Lie group classification of second-order ordinary difference equations

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Abstract

A group classification of invariant difference models, i.e. difference equations and meshes, is presented. In the continuous limit the results go over into Lie's classification of second order ordinary differential equations. The discrete model is a three point one and we show that it can be invariant under Lie groups of dimension $0 \leq n \leq 6$.

Résumé

Une classification de schémas aux différences finies est présentées, c.-à-d. une classification des équations aux différences finies et des réseaux correspondants. Dans la limite continue les résultats coïncident avec la classification des équations différentielles ordinaires de second ordre due à S. Lie. Les modèles discrets considérés sont des modèles à 3 points sur le réseau et nous montrons que leur groupe de symétrie est de dimension n avec $0 \leq n \leq 6$.

1 Introduction

Lie group theory started out as a theory of transformations of solutions of sets of differential equations [1, 2, 3, 4]. Over the years it has developed into a powerful tool for classifying differential equations and for solving them. These aspects of Lie group theory have been described in many books and lecture notes [5, 6, 7, 8, 9, 10].

Applications of Lie group theory to difference equations are much more recent [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. Essentially there are two different points of view that have been adopted when studying continuous symmetries of equations involving discrete or discretely varying independent variables.

One point of view is that a difference equation is a priori given on some fixed lattice and the task is to determine a group of transformations, leaving the solution set invariant. Different approaches differ in their treatment of independent variables, and in the assumed form of the global and infinitesimal transformations considered. In any case, the distinction between point symmetries and generalized symmetries becomes blurred. In order to obtain symmetries that go into dilations, rotations, or Lorentz transformations in the continuous limit, it is necessary to significantly modify the Lie techniques used in the continuous case [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

An alternative point of view [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34] is to pose the question: How does one discretize a differential equation while preserving all of its Lie point symmetries? Here one starts from a differential equation and finds its Lie point symmetries, using well known techniques [5, 6, 7, 8, 9, 10]. Thus a symmetry group G and its Lie algebra are a priori given. One then looks for a difference scheme, i.e. a difference equation and a mesh that have the same symmetry group and the same symmetry algebra L. In particular the Lie algebra L is realized by the same vector fields in the continuous and in the discrete case.

In this article we adopt the second point of view. We start out from Lie's classification of second order ordinary differential equations (ODEs) according to their point symmetries. Our aim is to provide a similar classification of second order difference schemes.

Thus, Lie considered equations of the form

$$E(x, y, y', y'') = 0, \qquad \frac{\partial E}{\partial y''} \neq 0, \tag{1.1}$$

where E is an arbitrary sufficiently smooth function. The Lie algebra L of the symmetry group G of eq. (1.1) is realized by vector fields of the form

$$X_\alpha = \xi_\alpha(x, y)\frac{\partial}{\partial x} + \eta_\alpha(x, y)\frac{\partial}{\partial y}, \qquad \alpha = 1, \ldots, n. \tag{1.2}$$

Lie showed [1,2] that the Lie algebra of eq. (1.1) can be of dimension $n = \dim L = 0, 1, 2, 3,$ or 8. Moreover, he classified equations with point symmetries into equivalence classes under the action of the infinite dimensional group $\mathrm{Diff}(2, \mathbb{C})$ of all local diffeomorphisms of a complex plane $\mathbb{C}^2$ (his analysis was over the field $\mathbb{C}$ of complex numbers). For each equivalence class he chose a simple representative equation and gave the corresponding realization of L. He showed that all equations with dim L ≥ 2 can be integrated in quadratures.

We shall provide a similar classification of discrete models of second order ODEs. We restrict ourselves to three point stencil, as shown in Fig. 1.
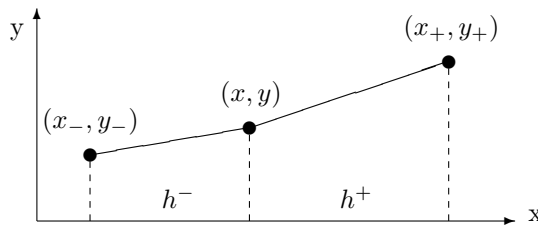

Figure 1: Elementary stencil for 3 point difference equation

Thus, we are considering a six-dimensional subspace $(x, x_-, x_+, y, y_-, y_+)$ of the space of independent and dependent variables. The discrete model under consideration can be presented in terms of a pair of difference equations

$$\begin{cases} F(x, x_-, x_+, y, y_-, y_+) = 0, \\ \Omega(x, x_-, x_+, y, y_-, y_+) = 0, \end{cases} \tag{1.3}$$

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such that

$$\det \frac{\partial(F,\Omega)}{\partial(x_+,y_+)} \neq 0, \quad \det \frac{\partial(F,\Omega)}{\partial(x_-,y_-)} \neq 0. \tag{1.4}$$

The conditions on the model are

1. In the continuous limit $h_+ = x_+ - x \to 0$ and $h_- = x - x_- \to 0$ the first equation should go into a second order ODE of the form (1.1). The second equation gives the lattice on which the first equation is considered. In the continuous limit the second equation becomes an identity $(0 = 0)$.

2. The difference system (1.3)–(1.4) is invariant under the same group as the ODE (1.1). That is, the difference scheme (1.3)–(1.4) and the ODE (1.1) are annihilated by the appropriate prolongations (see below) of the same vector fields (1.2).

Lie's classification of ODEs was performed over the field of complex numbers $\mathbb{C}$. He made use of his classification of all finite-dimensional Lie algebras that can be realized in terms of vector fields of the form (1.2). Thus, he classified all finite-dimensional subalgebras $L \subset \mathrm{diff}(2,\mathbb{C})$ [3].

Our classification of the difference models will be over $\mathbb{R}$; that over $\mathbb{C}$ will be obtained as a byproduct. We shall make use of a much more recent classification of finite dimensional subalgebras of $\mathrm{diff}(2,\mathbb{R})$ [37].

The article is organized as follows. In Section 2 we present the general theory and outline the classification method. Difference models invariant under 1 and 2–dimensional groups are analyzed in Section 3. Section 4 is devoted to groups of dimension 3. The dimensions $6 \leq n \leq 8$ are discussed in Section 5. Section 6 is devoted to the free particle equation $\ddot{y} = 0$ and its discretization. Finally, the results are summed up in two tables and the conclusions are presented in Section 7.

2 Construction of invariant difference schemes

As stated in the Introduction, we wish to construct all three-point difference schemes invariant under some transformation group. The Lie algebra of this group is realized by vector fields of the form (1.2), i.e. the Lie algebra, and the group action is the same as for differential equations.

An essential tool for studying Lie symmetries is prolongation theory. For a second order ODE we must prolong the action of a vector field (1.2) from the space (x,y) of independent and dependent variables to a four–dimensional space (x, y, y', y''). The prolongation formula for vector fields is [5, 6, 7, 8, 9, 10]:

$$\mathrm{pr}^{(2)} X_\alpha = \xi_\alpha(x,y)\frac{\partial}{\partial x} + \eta_\alpha(x,y)\frac{\partial}{\partial y} + \eta_\alpha^1(x,y,y')\frac{\partial}{\partial y'} + \eta_\alpha^2(x,y,y',y'')\frac{\partial}{\partial y''} \tag{2.1}$$

with

$$\eta_\alpha^1 = D_x(\eta_\alpha(x,y)) - y' D_x(\xi_\alpha(x,y)), \qquad \eta_\alpha^2 = D_x(\eta_\alpha^1) - y'' D_x(\xi_\alpha(x,y)), \tag{2.2}$$

where D_x is the total differentiation operator.

In the discrete case we prolong the operators X_α to a six–dimensional space $(x, x_-, x_+, y, y_-, y_+)$. The prolongation formula is

$$\mathrm{pr}^{(2)} X_\alpha = X_\alpha + \xi_\alpha(x_-,y_-)\frac{\partial}{\partial x_-} + \xi_\alpha(x_+,y_+)\frac{\partial}{\partial x_+} +$$

$$+\eta_\alpha(x_-,y_-)\frac{\partial}{\partial y_-} + \eta_\alpha(x_+,y_+)\frac{\partial}{\partial y_+}. \tag{2.3}$$

Let us assume that a Lie group G is given and that its Lie algebra is realized by vector fields of the form (1.2). If we wish to construct a second order ODE that is invariant under G, we proceed as follows. We choose a basis of L, namely $\{X_\alpha, \ \alpha = 1, \ldots, n\}$, and impose the equations

$$\mathrm{pr}^{(2)} X_\alpha \Phi(x, y, y', y'') = 0, \qquad \alpha = 1, \ldots, n, \tag{2.4}$$

with $\mathrm{pr}^{(2)} X_\alpha$ as in eq. (2.1). Using the method of characteristics, we obtain a set of elementary invariants $I_1, \ldots, I_k$. Their number is

$$k = \dim M - (\dim G - \dim G_0), \tag{2.5}$$

where M is the manifold that G acts on and G_0 is the stabilizer of a generic point on M. In our case we have $M \sim \{x, y, y', y''\}$ and hence $\dim M = 4$. An equivalent, but more practical formula for the number of invariants is

$$k = \dim M - \mathrm{rank} Z, \qquad k \geq 0,, \tag{2.6}$$

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where Z is the matrix

$$Z = \begin{pmatrix} \xi_1 & \eta_1 & \eta_1^1 & \eta_1^2 \\ \vdots & & & \\ \xi_n & \eta_n & \eta_n^1 & \eta_n^2 \end{pmatrix}. \tag{2.7}$$

The rank of Z is calculated at a generic point of M. The invariant equation is written as

$$E(I_1, \ldots, I_k) = 0, \tag{2.8}$$

where E must satisfy the condition from (1.1). Equation (2.8) obtained in this manner is "strongly invariant", i.e. $\mathrm{pr}^{(2)} X_\alpha E = 0$ is satisfied identically [38].

Further invariant equations are obtained if the rank of Z is less than maximum on some manifold described by the equation $E(x, y, y', y'') = 0$, itself satisfying the condition

$$\mathrm{pr}^{(2)} X_\alpha E \bigg|_{E = 0} = 0, \qquad \alpha = 1, \ldots, n. \tag{2.9}$$

We then obtain a "weakly invariant" equation of the form (1.1), i.e. eq. (2.9) is satisfied on the solution set of the equation $E = 0$.

