All integrable systems in two- and three-dimensional real Euclidean spaces with second order integrals of motion allow the separation of variables in at least one of the coordinate systems in which the free Schrödinger equation (or free Hamilton-Jacobi equation) allows separation. All superintegrable systems allow the separation of variables in at least two such coordinate systems.

Integrable Hamiltonian systems with velocity dependent potentials have also been studied [11] for $n = 2$, i.e. in a Euclidean plane. The classical Hamiltonian was assumed to have the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x,y)p_x + B(x,y)p_y + W(x,y), \tag{1.1}$$

and the corresponding integral of motion by assumption was

$$C = g_0\dot{x}^2 + g_1\dot{x}\dot{y} + g_2\dot{y}^2 + k_0\dot{x} + k_1\dot{y} + h, \tag{1.2}$$

where g_0, g_1, g_2, k_0, k_1 and h are functions of x and y. They are determined from the requirement that C be constant on solutions of the equations of motion corresponding to the Hamiltonian (1.1). For any Hamiltonian of the form (1.1), the condition

$$\frac{dC}{dt} = 0 \tag{1.3}$$

implies that the integral C will have the form

$$\begin{aligned} C =& \alpha(x\dot{y} - y\dot{x})^2 + \beta\dot{x}(x\dot{y} - y\dot{x}) + \gamma\dot{y}(x\dot{y} - y\dot{x}) \\ &+ \delta\dot{x}^2 + \zeta\dot{y}^2 + \xi\dot{x}\dot{y} + k_0(x,y)\dot{x} + k_1(x,y)\dot{y} + h(x,y), \end{aligned} \tag{1.4}$$

where α, β, γ, δ, ζ and ξ are real constants. The dots denote time derivatives. The quadratic part of the invariant C can thus be interpreted as a second order element in the Euclidean Lie algebra $e(2)$, with basis elements

$$\begin{aligned} L_3 &= (x\dot{y} - y\dot{x}) && \sim y\partial x - x\partial y, \\ P_1 &= \dot{x} && \sim \partial x, \\ P_2 &= \dot{y} && \sim \partial y. \end{aligned} \tag{1.5}$$

Euclidean transformations of the plane will change the potentials W, A and B, but leave the form of the Hamiltonian (1.1) invariant. The integral C can be simplified by these transformations and taken into one of the following standard forms:

$$C_C = \dot{x}^2 + k_0\dot{x} + k_1\dot{y} + h, \tag{1.6}$$

$$C_R = (x\dot{y} - y\dot{x})^2 + k_0\dot{x} + k_1\dot{y} + h, \tag{1.7}$$

$$C_P = \dot{x}(x\dot{y} - y\dot{x}) + k_0\dot{x} + k_1\dot{y} + h, \tag{1.8}$$

$$C_E = (x\dot{y} - y\dot{x})^2 + \sigma(\dot{x}^2 - \dot{y}^2) + k_0\dot{x} + k_1\dot{y} + h, \tag{1.9}$$

where k_0, k_1 and h are functions of x and y, and σ is a constant (related to the focal distance in elliptic coordinates). For purely scalar potentials, we have $A = B = 0$ and $k_0 = k_1 = 0$. The existence of the integrals C_C, C_R, C_P and C_E then implies the separation of variables in the Hamilton-Jacobi equation (and in the Schrödinger equation) for the corresponding Hamiltonian, in Cartesian, polar, parabolic or elliptic coordinates, respectively [6, 7].

The case of the "Cartesian integral" (1.6) in the presence of a magnetic field was studied earlier [11]. The potentials were completely determined in terms of two functions, $f(x)$ ang $g(y)$ that are either elliptic functions, or the elementary ones arising as special cases of elliptic functions. The equations of motion no longer separate, but all the attributes of integrability remain.

The purpose of this article is to analyse the integrable Hamiltonian system (1.1) and (1.7), i.e. the case of a second order integral of motion of the "polar type". The corresponding scalar and vector potentials are found in section II, together with the integral C_R. The possibility of superintegrability is studied in section III, where it is shown that the only system allowing simultaneously a Cartesian and a polar integral of motion is that of a particle in a constant magnetic field. Section IV is devoted to analytical and numerical solutions of the equations of motion. Some conclusions are drawn in section V.