The procedure for obtaining an invariant second order difference model for a given group G is quite analogous. Instead of eq. (2.4) we write

$$\mathrm{pr}^{(2)} X_\alpha \Phi(x, y, x_-, y_-, x_+, y_+) = 0, \qquad \alpha = 1, \ldots, n, \tag{2.10}$$

with $\mathrm{pr}^{(2)} X_\alpha$ as in eq. (2.3). We use the method of characteristics to obtain the elementary invariants $I_1, \ldots, I_k$ in the space $M \sim \{x, y, x_-, y_-, x_+, y_+\}$. The number of invariants k satisfies eq. (2.5) and (2.6). However, in this case we have $\dim M = 6$ and the matrix Z is

$$Z = \begin{pmatrix} \xi_1 & \eta_1 & \xi_{1,-} & \eta_{1,-} & \xi_{1,+} & \eta_{1,+} \\ \vdots & & & & & \\ \xi_n & \eta_n & \xi_{n,-} & \eta_{n,-} & \xi_{n,+} & \eta_{n,+} \end{pmatrix}, \tag{2.11}$$

where

$$\xi_{j,\pm} = \xi_j(x_\pm, y_\pm), \qquad \eta_{j,\pm} = \eta_j(x_\pm, y_\pm), \qquad 1 \leq j \leq n. \tag{2.12}$$

The "strongly invariant" difference scheme is then given by the equations

$$F(I_1, \ldots, I_k) = 0, \qquad \Omega(I_1, \ldots, I_k) = 0, \tag{2.13}$$

satisfying condition (1.4).

"Weakly invariant" difference schemes are obtained by finding invariant manifolds in M, i.e. finding surfaces $S(x, y, x_-, y_-, x_+, y_+) = 0$ on which the rank of Z is less than maximal. The system (2.13) represents both the difference equation and a mesh. In general, for the same group G we expect to have more difference invariants, than differential ones (since we have $\dim M = 6$ in the first case and $\dim M = 4$ in the second). We need two equations for a difference scheme, just one for a differential equation. That still leaves us with one more degree of freedom in the discrete case.

We shall run through subalgebras of $\mathrm{diff}(2, \mathbb{R})$ by dimension. For each representative subalgebra we shall construct the most general difference scheme. We shall then specialize it by specific choices of the arbitrary functions involved, so as to obtain a scheme that has the appropriate ODE as its continuous limit.

To do this it is often convenient to use different coordinates in the six–dimensional space of a three–point model. For instance, we may use

$$x, \quad h_+ = x_+ - x, \quad h_- = x - x_-, \quad y_x = \frac{y_+ - y}{h_+}, \quad y_{\bar{x}} = \frac{y - y_-}{h_-}$$

$$\tag{2.14}$$

$$y_{x\bar{x}} = \frac{2}{h_+ + h_-} \left(\frac{y_+ - y}{h_+} - \frac{y - y_-}{h_-} \right).$$

We shall call y_x and $y_{\bar{x}}$ discrete right and left derivatives respectively, and $y_{x\bar{x}}$ a discrete second derivative.

In the continuous limit we have

$$h_+ \to 0, \quad h_- \to 0, \quad y_x \to y', \quad y_{\bar{x}} \to y', \quad y_{x\bar{x}} \to y''. \tag{2.15}$$

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The most common way of discretizing independent variables is to introduce a regular (uniform) mesh, i.e. to put

$$h_+ = h_-. \tag{2.16}$$

Using the prolongation formula (2.3) it is easy to determine the class of transformations which preserves the uniformity of the mesh. The invariance condition of eq. (2.16) is

$$\xi(x_+, y_+) - 2\xi(x, y) + \xi(x_-, y_-) \Big|_{h_+ = h_-} = 0. \tag{2.17}$$

We shall see below that condition (2.17) is not satisfied for most transformations in the (x, y) plane. Hence, the use of nonregular lattices is essential in the construction of invariant difference schemes.

Below we shall make use of two types of classifications of low–dimensional Lie algebras, The first is a classification of abstract Lie algebras. Such classification exists for all Lie algebras of dim $L \leq 6$ [39, 40, 41, 42], both over $\mathbb{R}$ and over $\mathbb{C}$. The second is a classification of finite dimensional subalgebras of $\text{diff}(2, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. This classification is known for all (finite) values of dim L [3, 37]. It is easy to see that for dim $L \geq 2$ the two classifications do not coincide. Indeed, let us consider the lowest dimensions.

dim $L = 1$:

Any vector field of the form (1.2) can be rectified (in the neighborhood of a nonsingular point (x, y)). That is, by a locally invertible change of variables we can transform

$$X \to X = \frac{\partial}{\partial y}. \tag{2.18}$$

In other words, every one–dimensional subalgebra of $\text{diff}(2, \mathbb{F})$ is conjugate to X of eq. (2.18).

dim $L = 2$:

Two isomorphism classes of two–dimensional Lie algebras exist (over $\mathbb{R}$ and over $\mathbb{C}$). They are represented by

$$L_{2,1} \; : \quad [X_1, X_2] = X_1 \tag{2.19}$$

$$L_1 \oplus L_1 \; : \quad [X_1, X_2] = 0 \tag{2.20}$$

Each of the isomorphism classes can be realized in two different ways as subalgebras of $\text{diff}(2, \mathbb{F})$. The realizations are represented by

$$\begin{aligned}
\text{D}_{2,1} : \quad & X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \\[2mm]
\text{D}_{2,2} : \quad & X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y}, \\[2mm]
\text{D}_{2,3} : \quad & X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\[2mm]
\text{D}_{2,4} : \quad & X_1 = \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial y}.
\end{aligned} \tag{2.21}$$

The algebras $\text{D}_{2,1}$ and $\text{D}_{2,2}$ are Abelian, $\text{D}_{2,3}$ and $\text{D}_{2,4}$ isomorphic to $L_{2,1}$ from (2.19). In the tangent space $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$ $\text{D}_{2,2}$ and $\text{D}_{2,4}$ generate a one–dimensional subspace, $\text{D}_{2,1}$ and $\text{D}_{2,3}$ the entire two–dimensional space. We say that X_1 and X_2 are "linearly connected" for $\text{D}_{2,2}$ and $\text{D}_{2,4}$, i.e. they are linearly dependent at any fixed generic point of $\mathbb{F}^2$. For $\text{D}_{2,1}$ and $\text{D}_{2,3}$ the vector fields X_1 and X_2 are "linearly nonconnected".

dim $L = 3$:

Six classes of indecomposable three–dimensional Lie algebras exist over $\mathbb{R}$, four over $\mathbb{C}$. Two classes of decomposable three–dimensional Lie algebras exist in both cases. We shall use the following notations for the isomorphism classes:

1. Nilpotent

$$L_{3,1} \; : \quad [e_2, e_3] = e_1, \quad [e_1, e_2] = 0, \quad [e_1, e_3] = 0 \tag{2.22}$$

2. Solvable

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A solvable three–dimensional Lie algebra has a two–dimensional Abelian ideal. We choose X_1 and X_2 as basis elements of the ideal. The commutation relations then are

$$\begin{pmatrix} [e_1, e_3] \\ [e_2, e_3] \end{pmatrix} = M \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \qquad [e_1, e_2] = 0, \tag{2.23}$$

where $M \in \mathbb{F}^2$ can be chosen in its Jordan canonical form. Over $\mathbb{R}$ we have

$$L_{3,2}: \quad M = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \qquad 0 < |a| \leq 1$$

$$L_{3,3}: \quad M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{2.24}$$

$$L_{3,4}: \quad M = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \quad 0 \leq a$$

Over $\mathbb{C}$ the algebras $L_{3,2}$ and $L_{3,4}$ are isomorphic, so $L_{3,4}$ is dropped.

3. Simple

$$L_{3,5} \sim \mathfrak{sl}(2, \mathbb{F}) \; : \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3, \quad [e_3, e_1] = -2e_2 \tag{2.25}$$

$$L_{3,6} \sim \mathfrak{o}(3) \; : \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2 \tag{2.26}$$

Over $\mathbb{C}$ these two algebras are isomorphic, so $L_{3,6}$ should be dropped. The decomposable three dimensional algebras are represented by $L_{2,1} \oplus L_1$ and $L_1 \oplus L_1 \oplus L_1$.

As subalgebras of diff$(2, \mathbb{R})$ $L_{3,1}$, $L_{3,6}$, $L_{2,1} \oplus L_1$ and $L_1 \oplus L_1 \oplus L_1$ can be realized in one way each, $L_{3,5}$ in four inequivalent ways, all others in two inequivalent ways. All these realizations will be presented below in Section 4. For $L_{3,2}$ it is convenient to treat the value $a = 1$ separately.

Let us now construct the invariant difference schemes, proceeding by dimension.

3 Equations invariant under one and two–dimensional groups

We start with the simplest case of a symmetry group, namely a one–dimensional one. Its Lie algebra is generated by one vector field of the form (1.2). By an appropriate change of variables we take this vector field into its rectified form. Thus we have

$$\mathrm{D}_{1,1}: \qquad X_1 = \frac{\partial}{\partial y}. \tag{3.1}$$

The most general second order ODE invariant under the corresponding group is

$$y'' = F(x, y'), \tag{3.2}$$

where F is an arbitrary function. In order to write a difference scheme invariant under the same group we need the difference invariants annihilated by the prolongation of X_1 to the prolonged space $(x, x_-, x_+, y, y_-, y_+)$. A basis for the invariants is

$$\{\eta_+ = y_+ - y, \quad \eta_- = y - y_-, \quad x, \quad x_-, \quad x_+\},$$

but a more convenient basis is

$$\left\{ y_{x\bar{x}}, \quad \frac{y_x + y_{\bar{x}}}{2}, \quad x, \quad h_-, \quad h_+ \right\}.$$

The general invariant model can be written as

$$\begin{cases} y_{x\bar{x}} = f\left(x, \dfrac{y_x + y_{\bar{x}}}{2}, h_-\right); \\ h_+ = h_- g\left(x, \dfrac{y_x + y_{\bar{x}}}{2}, h_-\right), \end{cases} \tag{3.3}$$

where f and g are arbitrary functions.