II The polar integrable system

The equations of motion corresponding to the Hamiltonian (1.1) can be written in the form

$$\begin{aligned} \ddot{x} &= -V_x + \Omega\dot{y}, \\ \ddot{y} &= -V_y - \Omega\dot{x}, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} V(x,y) &= W - \frac{1}{2}(A^2 + B^2), \\ \Omega(x,y) &= A_y - B_x. \end{aligned} \tag{2.2}$$

Notice that the equations of motion (2.1) are invariant with respect to a gauge transformation of the potentials

$$\begin{aligned} \tilde{W} &= W + (\vec{A}, \vec{\nabla}\Phi) + \frac{1}{2}(\vec{\nabla}\Phi)^2, \\ \tilde{\vec{A}} &= \vec{A} + \vec{\nabla}\Phi, \end{aligned} \tag{2.3}$$

where we have put $\vec{A} \equiv (A, B)$ and $\Phi(x, y)$ is an arbitrary function.

Now, let us assume that the Hamiltonian (1.1) allows a "polar integral of motion" C_R of the form (1.7). We wish to determine the physical quantities $V(x, y)$, $\Omega(x, y)$ and $k_0(x, y)$, $k_1(x, y)$, $h(x, y)$ that arise in this case.

Let us first transform to polar coordinates: $x = r\cos\phi$, $y = r\sin\phi$. The integral C_R is transformed into

$$C_R = r^4\dot{\phi}^2 + P(r, \phi)\dot{r} + Q(r, \phi)r\dot{\phi} + h(r, \phi), \tag{2.4}$$

$$P = k_0\cos\phi + k_1\sin\phi, \quad Q = -k_0\sin\phi + k_1\cos\phi. \tag{2.5}$$

The equations of motion (2.1) in polar coordinates are

$$\ddot{r} - r\dot{\phi}^2 = -V_r + \Omega r\dot{\phi}, \tag{2.6}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\frac{1}{r}V_\phi - \Omega\dot{r}. \tag{2.7}$$

The condition that C_R be an integral of motion is imposed by differentiating C_R with respect to time and replacing $\ddot{r}$ and $\ddot{\phi}$ using eqs. (2.6) and (2.7). Setting the coefficients of $\dot{\phi}^2$, $\dot{r}^2$, $\dot{r}$, $\dot{r}\dot{\phi}$, $\dot{\phi}$, $\dot{r}$ and 1 to zero, we obtain an overdetermined system of equations for the functions V, Ω, P, Q and h. These determining equations are

$$P_r = 0, \quad P + Q_\phi = 0, \tag{2.8}$$

$$2r^3\Omega - P_\phi - rQ_r + Q = 0, \tag{2.9}$$

$$h_\phi - 2r^2V_\phi + rP\Omega = 0, \quad h_r - Q\Omega = 0, \tag{2.10}$$

$$PV_r + \frac{1}{r}QV_\phi = 0. \tag{2.11}$$

From eq. (2.8), we obtain

$$P = -f'(\phi), \quad Q = f(\phi) + R(r), \tag{2.12}$$

where f and R are so far arbitrary. Eq. (2.11) can now be solved by the method of characteristics and we have

$$\begin{gathered} V = V(\xi), \quad \xi = rf(\phi) + \psi(r), \\ \psi_r = R(r), \quad (P, Q) \neq (0, 0). \end{gathered} \tag{2.13}$$

We mention that the case of a purely scalar potential is recovered if we set $\Omega = P = Q = 0$. Then eqs. (2.8), (2.9) and (2.11) are satisfied trivially and eq. (2.10) implies

$$V = V_0(r) + \frac{1}{r^2}h(\phi), \tag{2.14}$$

i.e. we recover a separable potential.

In the following, we will assume $\Omega \neq 0$. From (2.9), we have

$$\Omega = -\frac{1}{2r^3}(f_{\phi\phi} + f + \psi_r - r\psi). \tag{2.15}$$

The remaining two equations (2.10) are compatible only if the potential $V(\xi)$ satisfies

$$V_{\xi\xi} + \frac{3}{rf}V_\xi + \frac{1}{4r^6ff'}\Big(ff''' + 3f'f'' + 4ff' + f'''\dot{\psi} + f'(r^2\dddot{\psi} - 3r\ddot{\psi} + 4\dot{\psi})\Big) = 0, \tag{2.16}$$

where primes denote derivatives with respect to ϕ, dots with respect to r. Eq. (2.16) requires that rf be a function of ξ. This is possible in two cases only. Let us investigate each separately.