The simplest invariant difference scheme that approximates the ODE (3.2) is obtained by restricting f to be independent of h_- and putting $g \equiv 1$. We have

$$\begin{cases} y_{x\bar{x}} = F\left(x, \dfrac{y_x + y_{\bar{x}}}{2}\right); \\ h_- = h_+. \end{cases} \tag{3.4}$$

We would like to stress that (3.4) is just a specified case of the difference model (3.3), involving two arbitrary functions. In other words, invariant difference schemes have much more freedom than invariant differential equations

$D_{2,1}$ The Abelian Lie algebra with nonconnected basis elements (see eq.(2.21)).
   In the continuous case the invariant ODE is

$$y'' = F(y'), \tag{3.5}$$

where F is an arbitrary function. A convenient set of difference invariants is

$$\left\{ h_+, \quad h_-, \quad \frac{y_x + y_{\bar{x}}}{2}, \quad y_{x\bar{x}} \right\}.$$

The most general invariant difference model can be written as

$$\begin{cases} y_{x\bar{x}} = f\left(\dfrac{y_x + y_{\bar{x}}}{2}, h_-\right); \\ \\ h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, h_-\right). \end{cases} \tag{3.6}$$

The simplest scheme approximating eq. (3.5) is again obtained by restricting f and putting $g = 1$:

$$\begin{cases} y_{x\bar{x}} = F\left(\dfrac{y_x + y_{\bar{x}}}{2}\right); \\ \\ h_- = h_+. \end{cases} \tag{3.7}$$

$D_{2,2}$ The Abelian Lie algebra with connected basis elements (see eq.(2.21)).
   The invariant differential equation is

$$y'' = F(x). \tag{3.8}$$

This equation can be transformed into the equation $u'' = 0$ by the change of variables

$$u = y - W(x), \quad \text{where} \quad W''(x) = F(x), \tag{3.9}$$

i.e. $W(x)$ is any solution of equation (3.8).
   A basis for the finite–difference invariants of the group corresponding to the Lie algebra $D_{2,2}$ is

$$\{y_{x\bar{x}}, \quad x, \quad h_+, \quad h_-\}$$

and the invariant model can be presented as

$$\begin{cases} y_{x\bar{x}} = f(x, h_-); \\ \\ h_+ = h_- g(x, h_-). \end{cases} \tag{3.10}$$

Restricting f and setting $g = 1$ we obtain the discrete system representing eq. (3.8):

$$\begin{cases} y_{x\bar{x}} = F(x); \\ \\ h_- = h_+. \end{cases} \tag{3.11}$$

The discrete model (3.10) can be taken into standard form

$$\begin{cases} u_{x\bar{x}} = 0; \\ \\ h_+ = h_- g(x, h_-) \end{cases} \tag{3.12}$$

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by putting

$$u = y - W(x, h_-, h_+), \qquad W_{x\bar{x}} = f(x, h_-), \tag{3.13}$$

i.e. W is any solution of the system (3.10) (just as in the continuous case).

$D_{2,3}$ The non–Abelian Lie algebra with nonconnected elements (see eq.(2.21)) yields the invariant ODE

$$y'' = \frac{1}{x} F(y'). \tag{3.14}$$

A convenient basis for the finite–difference invariants of the group corresponding to $D_{2,3}$ is

$$\left\{ xy_{x\bar{x}}, \quad \frac{y_x + y_{\bar{x}}}{2}, \quad \frac{h_+}{h_-}, \quad \frac{h_-}{x} \right\}.$$

The general invariant difference scheme can be written as

$$\begin{cases} y_{x\bar{x}} = \dfrac{1}{x} f\left(\dfrac{y_x + y_{\bar{x}}}{2}, \dfrac{h_-}{x} \right); \\[3mm] h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, \dfrac{h_-}{x} \right). \end{cases} \tag{3.15}$$

Restricting f and setting $g = 1$, we obtain an invariant approximation of eq. (3.14), namely

$$\begin{cases} y_{x\bar{x}} = \dfrac{1}{x} F\left(\dfrac{y_x + y_{\bar{x}}}{2} \right); \\[3mm] h_- = h_+. \end{cases} \tag{3.16}$$

$D_{2,4}$ The non–Abelian Lie algebra with linearly connected basis elements (see eq.(2.21)) leads to the invariant ODE

$$y'' = F(x)y'. \tag{3.17}$$

This equation can be taken to its standard form $v'' = 0$ by a transformation of the independent variable $t = g(x), \quad v(t) = y(x)$, where $g(x)$ is any particular solution of eq. (3.17).

Finite–difference invariants for $D_{2,4}$ are

$$\left\{ \frac{y_{x\bar{x}}}{y_x + y_{\bar{x}}}, \quad x, \quad h_-, \quad h_+ \right\}.$$

The general invariant difference model can be written as

$$\begin{cases} y_{x\bar{x}} = \dfrac{y_x + y_{\bar{x}}}{2} f(x, h_-); \\[3mm] h_+ = h_- g(x, h_-). \end{cases} \tag{3.18}$$

An invariant difference approximation of the ODE (3.17) is obtained by putting $f(x, h_-) = F(x)$, $g(x, h_-) = 1$, i.e.

$$\begin{cases} y_{x\bar{x}} = \dfrac{y_x + y_{\bar{x}}}{2} F(x); \\[3mm] h_- = h_+. \end{cases} \tag{3.19}$$

The discrete model (3.18) can be simplified by a transformation of the independent variable, just as in the continuous case. Indeed, let $\phi(x)$ be a solution of eq. (3.18) and transform

$$(x, y) \rightarrow (t = \phi(x), u(t) = y(x)). \tag{3.20}$$

We obtain the difference scheme (3.12)

The results on two–dimensional symmetry algebras can be summed up as a theorem.

Theorem 1. The two subalgebras $D_{2,1}$ and $D_{2,3}$ of $\text{diff}(2, \mathbb{F})$ with linearly nonconnected basis elements provide invariant difference schemes involving two arbitrary functions of two variables each, namely (3.6) and (3.15) respectively. The subalgebras $D_{2,2}$ and $D_{2,4}$ with linearly connected elements lead to the difference schemes (3.10) and (3.18), respectively. Both of them can be transformed into the scheme (3.12) if one solution of the original scheme is known.

3.0.1 Comments

1. We have obtained the discrete analog of a well known result for ODEs. Namely, any second order ODE invariant under a two–dimensional Lie group with linearly connected generators can be transformed by a point transformation into $y'' = 0$ [1, 2].

2. The transformations (3.13) and (3.20) will be used below for equations with higher dimensional symmetry algebras, containing $D_{2,2}$ and $D_{2,4}$ as subalgebras.

4 Equations invariant under three-dimensional Lie groups

As stated in Section 2, eight isomorphism classes of three–dimensional Lie algebras exist over $\mathbb{R}$. Two of them, $L_{3,2}$ and $L_{3,4}$, depend on a continuous parameter called a in eq. (2.24). For our purpose it is sometimes convenient to separate out some special values of this parameter for the algebra $L_{3,2}$. All together we must consider 16 classes of three dimensional subalgebras $D_{3,j} \subset \mathrm{diff}(2, \mathbb{R})$, $j = 1, \ldots, 16$.

For differential equations, seven of these algebras, we shall call them $D_{3,1}, \ldots, D_{3,7}$ lead to equations equivalent to

$$y'' = 0. \tag{4.1}$$

This ODE is invariant under $\mathrm{SL}(3, \mathbb{F})$, so none of of the groups corresponding to $D_{3,1}, \ldots, D_{3,7}$ is a maximal symmetry group of eq. (4.1), i.e. we have $D_{3,j} \subset \mathfrak{sl}(3, \mathbb{F})$, $j = 1, \ldots, 7$.

Two of the subalgebras, we call them $D_{3,15}$ and $D_{3,16}$, do not allow any invariant second order ODE. The remaining 7 algebras lead to specific invariant ODEs, not involving any arbitrary functions.

We shall run through all the algebras $D_{3,j}$, $j = 1, \ldots, 16$ and construct the invariant ODEs and the invariant difference schemes whenever they exist. The difference schemes in general involve arbitrary functions. Whenever possible, we specialize these functions so as to obtain invariant difference schemes, approximating the invariant ODEs. This last step is of course not unique; different discrete schemes can approximate the same ODE.

We start from the six algebras that contain $D_{2,2}$ or $D_{2,4}$ as subalgebras. The invariant ODE will hence be equivalent to eq. (4.1) and the invariant difference schemes can always be transformed to the form (3.12), though the function $g(x, h_-)$ may differ from case to case.

$D_{3,1}$ The nilpotent Lie algebra isomorphic to $L_{3,1}$ can, up to equivalence, be realized in one way only:

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial y}. \tag{4.2}$$

Notice that X_2 and X_3 commute and are linearly connected. The invariant ODE $y'' = C$ is equivalent to eq. (4.1).

The difference invariants are

$$\{y_{x\bar{x}}, \quad h_-, \quad h_+\}.$$

The most general invariant difference scheme is

$$\begin{cases} y_{x\bar{x}} = f(h_-); \\ h_+ = h_- g(h_-). \end{cases} \tag{4.3}$$

The ODE $y'' = C$ is approximated if we set $f = C$, $g = 1$. The scheme (4.3) is equivalent to that of eq. (3.12), however the function g is independent of x.

$D_{3,2}$ The solvable Lie algebra $L_{3,3}$ can be represented as

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \tag{4.4}$$

with X_1 and X_2 linearly connected. The invariant ODE is

$$y'' = C \exp(x). \tag{4.5}$$

A basis for the difference invariants is

$$\{y_{x\bar{x}} \exp(-x), \quad h_-, \quad h_+.\}$$

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The general invariant difference scheme is

$$\begin{cases} y_{x\bar{x}} = f(h_-)\exp(x); \\ \\ h_+ = h_- g(h_-); \end{cases} \tag{4.6}$$

and eq. (4.5) is approximated if we put $f = C$, $g = 1$. The scheme (4.6) can be transformed into (3.12) with g independent of x.

D$_{3,3}$ The decomposable algebra $L_{2,1} \oplus L_1$ can be represented by

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = y\frac{\partial}{\partial y}. \tag{4.7}$$

The invariant ODE is

$$y'' = Cy'. \tag{4.8}$$

The simplest set of difference invariants is $\left\{ h_+, \ h_-, \ \frac{y_+ - y}{y - y_-} \right\}$, but an equivalent and more convenient set is

$$\left\{ \frac{y_{x\bar{x}}}{y_x}, \quad \frac{y_x + y_{\bar{x}}}{y_x}, \quad h_- \right\}.$$

We write the general invariant difference scheme as

$$\begin{cases} y_{x\bar{x}} = \dfrac{y_x + y_{\bar{x}}}{2} f(h_-); \\ \\ h_+ = h_- g(h_-). \end{cases} \tag{4.9}$$

The ODE (4.8) is approximated by putting $f = C$, $g = 1$.

The transformation (3.20) will take (4.9) into (3.12) with g independent of x.

D$_{3,4}$ The Lie algebra $L_{3,2}$ with $a = 1$ is realized by

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = y\frac{\partial}{\partial y}. \tag{4.10}$$

In this case all three operators are linearly connected. The only differential invariant is x, but $y'' = 0$ is an invariant manifold, so $y'' = 0$ is a weakly invariant ODE.