The first possibility is $\dot{\psi} = 0$. We obtain

$$\xi = rf(\phi), \quad P = -f'(\phi), \quad Q = f(\phi), \tag{2.17}$$

$$\Omega = -\frac{1}{2r^3}(f'' + f), \tag{2.18}$$

$$V = \frac{\alpha}{\xi^4} + \frac{\beta}{\xi^2}, \tag{2.19}$$

where α and β are constants. Substituting eq. (2.19) into (2.16), we find that $f(\phi)$ must satisfy the following equation:

$$ff''' + 3f'f'' + 4ff' + \frac{32\alpha f'}{f^5} = 0. \tag{2.20}$$

Eq. (2.20) can be integrated twice to obtain

$$f'^2 + f^2 + \frac{8\alpha}{f^4} + \frac{K}{f^2} + C = 0, \tag{2.21}$$

where K and C are constants.

The function $h(r, \phi)$ is obtained from eq. (2.9) and is

$$h = \frac{1}{4r^2}f(f'' + f) + \frac{2\beta}{f^2}. \tag{2.22}$$

Thus, V, Ω, P, Q and h are all expressed in terms of the function $f(\phi)$, itself satisfying eq. (2.21). This equation is a first order differential equation with constant coefficients and is thus already reduced to a quadrature, expressible in terms of elliptic integrals.

Here, we just discuss some special cases when $f(\phi)$ is an elementary function.

a. $\alpha = 0$.

$$f = (a + b\cos 2\phi)^{\frac{1}{2}}. \tag{2.23}$$

The constants a and b are related to K and C. A second solution, $f = a\sin\phi$, is of no interest since it implies $\Omega = 0$.

b. $\alpha \neq 0$.

$$f = \sqrt{(-8\alpha)}\cos^{\frac{1}{3}}\phi, \qquad \alpha < 0. \tag{2.24}$$

This corresponds to $K = C = 0$ in eq. (2.21).

The second possibility of satisfying eq. (2.16) is to have $f'(\phi) = 0$ and hence $\xi = \xi(r)$, yielding

$$\begin{aligned} V = V(r), \quad \Omega = \frac{rQ_r - Q}{2r^3}, \\ P = 0, \quad Q = Q(r), \quad h = \frac{Q^2}{4r^2}. \end{aligned} \tag{2.25}$$

Thus both V and Ω depend on r alone. In this case, a first order integral also exists, namely

$$C_{1,R} = r^2\dot{\phi} + H(r), \quad V = V(r), \quad \Omega = \frac{H_r}{r}, \tag{2.26}$$

The second order integral C_R in this case is simply

$$C_R = (C_{1,R})^2, \quad Q = 2Hr. \tag{2.27}$$

III Superintegrability in a magnetic field

The Hamiltonian system with Hamiltonian (1.1) will be superintegrable if it allows at least two functionally independent integrals of motion. Here, we will restrict ourselves to systems that are not only superintegrable, but actually "doubly separable": separable in at least two coordinate systems. This means that both integrals are at most quadratic in the velocities. We shall moreover require that separation should occur in Cartesian and polar coordinates.

In the scalar case ($\Omega = 0$, i.e. no magnetic field), there is such a superintegrable system, namely

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \alpha(x^2 + y^2) + \frac{\beta}{x^2} + \frac{\gamma}{y^2} \tag{3.1}$$

We shall now require $\Omega \neq 0$ and that a Cartesian and a polar invariant exist. With no loss of generality, we can take the polar invariant C_R in the standard form (1.7) with V, Ω, P, Q and h as in (2.17), (2.18), (2.19) or as in (2.25). The Cartesian invariant can also be chosen in the standard form (the rotation needed for the standardiztion will not change C_R):

$$C_C = \frac{1}{2}\dot{x}^2 + k_0(x,y)\dot{x} + k_1(x,y)\dot{y} + h(x,y). \tag{3.2}$$