The difference invariants are only $\{x_-, \ x, \ x_+\}$, but $y_{x\bar{x}} = 0$ is again an invariant manifold. The general (weakly) invariant difference scheme is

$$\begin{cases} y_{x\bar{x}} = 0; \\ \\ h_+ = h_- g(x, h_-). \end{cases} \tag{4.11}$$

The simplest approximation of the invariant ODE is obtained if we set $g = 1$ in eq. (4.11).

D$_{3,5}$ The Lie algebra $L_{3,2}$ with $a \neq 1$ can be realized as

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = (1-a)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad a \neq 1. \tag{4.12}$$

The invariant ODE is

$$y'' = Cx^{\frac{2a-1}{1-a}}, \quad a \neq 1. \tag{4.13}$$

A convenient set of invariants is

$$\left\{ y_{x\bar{x}} x^{\frac{2a-1}{a-1}}, \quad \frac{h_-}{x}, \quad \frac{h_+}{x} \right\}.$$

The general invariant difference scheme is

$$\begin{cases} y_{x\bar{x}} = x^{\frac{2a-1}{1-a}} f\left(\dfrac{h_-}{x}\right); \\ \\ h_+ = h_- g\left(\dfrac{h_-}{x}\right). \end{cases} \tag{4.14}$$

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Equation (4.13) is approximated by setting $f = C$, $g = 1$. The difference scheme can again be transformed into (3.12) with g as in eq. (4.14).

D$_{3,6}$ The algebra $L_{3,4}$ can be represented as

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = (1+x^2)\frac{\partial}{\partial x} + (x+b)y\frac{\partial}{\partial y}; \tag{4.15}$$

and corresponds to the invariant ODE

$$y'' = C(1+x^2)^{-3/2}\exp(b\arctan(x)), \tag{4.16}$$

which can be transformed into $y'' = 0$.

The expressions

$$\left\{ \frac{h_+}{1+xx_+}, \quad \frac{h_-}{1+xx_-}, \quad (y_x - y_{\bar{x}})\sqrt{1+x^2}\exp(-b\arctan(x)), \right\}$$

form a complete set of difference invariants for the group corresponding to D$_{3,6}$. The invariant difference scheme can be written as

$$\begin{cases} y_{x\bar{x}} = \dfrac{\exp(b\arctan(x))}{\sqrt{1+x^2}}\left(\dfrac{h_+}{h_- + h_+}\dfrac{1}{1+xx_+} + \dfrac{h_-}{h_- + h_+}\dfrac{1}{1+xx_-} \right) f\left(\dfrac{h_-}{1+xx_-}\right); \\[2mm] h_+ = h_-\dfrac{1+xx_+}{1+xx_-}g\left(\dfrac{h_-}{1+xx_-}\right). \end{cases} \tag{4.17}$$

Putting $f = g = 1$ we obtain an invariant discrete approximation of the ODE (4.16).

When considered over $\mathbb{C}$, the case D$_{3,6}$ is equivalent to D$_{3,5}$. Indeed, if we put

$$x = i\frac{t-i}{t+i}, \qquad y = \frac{\sqrt{2}u}{1-it}, \tag{4.18}$$

the vector fields (4.15) go into a linear combination of the fields (4.12) and eq. (4.16) goes into

$$u_{tt} = \frac{\sqrt{2}}{4}\exp\left(i\pi\frac{3-ib}{4}\right)Ct^{-\frac{3+ib}{2}}, \tag{4.19}$$

i.e. (4.13) with $a = (-1+ib)(1+ib)^{-1}$. Similarly, the model (4.17) goes into (4.14).

D$_{3,7}$ The Lie algebra $L_{3,2}$ with $a = 1$ was already realized as D$_{3,4}$. A different, inequivalent realization of $L_{3,2}$ ($a = 1$) is given by

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{4.20}$$

Notice that the Abelian ideal $\{X_1, X_2\}$ is realized by linearly nonconnected vector fields. The only differential invariant is y', however $y'' = 0$ is an invariant manifold. Hence the equation $y'' = 0$ is weakly invariant.

A basis for the difference invariants is

$$\left\{ h_-y_{x\bar{x}}, \quad \frac{y_x + y_{\bar{x}}}{2}, \quad \frac{h_+}{h_-} \right\}$$

We obtain a strongly invariant difference scheme, namely

$$\begin{cases} h_-y_{x\bar{x}} = f\left(\dfrac{y_x + y_{\bar{x}}}{2}\right); \\[2mm] h_+ = h_-g\left(\dfrac{y_x + y_{\bar{x}}}{2}\right). \end{cases} \tag{4.21}$$

In general, this model does not have a continuous limit. That exists only for $f = 0$. The equation $y'' = 0$ is approximated by eq. (4.21) with $f = 0$, $g = 1$.

All different schemes obtained so far are equivalent to special cases of the model

$$\begin{cases} y_{x\bar{x}} = 0; \\[2mm] g(x, h_-, h_+) = 0. \end{cases} \tag{4.22}$$

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The following algebras $D_{3,8}, \ldots, D_{3,14}$ are different in that they lead to equations that can not be reduced to the form (4.22).

$D_{3,8}$ The Lie algebra $L_{3,3}$ was already realized as $D_{3,2}$. A second realization, not equivalent to (4.4), is given by

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}. \tag{4.23}$$

Notice that the elements of the ideal $\{X_1, X_2\}$ are linearly nonconnected. The corresponding invariant ODE is

$$y'' = \exp(-y'). \tag{4.24}$$

Its general solution is

$$y = -x + (x + B)\ln(x + B) + A,$$

where A and B are integration constants.

A basis for the difference invariants is

$$\left\{ \frac{h_+}{h_-}, \quad h_+ \exp\left(-y_x\right), \quad h_- \exp\left(-y_{\bar{x}}\right) \right\}.$$

The general invariant difference scheme can be written as

$$\begin{cases} \dfrac{2}{h_- + h_+} \left(\exp\left(y_x\right) - \exp\left(y_{\bar{x}}\right)\right) = f\left(h_- \exp\left(-y_{\bar{x}}\right)\right); \\[4mm] h_+ = h_- g\left(h_- \exp\left(-y_{\bar{x}}\right)\right); \end{cases} \tag{4.25}$$

An invariant approximation of the ODE (4.24) is obtained by setting $f = g = 1$. An equivalent alternative to the scheme (4.25) is

$$\begin{cases} y_{x\bar{x}} = \exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right) f\left(\sqrt{h_- h_+} \exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right)\right); \\[4mm] h_+ = h_- g\left(\sqrt{h_- h_+} \exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right)\right). \end{cases} \tag{4.26}$$

An approximation of eq. (4.24) is again obtained by setting $f = g = 1$. The two above approximations are not equivalent (they correspond to different choices of the arbitrary functions).

$D_{3,9}$ The algebra $L_{3,2}$ with $a \neq 1$ was realized as $D_{3,5}$. A second realization is

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \qquad k \neq 0, 1. \tag{4.27}$$

In this case $\{X_1, X_2\}$ are not linearly connected, hence $D_{3,9}$ and $D_{3,5}$ are not equivalent. The corresponding invariant ODE is

$$y'' = y'^{\frac{k-2}{k-1}}. \tag{4.28}$$

and its general solution is

$$y = \left(\frac{1}{k-1}\right)^{k-1} \frac{1}{k}(x - x_0)^k + y_0 \tag{4.29}$$

A basis for finite–difference invariants is given by

$$\left\{ \frac{h_+}{h_-}, \quad y_x h_+^{(1-k)}, \quad y_{\bar{x}} h_-^{(1-k)} \right\}.$$

An invariant difference scheme is given by

$$\begin{cases} \dfrac{2(k-1)}{h_- + h_+} \left((y_x)^{\frac{1}{k-1}} - (y_{\bar{x}})^{\frac{1}{k-1}}\right) = f\left(y_{\bar{x}} h_-^{(1-k)}\right); \\[4mm] h_+ = h_- g\left(y_{\bar{x}} h_-^{(1-k)}\right). \end{cases} \tag{4.30}$$

An approximation of eq. (4.28) is obtained if we put $f = g = 1$ (this can be verified by setting $y_x = y_{\bar{x}} + \varepsilon$ and expanding in terms of powers of ε). An alternative invariant difference model is

$$
\begin{cases}
y_{x\bar{x}} = \left(\dfrac{y_x + y_{\bar{x}}}{2} \right)^{\frac{k-2}{k-1}} f \left(\dfrac{y_x + y_{\bar{x}}}{2} h_-^{(1-k)} \right); \\[4mm]
h_+ = h_- g \left(\dfrac{y_x + y_{\bar{x}}}{2} h_-^{(1-k)} \right).
\end{cases}
\tag{4.31}
$$

The ODE (4.28) is approximated if we put $f = g = 1$.

$D_{3,10}$ The Lie algebra $L_{3,4}$, already realized as $D_{3,6}$, can also be realized with the ideal $\{X_1, X_2\}$ linearly unconnected. We have

$$
X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = (kx + y)\frac{\partial}{\partial x} + (ky - x)\frac{\partial}{\partial y}.
\tag{4.32}
$$

Over $\mathbb{C}$ this algebra is equivalent to $D_{3,9}$. The corresponding transformation of variables is

$$
t = \frac{x - iy}{\sqrt{2}}, \qquad w = -\frac{i}{\sqrt{2}}(x + iy).
\tag{4.33}
$$

The invariant ODE is

$$
y'' = (1 + (y')^2)^{3/2} \exp(k \arctan(y')).
\tag{4.34}
$$

and its solution is

$$
\exp\left(2k \arctan\left(\frac{-(x - x_0) + k(y - y_0)}{k(x - x_0) + (y - y_0)} \right) \right) (1 + k^2)\left((x - x_0)^2 + (y - y_0)^2 \right) = 1
\tag{4.35}
$$

A basis for the finite–difference invariants can be chosen in the form

$$
\left\{ h_+ \sqrt{1 + y_x^2} \exp\left(k \arctan y_x \right), \quad h_- \sqrt{1 + y_{\bar{x}}^2} \exp\left(k \arctan y_{\bar{x}} \right), \quad \frac{y_x - y_{\bar{x}}}{1 + y_x y_{\bar{x}}} \right\}.
$$

The general form of the invariant difference model is

$$
\begin{cases}
y_{x\bar{x}} = (1 + y_x y_{\bar{x}}) \left(\dfrac{h_+}{h_- + h_+} \sqrt{1 + y_x^2} \exp\left(k \arctan y_x \right) + \right. \\[4mm]
\left. + \dfrac{h_-}{h_- + h_+} \sqrt{1 + y_{\bar{x}}^2} \exp\left(k \arctan y_{\bar{x}} \right) \right) f\left(h_- \sqrt{1 + y_{\bar{x}}^2} \exp\left(k \arctan(y_{\bar{x}}) \right) \right); \\[4mm]
h_+ \sqrt{1 + y_x^2} \exp\left(k \arctan(y_x) \right) = h_- \sqrt{1 + y_{\bar{x}}^2} \exp\left(k \arctan(y_{\bar{x}}) \right) + \\[4mm]
+ g\left(h_- \sqrt{1 + y_{\bar{x}}^2} \exp\left(k \arctan(y_{\bar{x}}) \right) \right).
\end{cases}
\tag{4.36}
$$

In order to approximate the ODE (4.34) we set $f = 1$, $g = 0$.