In the Cartesian case [7], all physical quantities are expressed in terms of the two functions $f(x)$ and $g(y)$, satisfying

$$f_{xx} = \alpha f^2 + \beta f + \gamma, \quad g_{yy} = -\alpha g^2 + \delta g + \xi. \tag{3.3}$$

In particular, we have

$$\begin{aligned} \Omega &= \alpha(f^2 - g^2) + \beta f + \delta g + \gamma + \xi, \\ V &= \frac{\alpha}{3}(g-f)^3 + \frac{\beta+\delta}{2}(g-f)^2 + (\gamma + \kappa - \xi)(g-f), \end{aligned} \tag{3.4}$$

where all greek letters denote constants. The question now is: when are equations (3.4) compatible with the existence of a polar invariant C_R ?

Let us first consider Ω and V as in eqs. (2.18) and (2.19). We must have

$$\frac{\partial}{\partial r} r^3 \Omega = 0. \tag{3.5}$$

This requires that f and g be constant, hence Ω and V are constant. However, if Ω is constant in eq. (2.18), it must vanish—i.e. we have $\Omega = 0$—which we are not interested in.

Now, let us consider the case $V = V(r)$ and $\Omega = \Omega(r)$ and require that a Cartesian invariant should exist, in addition to the polar one. From equations (3.3) and (3.4), we obtain in this case that both $V(r)$ and $\Omega(r)$ are constant.

The result is that the only superintegrable system in a magnetic field with a polar and Cartesian invariant is that of a zero scalar potential and a constant magnetic field:

$$V = 0, \quad \Omega = \Omega_0 \neq 0. \tag{3.6}$$

It is easy to verify that in this case, three invariants exist that are linear in the momenta, namely

$$C_1 = \dot{x} - \Omega y, \quad C_2 = \dot{y} + \Omega x, \quad C_3 = y\dot{x} - x\dot{y} - \frac{1}{2}\Omega(x^2 + y^2). \tag{3.7}$$

Out of these, we can form all quadratic integrals that exist in this highly degenerate, but physically important case. Thus, we have

$$C_C = C_1^2, \quad C_R = C_3^2, \quad C_P = C_1C_3, \quad C_E = C_3^2 + \sigma(C_1^2 - C_2^2). \tag{3.8}$$

We mention that the original Hamiltonian (1.1) corresponding to $V = 0$, $\Omega = const$ can be written in the standard form

$$H = \frac{1}{2}(p_x + \Omega y)^2 + \frac{1}{2}p_y^2. \tag{3.9}$$

In the quantum mechanical case, we obtain three first order operators commuting with the quantum Hamiltonian corresponding to H of eq. (3.9), namely

$$P_1 = -i\partial x, \quad P_2 = -i\partial y, \quad L_3 = -i(y\partial x - x\partial y) - \frac{1}{2}\Omega(x^2 + y^2). \tag{3.10}$$

They generate a Lie algebra isomorphic to the Galilei algebra with a central extension

$$[L_3, P_1] = iP_2, \quad [L_3, P_2] = -iP_1, \quad [P_1, P_2] = -i\Omega. \tag{3.11}$$

While the case of a classical, or quantum mechanical particle in a constant magnetic field is well studied, its superintegrability, to our knowledge has not been noticed. The Schrödinger equation obviously separates in Cartesian coordinates. Less obviously, it also "R-separates" [12, 13] in polar coordinates, that is the wave function can be written as

$$\psi(r, \phi) = R(r, \phi)A(r)B(\phi) \tag{3.12}$$

where R is an overall multiplier that does not depend on the separation constants. More specifically, we have

$$\psi(r, \phi) = e^{-\frac{i}{4}\Omega r^2 \sin 2\phi} J_m(kr)e^{im\phi}, \quad k^2 = 2E + m\Omega \tag{3.13}$$

where E is the energy, and $J_m(z)$ a Bessel function.

IV Examples of solutions

IV.1 The superintegrable case

We have $V = 0$, $\Omega = const$. We integrate by setting two integrals C_1 and C_2 of eq. (3.7) equal to constants:

$$\dot{x} - \Omega y = -\Omega y_0, \quad \dot{y} + \Omega x = \Omega x_0. \tag{4.1}$$

The solution is

$$x = A\sin(\Omega t + \phi) + x_0, \quad y = A\cos(\Omega t + \phi) + y_0, \tag{4.2}$$

where A, ϕ, x_0 and y_0 are integration constants. We have of course obtained a well known result: the trajectories are circles.