$D_{3,11}$ This is the first of four inequivalent realizations of $\mathfrak{sl}(2, \mathbb{F})$. We have

$$
X_1 = \frac{\partial}{\partial x}; \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.
\tag{4.37}
$$

The invariant ODE is

$$
y'' = y^{-3},
\tag{4.38}
$$

with the general solution

$$
y^2 = A(x - x_0)^2 + \frac{1}{A}, \qquad A \neq 0.
$$

A convenient set of difference invariants is

$$
\left\{ y(y_x - y_{\bar{x}}), \quad \frac{1}{y}\left(\frac{h_+}{y_+} + \frac{h_-}{y_-} \right), \quad \frac{1}{y^2}\frac{h_+ h_-}{h_+ + h_-} \right\}.
$$

The invariant difference scheme can be written as

$$
\begin{cases}
y_{x\bar{x}} = \dfrac{1}{y^2}\left(\dfrac{h_+}{h_+ + h_-}\dfrac{1}{y_+} + \dfrac{h_-}{h_+ + h_-}\dfrac{1}{y_-}\right) f\left(\dfrac{1}{y^2}\dfrac{h_+ h_-}{h_+ + h_-}\right); \\[4mm]
\dfrac{1}{y}\left(\dfrac{h_+}{y_+} + \dfrac{h_-}{y_-}\right) = 4\dfrac{1}{y^2}\dfrac{h_+ h_-}{h_+ + h_-}\, g\left(\dfrac{1}{y^2}\dfrac{h_+ h_-}{h_+ + h_-}\right).
\end{cases}
\tag{4.39}
$$

We approximate the ODE (4.38) by setting $f = g = 1$.

We mention that this particular realization of $\mathfrak{sl}(2,\mathbb{F})$ is not maximal in $\mathrm{diff}(2,\mathbb{F})$. Indeed, we have an embedding

$$
\mathfrak{sl}(2,\mathbb{F}) \subset \mathfrak{gl}(2,\mathbb{F}) \subset \mathfrak{sl}(3,\mathbb{F}) \subset \mathrm{diff}(2,\mathbb{F})
\tag{4.40}
$$

and the centralizer of $D_{3,11}$ is $Y = y\frac{\partial}{\partial y}$. Note also that the coefficients of $\frac{\partial}{\partial x}$ in X_i depend only on the variable x. The realization is hence imprimitive.

$D_{3,12}$ A second realization of $\mathfrak{sl}(2,\mathbb{F})$ is

$$
X_1 = \frac{\partial}{\partial x}; \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y};
\tag{4.41}
$$

The invariant ODE is

$$
yy'' = C(1 + (y')^2)^{3/2} - (1 + (y')^2), \quad C = const.
\tag{4.42}
$$

with the general solution

$$
(Ax - B)^2 + (Ay - C)^2 = 1.
$$

The difference invariants can be chosen in the form

$$
I_1 = \frac{h_-^2 + (y - y_-)^2}{yy_-}, \qquad I_2 = \frac{h_+^2 + (y_+ - y)^2}{yy_+},
$$

$$
I_3 = \frac{2y(h_+ + h_- + h_+ y_x^2 + h_- y_{\bar{x}}^2 + 2y(y_x - y_{\bar{x}}))}{4y^2 - (h_+(1 + y_x^2) + 2yy_x)(h_-(1 + y_{\bar{x}}^2) - 2yy_{\bar{x}})}.
$$

We write the invariant difference scheme as

$$
\begin{cases}
I_3 = \dfrac{1}{2}(\sqrt{I_1} + \sqrt{I_2})f(I_1); \\[4mm]
I_2 = I_1 g(I_1).
\end{cases}
\tag{4.43}
$$

A discrete approximation of the ODE (4.42) is obtained by setting $f = C$, $g = 1$ in eq. (4.43).

The subalgebras $D_{3,11}$ and $D_{3,12}$ are not equivalent, neither for $\mathbb{F} = \mathbb{C}$, nor for $\mathbb{F} = \mathbb{R}$. To see this, it is sufficient to notice that there is no nonzero element of $\mathrm{diff}(2,\mathbb{F})$ that commutes with all elements X_i from (4.41), i.e. the centralizer of $D_{3,12}$ in $\mathrm{diff}(2,\mathbb{F})$ is zero, whereas that of $D_{3,11}$ is $Y = y\frac{\partial}{\partial y}$. Over $\mathbb{R}$ the algebra (4.41) is primitive. However, over $\mathbb{C}$ we can put

$$
u = x + iy, \qquad v = x - iy.
\tag{4.44}
$$

The algebra (4.41) in terms of the coordinates u and v is transformed into the algebra $D_{3,13}$, which we turn to now.

$D_{3,13}$ A third realization of $\mathfrak{sl}(2,\mathbb{F})$ is

$$
X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.
\tag{4.45}
$$

Over $\mathbb{R}$ $D_{3,13}$ is a new inequivalent imprimitive realization of $\mathfrak{sl}(2,\mathbb{F})$. As mentioned above $D_{3,13}$ and $D_{3,12}$ are equivalent over $\mathbb{C}$. The realization is imprimitive.

The invariant ODE is

$$
y'' + \frac{2}{x - y}(y' + y'^2) = \frac{2C}{x - y}y'^{3/2}.
\tag{4.46}
$$

Its general solution is

$$
y = \frac{1}{A\left(B + \frac{1}{2}C\right) - Ax} + \frac{2B - C}{2A}, \qquad A \neq 0.
$$

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A special solution is

$$y = ax,$$

where the constant a is a solution of the following algebraic equation

$$a - Ca\sqrt{a} + a^2 = 0.$$

A complete set of difference invariants is

$$I_1 = \frac{h_+^2 y_x}{(x - y_+)(x_+ - y)}, \qquad I_2 = \frac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}, \qquad I_3 = \frac{x_+ - y}{x - y} \frac{h_-}{h_- + h_+}.$$

An invariant difference scheme can be written as

$$\begin{cases} \dfrac{I_1}{(1 - I_3)^2} - \dfrac{I_2}{(I_3)^2} = f(I_3) \left(\dfrac{I_1}{(1 - I_3)^2} + \dfrac{I_2}{(I_3)^2} \right)^{3/2}; \\[4mm] I_1 = I_2 g(I_3). \end{cases} \tag{4.47}$$

It approximates eq. (4.46) if we put $f = C$, $g = 1$.

We mention that $D_{3,13}$ is not maximal in $\mathrm{diff}(2, \mathbb{F})$. For $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ we have

$$\begin{cases} D_{3,13} \subset \mathfrak{o}(2, 2) \subset \mathrm{diff}(2, \mathbb{R}), \quad \text{or} \\[3mm] D_{3,13} \subset \mathfrak{o}(4, \mathbb{C}) \subset \mathrm{diff}(2, \mathbb{C}), \end{cases} \tag{4.48}$$

respectively. The algebras $\mathfrak{o}(2, 2)$ and $\mathfrak{o}(4, \mathbb{C})$ are both realized by the vector fields

$$\left\{ \frac{\partial}{\partial x}, \ x \frac{\partial}{\partial x}, \ x^2 \frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}, \ y \frac{\partial}{\partial y}, \ y^2 \frac{\partial}{\partial y} \right\} \tag{4.49}$$

$D_{3,14}$ There exists just one (up to equivalence) realization of $\mathfrak{o}(3)$ as a subalgebra of $\mathrm{diff}(2, \mathbb{F})$. We choose it in the form

$$X_1 = (1 + x^2) \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}; \qquad X_2 = xy \frac{\partial}{\partial x} + (1 + y^2) \frac{\partial}{\partial y}; \tag{4.50}$$

$$X_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}.$$

The corresponding invariant ODE is

$$y'' = C \left(\frac{1 + y'^2 + (y - xy')^2}{1 + x^2 + y^2} \right)^{3/2}. \tag{4.51}$$

The general solution of this equation can be presented in the from

$$\left(Bx - Ay + C\sqrt{1 + x^2 + y^2} \right)^2 = 1 + C^2 - A^2 - B^2.$$

The discrete invariants can be chosen to be

$$I_1 = \frac{h_+^2 (1 + y_x^2 + (y - xy_x)^2)}{(1 + x^2 + y^2)(1 + x_+^2 + y_+^2)}, \qquad I_2 = \frac{h_-^2 (1 + y_{\bar{x}}^2 + (y - xy_{\bar{x}})^2)}{(1 + x_-^2 + y_-^2)(1 + x^2 + y^2)},$$

$$I_3 = \frac{h_+ h_- (y_x - y_{\bar{x}})}{\sqrt{1 + x_-^2 + y_-^2} \sqrt{1 + x^2 + y^2} \sqrt{1 + x_+^2 + y_+^2}}$$

The general form of the invariant difference model can be written as

$$\begin{cases} \dfrac{h_+ h_- (y_x - y_{\bar{x}})}{\sqrt{1 + x_-^2 + y_-^2} \sqrt{1 + x^2 + y^2} \sqrt{1 + x_+^2 + y_+^2}} = f \left(\dfrac{h_-^2 (1 + y_{\bar{x}}^2 + (y - xy_{\bar{x}})^2)}{(1 + x_-^2 + y_-^2)(1 + x^2 + y^2)} \right); \\[5mm] \dfrac{h_+^2 (1 + y_x^2 + (y - xy_x)^2)}{(1 + x^2 + y^2)(1 + x_+^2 + y_+^2)} = g \left(\dfrac{h_-^2 (1 + y_{\bar{x}}^2 + (y - xy_{\bar{x}})^2)}{(1 + x_-^2 + y_-^2)(1 + x^2 + y^2)} \right). \end{cases} \tag{4.52}$$

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As an invariant discrete model that approximates the ODE (4.51) we can consider the discrete equation

$$\frac{h_+ h_- (y_x - y_{\bar{x}})}{\sqrt{1 + x_-^2 + y_-^2}\sqrt{1 + x^2 + y^2}\sqrt{1 + x_+^2 + y_+^2}} =$$

$$= C \left(\left(\frac{h_+^2 (1 + y_x^2 + (y - xy_x)^2)}{(1 + x^2 + y^2)(1 + x_+^2 + y_+^2)} \right)^{3/2} + \left(\frac{h_-^2 (1 + y_{\bar{x}}^2 + (y - xy_{\bar{x}})^2)}{(1 + x_-^2 + y_-^2)(1 + x^2 + y^2)} \right)^{3/2} \right)$$

on the grid

$$\frac{h_+^2 (1 + y_x^2 + (y - xy_x)^2)}{(1 + x^2 + y^2)(1 + x_+^2 + y_+^2)} = \frac{h_-^2 (1 + y_{\bar{x}}^2 + (y - xy_{\bar{x}})^2)}{(1 + x_-^2 + y_-^2)(1 + x^2 + y^2)} = \varepsilon^2$$

Over $\mathbb{C}$ the algebra $D_{3,14}$ is equivalent to the $\mathfrak{sl}(2, \mathbb{R})$ algebra $D_{3,13}$. The transformation of variables that takes one algebra into the other one is quite complicated and we shall not reproduce it here. The algebra $D_{3,14}$ does not have a nontrivial centralizer in $\text{diff}(2, \mathbb{R})$.