IV.2 Rotationally symmetric case

We have $V = V(r)$, $\Omega = \Omega(r)$. The first integral (2.26) gives us the general solution of eq. (2.7):

$$r^2\dot{\phi} + H(r) = C, \quad \Omega = \frac{H_r}{r}. \tag{4.3}$$

Substituting $\dot{\phi}$ from eq. (4.3) into eq. (2.6), we can integrate once to obtain:

$$\dot{r}^2 + 2V + \frac{(C-H)^2}{r^2} = K, \quad \Omega = \frac{H_r}{r}, \tag{4.4}$$

where C and K are constants. Since there is no explicit t dependence, eq. (4.4) is integrated in quadratures. For instance, if we consider a potential and magnetic field of the form

$$\begin{aligned} V &= \frac{\gamma_1}{r^2} + \gamma_2 r^2 + \gamma_3 r^4 + \gamma_4 r^6, \\ \Omega &= \Omega_0 + \Omega_1 r^2, \end{aligned} \tag{4.5}$$

where γ_i and Ω_j are constants, we can express r in terms of elliptic functions, or elementary ones in special cases.

The special case $V = \frac{1}{2}\omega^2(x^2+y^2)$, $\Omega = const$ was considered earlier [11]. It can easily be solved in Cartesian coordinates. Equations (2.1) in this case yield

$$\begin{aligned} x &= \frac{\omega^2 - \alpha_1^2}{\alpha_1 \Omega} A\cos(\alpha_1 t + \phi_1) + \frac{\omega^2 - \alpha_2^2}{\alpha_2 \Omega} B\cos(\alpha_2 t + \phi_2), \\ y &= A\sin(\alpha_1 t + \phi_1) + B\sin(\alpha_2 t + \phi_2), \end{aligned} \tag{4.6}$$

$$\alpha_{1,2}^2 = \frac{2\omega^2 + \Omega^2 \pm \sqrt{(2\omega^2+\Omega^2)^2 - 4\omega^4}}{2}, \tag{4.7}$$

where A, B, ϕ_1 and ϕ_2 are arbitrary constants. Thus the motion is always bounded and quasiperiodic. It is periodic if the ratio α_1/α_2 is rational.

Two such periodic cases are shown on Fig. 1, with $\alpha_1/\alpha_2 = 1/2$ and $\alpha_1/\alpha_2 = 1/50$, respectively.

IV.3 Potential and magnetic field with azimuthal dependence

Let us consider the magnetic field and potential of eqs. (2.17) and (2.18), with the function $f(\phi)$ as in eq. (2.23). We then have

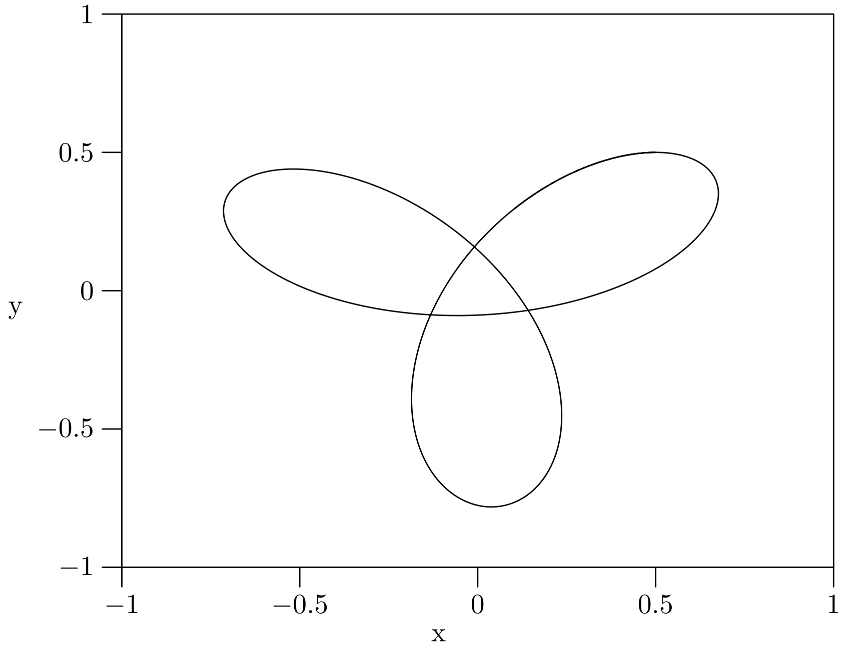
$$V = \frac{\beta}{r^2(a + b\cos 2\phi)}, \quad \Omega = \frac{b^2 - a^2}{r^3(a+b\cos 2\phi)^{3/2}}. \tag{4.8}$$