$D_{3,15}$ The fourth realization of $\mathfrak{sl}(2, \mathbb{R})$ is represented by

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = y \frac{\partial}{\partial y}; \qquad X_3 = y^2 \frac{\partial}{\partial y}; \tag{4.53}$$

so that all three elements are linearly connected. The independent variable x is the only invariant in the continuous case, i.e. in the space (x, y, y', y''). The only invariant manifold is $y' = 0$, so it does not provide an invariant second order ODE.

In the discrete case we have three invariants x, x_-, x_+. The variables y, y_- and y_+ are not involved and hence we can not form an invariant difference scheme.

$D_{3,16}$ The Abelian Lie algebra can be presented as

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x \frac{\partial}{\partial y}; \qquad X_3 = \phi(x) \frac{\partial}{\partial y}; \qquad \phi''(x) \neq 0. \tag{4.54}$$

Again, all the elements of the Lie algebra $D_{3,16}$ are linearly connected. There is neither a second order ODE, nor a second order difference scheme, invariant under this group.

5 Equations invariant under higher dimensional Lie groups

The symmetry group of a second order ODE can be at most 8–dimensional. Moreover, it is 8–dimensional only if the equation can be transformed into $y'' = 0$ by a point transformation. The symmetry group is that case is $\text{SL}(3, \mathbb{F})$. Any second order ODE invariant under a Lie group of dimension 4, 5, or 6 is also invariant under $\text{SL}(3, \mathbb{F})$. No such ODE invariant under a Lie group of dimension 7 exists.

Now let us consider the case of invariant difference models.

5.1 Four–dimensional Lie algebras

Twelve isomorphism classes of indecomposable Lie algebras with $\dim L = 4$ exist, as well as ten decomposable ones [39]. Many of them can be ruled out immediately as symmetry algebras of three–point difference schemes. We already know that no difference schemes invariant under the group corresponding to the algebras $D_{3,15}$ and $D_{3,16}$ exist. Hence we can rule out all algebras containing one of these as a subalgebra.

This rules out Abelian and nilpotent Lie algebras, solvable Lie algebras with three–dimensional Abelian ideals and all decomposable Lie algebras of the type $L_{3,j} \oplus L_1$, where $L_{3,j}$ is nilpotent, solvable, or $\mathfrak{sl}(2, \mathbb{R})$ realized as $D_{3,15}$. We can also rule out the $\mathfrak{sl}(2, \mathbb{F})$ algebras with no centralizer in $\text{diff}(2, \mathbb{F})$ and the $\mathfrak{o}(3)$ algebra $D_{3,14}$ for the same reason. Indeed, L_1 must be in the centralizer of $L_{3,j}$ in $\text{diff}(2, \mathbb{F})$.

This leaves us with the following eight Lie algebras to consider.

A. Solvable, indecomposable, with nilradical $\text{NR}(L) \sim 2L_1$. There is just one class of such algebras, isomorphic to the similitude algebra of the Euclidean plane (two translations, a rotation and a uniform dilation. Over $\mathbb{R}$ the algebra is indecomposable, over $\mathbb{C}$ decomposable according to the pattern $4 = 2 + 2$. Since $2L_1$ can be realized in two ways as a subalgebra of $\text{diff}(2, \mathbb{R})$ we have two realizations of this algebra.

$D_{4,1}$ The algebra is represented as

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{5.1}$$

There are no differential invariants in the space (x, y, y', y''), but $y'' = 0$ is an invariant manifold.

There are two independent difference invariants, namely

$$I_1 = \frac{y_x - y_{\bar{x}}}{1 + y_x y_{\bar{x}}}, \qquad I_2 = \frac{h_+}{h_-}\left(\frac{1 + y_x^2}{1 + y_{\bar{x}}^2}\right)^{1/2}.$$

Hence we can write an invariant difference scheme as

$$\begin{cases} y_x - y_{\bar{x}} = C_1(1 + y_x y_{\bar{x}}); \\[2mm] h_+ = C_2 h_- \sqrt{\dfrac{1 + y_{\bar{x}}^2}{1 + y_x^2}}; \end{cases} \tag{5.2}$$

where C_1 and C_2 are arbitrary constants. However, a continuous limit exists only if we have $C_1 = 0$. The limit is then $y'' = 0$.

$D_{4,2}$ The group with infinitesimal operators

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = y\frac{\partial}{\partial y}; \qquad X_4 = (1 + x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}; \tag{5.3}$$

has no invariants in the space (x, y, y', y''). There is however an invariant manifold $y'' = 0$.

Finite–difference invariants are

$$I_1 = \frac{h_+}{1 + xx_+}, \qquad I_2 = \frac{h_-}{1 + xx_-}$$

and $y_{x\bar{x}} = 0$ is an invariant manifold. We thus have an invariant difference model

$$\begin{cases} y_{x\bar{x}} = 0; \\[2mm] h_+ = h_- \dfrac{1 + xx_+}{1 + xx_-} g\left(\dfrac{h_-}{1 + xx_-}\right). \end{cases} \tag{5.4}$$

For $g = 1$ it approximates the ODE $y'' = 0$.

B. Solvable, indecomposable with $NR(L) \sim A_{3,1}$. The nilradical will be $\{X_1, X_2, X_3\}$ (the Heisenberg algebra). The additional nonnilpotent element is X_4. Depending on the form of X_4, we obtain three mutually nonisomorphic Lie algebras.

$D_{4,3}$ The group with infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial y}; \qquad X_4 = x\frac{\partial}{\partial x} + ay\frac{\partial}{\partial y} \tag{5.5}$$

has a differential invariant, namely y'', only if $a = 2$ in X_4. For $a \neq 2$ there is an invariant manifold, $y'' = 0$.

The expressions

$$I_1 = \frac{h_+}{h_-}, \qquad I_2 = y_{x\bar{x}} h_+^{2-a}$$

form the entire set of difference invariants of the corresponding group. The general form of the invariant model is

$$\begin{cases} y_{x\bar{x}} = C_1 h_+^{a-2}; \\[2mm] h_+ = C_2 h_-. \end{cases} \tag{5.6}$$

For $a > 2$ the continuous limit is $y'' = 0$. For $\alpha < 2$ a continuous limit exists only if we choose $C_1 = 0$.

$D_{4,4}$ The group with infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial y}; \qquad X_4 = x\frac{\partial}{\partial x} + (2y + x^2)\frac{\partial}{\partial y} \tag{5.7}$$

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has neither invariants, nor invariant manifolds in the continuous case.

The expressions

$$I_1 = \frac{h_+}{h_-}, \qquad I_2 = y_{x\bar{x}} - \ln(h_-h_+)$$

form the entire set of difference invariants in this case. We can hence write an invariant difference scheme

$$\begin{cases} y_{x\bar{x}} = \ln(h_-h_+) + C_1; \\ h_+ = C_2 h_-; \end{cases} \tag{5.8}$$

but it does not have a continuous limit.

$D_{4,5}$ The group with infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial y}; \qquad X_4 = y\frac{\partial}{\partial y}; \tag{5.9}$$

has no invariants in the space (x, y, y', y''), but there is the invariant manifold $y'' = 0$.

The step lengths h_+ and h_- form a basis of difference invariants and $y_{x\bar{x}} = 0$ is an invariant manifold.

The general form of the invariant difference model is

$$\begin{cases} y_{x\bar{x}} = 0; \\ h_+ = h_- g(h_-). \end{cases} \tag{5.10}$$

C. The decomposable Lie algebra $A_2 \oplus A_2$ can be realized in two inequivalent manners.

$D_{4,6}$ The Lie algebra is

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x}; \qquad X_4 = y\frac{\partial}{\partial y}; \tag{5.11}$$

and exist both over $\mathbb{C}$ and $\mathbb{R}$. Over $\mathbb{C}$ $D_{4,1}$ and $D_{4,6}$ are equivalent.

There are no differential invariants, but $y'' = 0$ is an invariant manifold. Difference invariants are

$$I_1 = \frac{h_+}{h_-}, \qquad I_2 = \frac{y_x}{y_{\bar{x}}}.$$

The general form of the invariant difference scheme can be written as

$$\begin{cases} y_{x\bar{x}} = C_1 \dfrac{y_{\bar{x}}}{h_-}; \\ h_+ = C_2 h_-; \end{cases} \tag{5.12}$$

however a continuous limit exists only for $C_1 = 0$.

$D_{4,7}$ The second realization of $A_2 \oplus A_2$ is

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x}; \qquad X_4 = y\frac{\partial}{\partial y} \tag{5.13}$$

Over $\mathbb{R}$ this is new, over $\mathbb{C}$ $D_{4,2}$ and $D_{4,7}$ are equivalent. There are no differential invariants in the space (x, y, y', y''), but $y'' = 0$ is an invariant manifold.

The ratios $\frac{h_+}{x}$ and $\frac{h_-}{x}$ are difference invariants and $y_{x\bar{x}} = 0$ is an invariant manifold. The general invariant difference scheme is

$$\begin{cases} y_{x\bar{x}} = 0; \\ h_+ = h_- g\left(\dfrac{h_-}{x}\right). \end{cases} \tag{5.14}$$

D. The decomposable Lie algebra $\mathfrak{sl}(2, \mathbb{F}) \oplus A_1$ can be realized in a single manner allowing invariant differential or difference equations.

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$D_{4,8}$ The group with infinitesimal generators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = x\frac{\partial}{\partial x}; \qquad X_3 = y\frac{\partial}{\partial y}; \qquad X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{5.15}$$

has no differential invariants, but does leave the manifold $y'' = 0$ invariant.

The expressions

$$I_1 = h_+ h_- \frac{y_{x\bar{x}}}{y}, \qquad I_2 = \frac{y_- h_+}{y_+ h_-}$$

generate the entire set of difference invariants. The general form of an invariant difference model is

$$\begin{cases} y_{x\bar{x}} = \dfrac{C_1}{h_+ h_-} y; \\[2ex] h_+ y_- = C_2 h_- y_+. \end{cases} \tag{5.16}$$

A continuous limit exists only for $C_1 = 0$.

5.2 Five–dimensional Lie algebras

The number of isomorphism classes of five–dimensional Lie algebras is quite large. We can immediately exclude all those that are not subalebras of $\mathrm{diff}(2, \mathbb{F})$, that contain a three–dimensional Abelian subalgebra, or the $\mathfrak{sl}(2, \mathbb{F})$ algebra $D_{3,15}$ of eq. (4.53).

Finally, only two five–dimensional Lie algebras provide invariant difference schemes.

$D_{5,1}$ A solvable Lie algebra with the Heisenberg algebra $A_{3,1}$ as its nilradical.

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x}; \qquad X_4 = x\frac{\partial}{\partial y}; \qquad X_5 = y\frac{\partial}{\partial y}. \tag{5.17}$$

The corresponding group has no differential invariants, but $y'' = 0$ is an invariant manifold.

The group has one difference invariant, namely $\xi = \frac{h_+}{h_-}$, and one invariant manifold, namely

$$\eta = (x - x_-)(y_+ - y) - (x_+ - x)(y - y_-). \tag{5.18}$$

The most general invariant difference scheme can hence be written as

$$\begin{cases} y_{x\bar{x}} = 0; \\[2ex] h_+ = Ch_-; \end{cases} \tag{5.19}$$

where C is an arbitrary constant.

$D_{5,2}$ The special affine Lie algebra $\mathfrak{saff}(2, \mathbb{F})$

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = y\frac{\partial}{\partial x}; \qquad X_4 = x\frac{\partial}{\partial y}; \qquad X_5 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{5.20}$$

There is no differential invariant, but $y'' = 0$ is an invariant manifold.

There are no difference invariants either, but $\eta = 0$ of eq. (5.18) is an invariant manifold. Eq. (5.19) again provide a (weakly) invariant difference scheme. Note that the relation $\xi = x_+ - x - C(x - x_-) = 0$ is invariant on the manifold $\eta = 0$ (It is strongly invariant under the group generated by $\{X_1, X_2, X_5\}$.

5.3 A six-dimensional symmetry algebra

$D_{6,1}$ The general affine Lie algebra $\mathfrak{gaff}(2, \mathbb{F})$

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x};$$

$$X_4 = y\frac{\partial}{\partial x}; \qquad X_5 = x\frac{\partial}{\partial y}; \qquad X_6 = y\frac{\partial}{\partial y}. \tag{5.21}$$

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This algebra contain $D_{5,1}$ and $D_{5,2}$ as subalgebras. Again, $y'' = 0$ is an invariant manifold in the continuous case and eq. (5.19), with C arbitrary, provides a weakly invariant difference scheme.

The ODE $y'' = 0$ is invariant under a larger group, namely $SL(3, \mathbb{F})$ of dimension 8. There are no three–point difference schemes invariant under Lie groups of dimension $d \geq 7$. We shall compare the continuous and discrete situations in the following section.

The obtained classification of invariant difference schemes is summed up in Table 1.

6   The free particle equation and its discretization

The free particle equation

$$y'' = 0. \tag{6.1}$$

is invariant (weakly) under the group $SL(3, \mathbb{F})$. The most general second order ODE invariant under $SL(3, \mathbb{F})$ is known [43, 44, 45, 46, 47] and is quite complicated. It can be transformed into eq. (6.1) by a point transformation. Every linear second order ODE is invariant under $SL(3, \mathbb{F})$.

A three–point discretization of the free particle equation (6.1) should have the form

$$y_{x\bar{x}} = 0;$$
$$\Omega(x, x_-, x_+, y_{\bar{x}}, y_x) = 0; \tag{6.2}$$

where $\Omega = 0$ determines the mesh.

In the continuous case the second prolongations of the 8 basis elements of $\mathfrak{sl}(3, \mathbb{F})$ are

$$\mathrm{pr}^{(2)} X_1 = \frac{\partial}{\partial x};$$

$$\mathrm{pr}^{(2)} X_2 = \frac{\partial}{\partial y};$$

$$\mathrm{pr}^{(2)} X_3 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'};$$

$$\mathrm{pr}^{(2)} X_4 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'};$$

$$\mathrm{pr}^{(2)} X_5 = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''}; \tag{6.3}$$

$$\mathrm{pr}^{(2)} X_6 = y\frac{\partial}{\partial x} - (y')^2\frac{\partial}{\partial y'} - 3y'y''\frac{\partial}{\partial y''};$$

$$\mathrm{pr}^{(2)} X_7 = x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + (y - xy')\frac{\partial}{\partial y'} - 3xy''\frac{\partial}{\partial y''};$$

$$\mathrm{pr}^{(2)} X_8 = y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + y'(y - xy')\frac{\partial}{\partial y'} - 3xy'y''\frac{\partial}{\partial y''}.$$

Thus $I = y''$ is an invariant of the group generated by the subalgebra $\{X_1, X_2, X_3, X_4\}$ and $y'' = 0$ is an invariant manifold for the entire group (i.e. the coefficients of $\frac{\partial}{\partial y''}$ vanish for $X_1, \ldots, X_4$ and are proportional to y'' for $X_5, \ldots, X_8$).

This realization of $\mathfrak{sl}(3,\mathbb{F})$ as a subalgebra of $\mathrm{diff}(2,\mathbb{F})$ will be called $D_{8,1}$. In the discrete case we have

$$
\begin{aligned}
\mathrm{pr}^{(2)}X_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}\\[4pt]
\mathrm{pr}^{(2)}X_2 &= \frac{\partial}{\partial y} + \frac{\partial}{\partial y_+} + \frac{\partial}{\partial y_-}\\[4pt]
\mathrm{pr}^{(2)}X_3 &= x\frac{\partial}{\partial y} + x_+\frac{\partial}{\partial y_+} + x_-\frac{\partial}{\partial y_-}\\[4pt]
\mathrm{pr}^{(2)}X_4 &= x\frac{\partial}{\partial x} + x_+\frac{\partial}{\partial x_+} + x_-\frac{\partial}{\partial x_-} + 2y\frac{\partial}{\partial y} + 2y_+\frac{\partial}{\partial y_+} + 2y_-\frac{\partial}{\partial y_-}\\[4pt]
\mathrm{pr}^{(2)}X_5 &= y\frac{\partial}{\partial y} + y_+\frac{\partial}{\partial y_+} + y_-\frac{\partial}{\partial y_-}\\[4pt]
\mathrm{pr}^{(2)}X_6 &= y\frac{\partial}{\partial x} + y_+\frac{\partial}{\partial x_+} + y_-\frac{\partial}{\partial x_-}\\[4pt]
\mathrm{pr}^{(2)}X_7 &= x^2\frac{\partial}{\partial x} + x_+^2\frac{\partial}{\partial x_+} + x_-^2\frac{\partial}{\partial x_-} + xy\frac{\partial}{\partial y} + x_+y_+\frac{\partial}{\partial y_+} + x_-y_-\frac{\partial}{\partial y_-}\\[4pt]
\mathrm{pr}^{(2)}X_8 &= xy\frac{\partial}{\partial x} + x_+y_+\frac{\partial}{\partial x_+} + x_-y_-\frac{\partial}{\partial x_-} + y^2\frac{\partial}{\partial y} + y_+^2\frac{\partial}{\partial y_+} + y_-^2\frac{\partial}{\partial y_-}
\end{aligned}
\tag{6.4}
$$

The subalgebra $\{X_1,\ X_2,\ X_3,\ X_4\}$ has two difference invariants, namely

$$
I_1 = y_{x\bar{x}}, \quad I_2 = \frac{h_+}{h_-}.
\tag{6.5}
$$

Moreover, the matrix of the coefficients of the derivatives in eq. (6.4) has rank 3, rather than rank 4, on the surface

$$
y_{x\bar{x}} = 0, \quad \text{i.e.} \quad (x-x_-)(y_+-y) - (x_+-x)(y-y_-) = 0.
\tag{6.6}
$$

Thus $y_{x\bar{x}} = 0$ is an invariant manifold, but it is the only one, and we need an independent equation to determine the mesh.

We have

$$
\mathrm{pr}^{(2)}X_i I_2 \bigg|_{I_1 = 0} = 0, \qquad \alpha = 1,\ldots,6.
\tag{6.7}
$$

However, we have

$$
\begin{aligned}
\mathrm{pr}^{(2)}X_7 I_2 \bigg|_{I_1 = 0} &= I_2(h_+ + h_-) \neq 0;\\[6pt]
\mathrm{pr}^{(2)}X_8 I_2 \bigg|_{I_1 = 0} &= I_2(\eta_+ + \eta_-) \neq 0;
\end{aligned}
\tag{6.8}
$$

so I_2 is not invariant on the surface $I_1 = 0$.

We are not allowed to set $I_2 = 0$ since h_+ and h_- by assumption satisfy $h_+ > 0$, $h_- > 0$.

Finally, we find that the difference scheme (5.19) is (weakly) invariant under the general affine group generated by $D_{6,1}$, but not under the larger group $SL(3,\mathbb{F})$.

The difference scheme (6.2) is invariant under subgroups of $D_{6,1}$ for more general meshes than those satisfying $h_+ = Ch_-$. These are obtained from the results of Sections 3, 4 and 5. The invariant mesh equations and maximal invariance algebras are given in Table 2.

Finally, while there is no difference scheme invariant under $SL(3,\mathbb{F})$, there are schemes of the type (5.19), invariant under subgroups of $SL(3,\mathbb{F})$, not contained in $D_{6,1}$. The corresponding algebras are $D_{3,11}$, $D_{3,14}$, $D_{4,2}$ and $D_{4,8}$.

7 Conclusions

Let us first of all compare Lie's classification of second order ordinary differential equations with the obtained classification of three point difference schemes.