The trajectories of a particle in this case will in general not be closed. For $|b| > |a|$, the potential V and the field Ω only have a point singularity at $r = 0$ (not along the line $\cos 2\phi = -\frac{a}{b}$.

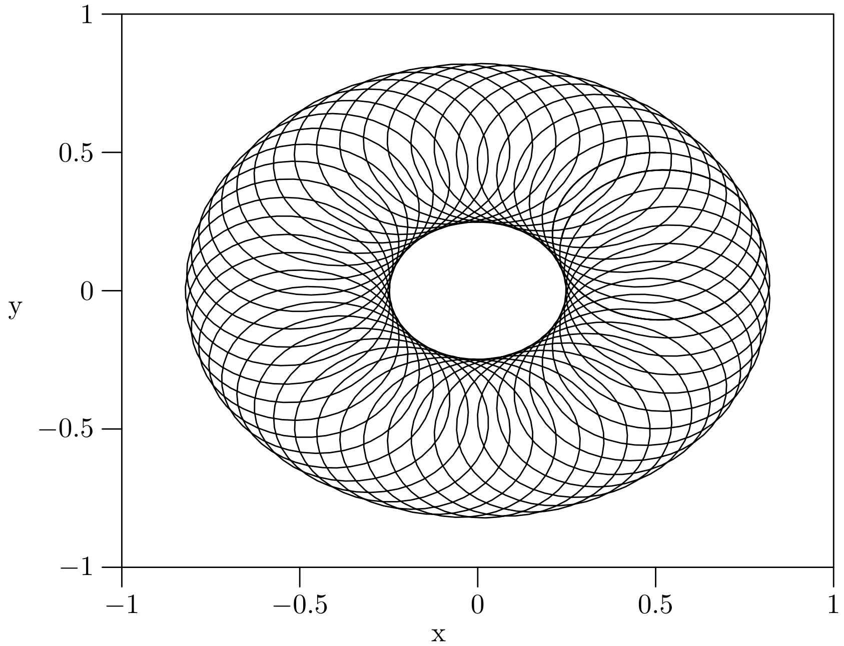
The integral of motion in this case is

$$\begin{aligned} C =& r^4\dot{\phi}^2 + \frac{b\sin 2\phi}{(a+b\cos 2\phi)^{1/2}}\dot{r} + (a+b\cos 2\phi)^{1/2} r\dot{\phi} \\ &+ \frac{1}{(a+b\cos 2\phi)}\left(\frac{a^2-b^2}{4r^2} + 2\beta\right) \end{aligned} \tag{4.9}$$

An example of a trajectory in this integrable case is shown on Fig. 2(a). For comparison, we show a trajectory in a nonintegrable case on Fig. 2(b). We see that, while the difference in the formulas seems slight (namely, r^{-3} in Ω is replaced by r^{-4}) the trajectory becomes quite irregular.

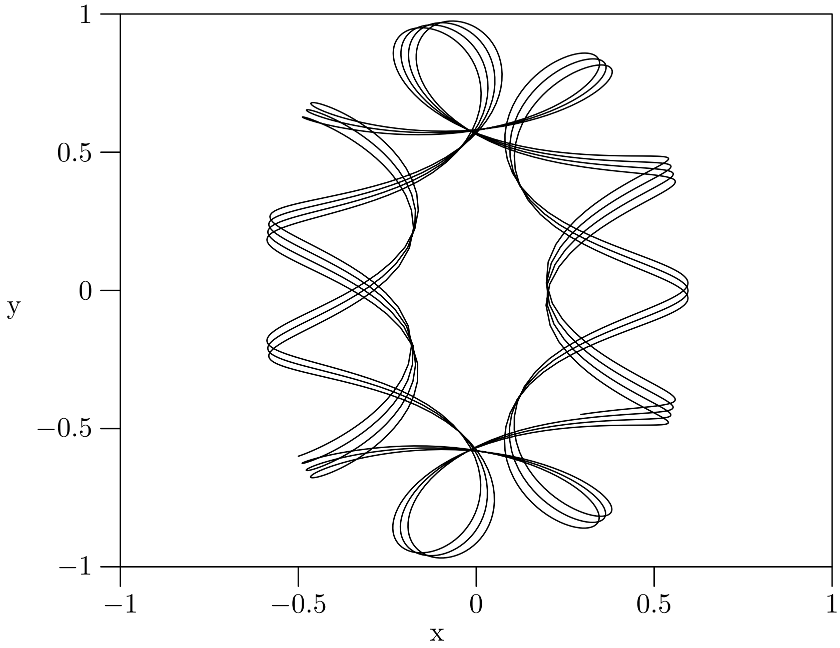


(a) $V = \frac{1}{2}r^2$, $\Omega = \frac{1}{\sqrt{2}}$, $\frac{\alpha_1}{\alpha_2} = \frac{1}{2}$

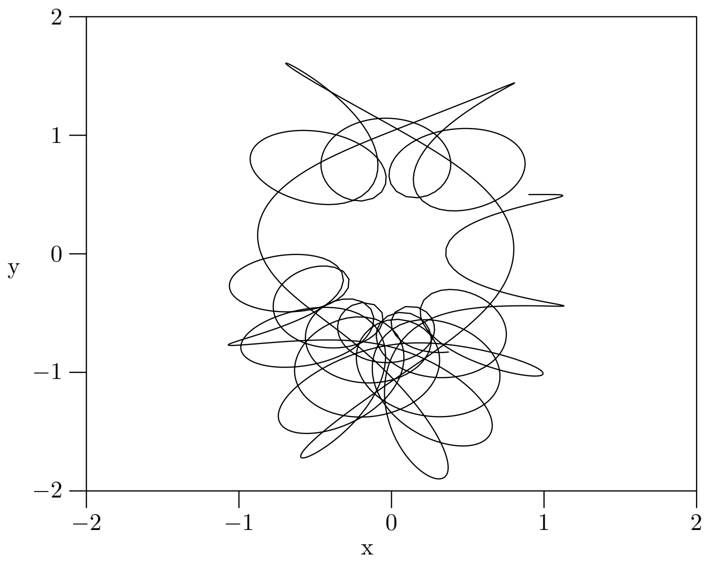


(b) $V = \frac{1}{2}r^2$, $\Omega = \frac{49}{\sqrt{50}}$, $\frac{\alpha_1}{\alpha_2} = \frac{1}{50}$

Figure 1: Trajectories in an integrable rotationally symmetric case.



(a) Integrable case:$V = \frac{\beta}{r^2(a+b\cos 2\phi)}$, $\Omega = \frac{b^2-a^2}{r^3(a+b\cos 2\phi)^{3/2}}$



(b) A non-integrable case:
$V = \frac{\beta}{r^2(a+b\cos 2\phi)}$, $\Omega = \frac{b^2-a^2}{r^4(a+b\cos 2\phi)^{3/2}}$

Figure 2: Trajectories in potentials with angular dependence.

V Conclusions

A very sizable literature exists on integrable and super-integrable finite-dimensional systems of the form (1.1) with a purely scalar potential, i.e. with A = B = 0. For recent reviews containing numerous references, see e.g. ref. [14, 15, 16].

A systematic search for such superintegrable systems in two- and three-dimensional Euclidean spaces has been conducted some time ago [6, 7, 8, 9, 10]. A more recent series of articles is devoted to superintegrable systems in space of constant curvature [17, 18, 19]. The emphasis is on special function aspects of these systems.

The obtained systems have been analyzed using algebraic techniques originally developed for the hydrogen atom [20, 21, 22, 23]. A different approach to these systems makes use of path integrals [14, 24].

In addition to being explixitly solvable, many superintegrable systems occur in applications. They include the ring-shaped Hartmann potential used in quantum chemistry [25, 26, 27], the Aharonov-Bohm potential [28], the Calogero-Moser system [29] and many others. Interesting mathematical objects arise in these studies, including quadratic algebras [30] and twisted Kac-Moody algebras [31, 32].

Systems invloving vector potenitals have been studied to a much lesser degree. A systematic search for integrable systems with vector potentials in two dimensions was initiated in ref. [11]. This article is a continuation of that search and, to our knowledge, the first in which the question of superintegrability is posed. The study of "parabolic" and "elliptic" integrals of motion is postponed to a future article. Further questions under study include a search for superintegrable velocity-dependent systems in 3 and, more generally, n dimensions.

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References

1. J. Bertrand, Comptes Rendus **77**, 849 (1873).
2. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, MA, 1980.
3. V. Bargmann, Z. Physik **99**, 576 (1936).
4. V. Fock, Z. Physik **98**, 145 (1935).
5. J. Jauch and E. Hill, Phys. Rev. **57**, 641 (1940).
6. I. Friš, V. Mandrosov, J. Smorodinsky, M. Uhliř, and P. Winternitz, Phys. Lett. **16**, 354 (1965).
7. P. Winternitz, Y. A. Smorodinsky, M. Uhliř, and I. Friš, Yad. Fiz. **4**, 625 (1966).
8. A. Makarov, Y. A. Smorodinsky, K. Valiev, and P. Winternitz, Nuovo Cim. **A52**, 158 (1967).
9. N. Evans, Phys. Rev. **A41**, 5666 (1990).
10. N. Evans, J. Math. Phys. **32**, 3369 (1991).
11. B. Dorizzi, B. Grammaticos, A. Ramani, and P. Winternitz, J. Math. Phys. **26**, 3070 (1985).
12. E. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Longman, Essex, 1986.
13. W. M. Jr, *Symmetry and Separation of Variables*, Addison-Wesley, Reading, MA, 1977.
14. C. Grosche, *Path Integrals, Hyperbolic Spaces and Selberg Trace Formulae*, World Scientific, Singapore, 1996.
15. J. Harnad, Isospectral flow and Liouville-Arnold integration in loop algebras, in *Geometric and Quantum Methods in Integrable Systems*, volume 424 of *Springer Lecture Notes in Physics*, Springer, Berlin, 1993.
16. A. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser Verlag, Basel, 1990.
17. E. Kalnins, W. M. Jr, and G. Pogosyan, J. Math. Phys. **37**, 6439 (1996).
18. E. Kalnins, G. Williams, W. M. Jr, and G. Pogosyan, J. Math. Phys. **40**, 708 (1999).
19. E. Kalnins, W. M. Jr, Y. M. Hakobian, and G. Pogosyan, J. Math. Phys. **40**, 2291 (1999).
20. A. Barut and H. Kleinert, Phys. Rev. **156**, 1541 (1967).
21. C. Fronsdal, Phys. Rev. **156**, 1665 (1967).
22. A. Frank and B. Wolf, J. Math. Phys. **25**, 973 (1983).
23. A. Barut, A. Inomata, and R. Wilson, J. Phys. **A20**, 4075 (1987).
24. C. Grosche, G. Pogosyan, and A. Sissakian, Fortichr. Phys. **43**, 453 (1995).
25. H. Hartmann, Theoret. Chim. Acta. **24**, 201 (1972).

26. M. Kibler and P. Winternitz, J. Phys. A. Math. Gen. **20**, 4097 (1987).

27. M. Kibler, G. Lamot, and P. Winternitz, Int. J. Quant. Chem. **43**, 338 (1990).

28. M. Kibler and T. Negadi, Phys. Lett. **A124**, 42 (1987).

29. S. Wojciechowski, Phys. Lett. **A95**, 279 (1983).

30. P. Létourneau and L. Vinet, Ann. Phys. **243**, 144 (1995).

31. J. Daboul, P. Slodowy, and C. Daboul, Phys. Lett. **B317**, 321 (1993).

32. C. Daboul and J. Daboul, Phys. Lett. **B425**, 135 (1998).