1. For every ODE invariant under a Lie group G of dimension n, $1 \leq n \leq 3$, there exist a family of different schemes invariant under the same group G. In particular, for $n = 3$, the invariant ODE is specified up to at most a constant. The invariant difference scheme in general involves two arbitrary functions.

2. All ODEs invariant under a Lie group G of dimension $n = 4$, 5, or 6 are also invariant under $\mathrm{SL}(3, \mathbb{F})$ and we have $G \subset \mathrm{SL}(3, \mathbb{F})$. Moreover, the ODE can be transformed into $y'' = 0$. Three point difference schemes invariant under groups of dimensions 4, 5, and 6 exist. If a continuous limit exists, it is $y'' = 0$. However, other invariant schemes exist that do not have continuous limits. (see $\mathrm{D}_{4,1}$, $\mathrm{D}_{4,4}$, $\mathrm{D}_{4,6}$, $\mathrm{D}_{4,8}$).

3. The "discrete free particle equation" $y_{x\bar{x}} = 0$ has a different symmetry behavior from its continuous limit. First of all, it is invariant at most under a six–dimensional Lie group, namely the group of all linear transformations of the space $\mathbb{R}^2$, or $\mathbb{C}^2$, i.e. $\{x, y\}$. This invariance occurs on the mesh $h_+ = Ch_-$, where $C > 0$ is a constant. We have shown in Section 6 that the equation $y_{x\bar{x}} = 0$ is invariant under groups of dimension $1 \leq n \leq 4$ for more general meshes (see Table 2).

Our main results are summed up in two tables. Table 1 presents all invariant three point difference schemes. In Column 1 we identify the Lie algebra of the invariance group, using the notations of Sections 2–5. The difference equations and meshes are given in Columns 2 and 3 for each algebra. The invariant ODE is given in Column 4. The ODEs involve arbitrary functions for $1 \leq \dim L \leq 2$. For $\dim L \geq 3$ they are either completely specified or involve arbitrary constants. The difference schemes in general involve two functions. In the limit $h_+ \to 0$, $h_- \to 0$ they can be specified to obtain the correct continuous limit. The difference schemes are linear whenever the ODEs are, and vice versa.

In Table 2 we sum up the results on the free particle equation $y_{x\bar{x}} = 0$. For each Lie algebra in Column 2 the invariant mesh is in Column 3. We list those algebras that are maximal for the given mesh.

Work is in progress on constructing invariant Lagrangians, first integrals and solutions of the obtained difference schemes.

Acknowledgments

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Table 1: Group classification of three-point difference schemes and their continuous limits. The functions f and g are nonsingular for $h_+ \to 0$, $h_- \to 0$, otherwise arbitrary.

Group	Difference Equation	Mesh	Differential Equation
$D_{1,1}$	$y_{x\bar{x}} = f\left(x, \dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$	$h_+ = h_- g\left(x, \dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$	$y'' = F(x, y')$
$D_{2,1}$	$y_{x\bar{x}} = f\left(\dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$	$y'' = F(y')$
$D_{2,2}$	$y_{x\bar{x}} = f(x, h_-)$	$h_+ = h_- g(x, h_-)$	$y'' = F(x)$
$D_{2,3}$	$y_{x\bar{x}} = \dfrac{1}{x} f\left(\dfrac{y_x + y_{\bar{x}}}{2}, \dfrac{h_-}{x}\right)$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, \dfrac{h_-}{x}\right)$	$y'' = \dfrac{1}{x} F(y')$
$D_{2,4}$	$y_{x\bar{x}} = \dfrac{y_x + y_{\bar{x}}}{2} f(x, h_-)$	$h_+ = h_- g(x, h_-)$	$y'' = F(x) y'$
$D_{3,1}$	$y_{x\bar{x}} = f(h_-)$	$h_+ = h_- g(h_-)$	$y'' = C$
$D_{3,2}$	$y_{x\bar{x}} = f(h_-) \exp(x)$	$h_+ = h_- g(h_-)$	$y'' = C\exp(x)$
$D_{3,3}$	$y_{x\bar{x}} = \dfrac{y_x + y_{\bar{x}}}{2} f(h_-)$	$h_+ = h_- g(h_-)$	$y'' = Cy'$
$D_{3,4}$	$y_{x\bar{x}} = 0$	$h_+ = h_- g(x, h_-)$	$y'' = 0$
$D_{3,5}$	$y_{x\bar{x}} = x^{(2a-1)/(1-a)} f\left(\dfrac{h_-}{x}\right)$	$h_+ = h_- g\left(\dfrac{h_-}{x}\right)$	$y'' = Cx^{(2a-1)/(1-a)}$
$D_{3,6}$	$y_{x\bar{x}} = \dfrac{\exp(b\arctan(x))}{\sqrt{1+x^2}} \dfrac{1}{\left(\dfrac{h_+}{h_- + h_+}\dfrac{1}{1+xx_+} + \dfrac{h_-}{h_- + h_+}\dfrac{1}{1+xx_-}\right)}$ $\times\, f\left(\dfrac{h_-}{1+xx_-}\right)$	$h_+ = h_- \dfrac{1+xx_+}{1+xx_-} g\left(\dfrac{h_-}{1+xx_-}\right)$	$y'' = C\dfrac{\exp(b\arctan(x))}{(1+x^2)^{3/2}}$
$D_{3,7}$	$h_- y_{x\bar{x}} = f\left(\dfrac{y_x + y_{\bar{x}}}{2}\right)$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}\right)$	$y'' = 0$

Table 1: (continued)

Group	Difference Equation	Mesh	Differential Equation
$D_{3,8}$	$y_{x\bar{x}} = \exp\left(-\frac{y_x+y_{\bar{x}}}{2}\right) f\left(\sqrt{h_-h_+}\exp\left(-\frac{y_x+y_{\bar{x}}}{2}\right)\right)$	$h_+ = h_- g\left(\sqrt{h_-h_+}\exp\left(-\frac{y_x+y_{\bar{x}}}{2}\right)\right)$	$y'' = \exp(-y')$
$D_{3,9}$	$y_{x\bar{x}} = \left(\frac{y_x+y_{\bar{x}}}{2}\right)^{(k-2)/(k-1)} f\left(\frac{y_x+y_{\bar{x}}}{2}h_-^{(1-k)}\right)$	$h_+ = h_- g\left(\frac{y_x+y_{\bar{x}}}{2}h_-^{(1-k)}\right)$	$y'' = y^{(k-2)/(k-1)}$
$D_{3,10}$	$y_{x\bar{x}} = \left(\frac{h_+}{h_-+h_+}\sqrt{1+y_x^2}\exp(k\arctan y_x)\right.$ $+\frac{h_-}{h_-+h_+}\sqrt{1+y_{\bar{x}}^2}\exp(k\arctan y_{\bar{x}})$ $\left.\times(1+y_xy_{\bar{x}})f(h_-\sqrt{1+y_{\bar{x}}^2}\exp(k\arctan(y_{\bar{x}})))\right.$	$h_+\sqrt{1+y_x^2}\exp(k\arctan(y_x))$ $=h_-\sqrt{1+y_{\bar{x}}^2}\exp(k\arctan(y_{\bar{x}}))$ $+g(h_-\sqrt{1+y_{\bar{x}}^2}\exp(k\arctan(y_{\bar{x}})))$	$y'' = (1+(y')^2)^{3/2}\exp(k\arctan(y'))$
$D_{3,11}$	$y_{x\bar{x}} = \frac{1}{y^2}\left(\frac{h_+}{h_++h_-}\frac{1}{y_+}+\frac{h_-}{h_++h_-}\frac{1}{y_-}\right)$ $\times f\left(\frac{1}{y^2}\frac{h_+h_-}{h_++h_-}\right)$	$\frac{1}{y}\left(\frac{h_+}{y_+}+\frac{h_-}{y_-}\right)$ $=4\frac{1}{y^2}\frac{h_+h_-}{h_++h_-}g\left(\frac{1}{y^2}\frac{h_+h_-}{h_++h_-}\right)$	$y'' = y^{-3}$
$D_{3,12}$	$\frac{2y(h_++h_-)+2y(h_+y_x^2+h_-y_{\bar{x}}^2+2yy_x)(h_-(1+y_{\bar{x}}^2)-2yy_{\bar{x}})+4y^2(y_x-y_{\bar{x}})}{4y^2-(h_+(1+y_x^2)+2yy_x)(h_-(1+y_{\bar{x}}^2)-2yy_{\bar{x}})}$ $=\frac{1}{2}\left(\sqrt{\frac{h_-^2+(y-y_-)^2}{yy_-}}+\sqrt{\frac{h_+^2+(y_+-y)^2}{yy_+}}\right)$ $\times f\left(\frac{h_-^2+(y-y_-)^2}{yy_-}\right)$	$\frac{h_+^2+(y_+-y)^2}{yy_+}$ $=\frac{h_-^2+(y-y_-)^2}{yy_-}g\left(\frac{h_-^2+(y-y_-)^2}{yy_-}\right)$	$yy'' = C(1+(y')^2)^{3/2}-(1+(y')^2)$
$D_{3,13}$	$\frac{y_x}{(x-y_+)(x_--y)}-\frac{y_{\bar{x}}}{(x-y_-)(x_+-y)}$ $=f\left(\frac{x_+-y}{x-y}\frac{h_-}{h_-+h_+}\right)(h_-+h_+)\sqrt{(x_--y)(x_+-y)}$ $\times\left(\frac{y_x}{(x-y_+)(x_--y)}+\frac{y_{\bar{x}}}{(x-y_-)(x_+-y)}\right)^{3/2}$	$\frac{h_+^2 y_x}{(x-y_+)(x_+-y)}$ $=\frac{h_-^2 y_{\bar{x}}}{(x-y_-)(x_+-y)}g\left(\frac{x_+-y}{x-y}\frac{h_-}{h_-+h_+}\right)$	$y''+\frac{2}{x-y}(y'+y'^2)=\frac{2C}{x-y}y'^{3/2}$
$D_{3,14}$	$\frac{h_+h_-(y_x-y_{\bar{x}})}{\sqrt{1+x_-^2+y^2}\sqrt{1+x^2+y^2}\sqrt{1+x_+^2+y^2}\sqrt{1+x^2+y^2}}$ $=f\left(\frac{h_-^2(1+y_{\bar{x}}^2+(y-xy_{\bar{x}})^2)}{(1+x_-^2+y^2)(1+x^2+y^2)}\right)$	$\frac{h_+^2(1+y_x^2+(y-xy_x)^2)}{(1+x^2+y^2)(1+x_+^2+y^2)}$ $=g\left(\frac{h_-^2(1+y_{\bar{x}}^2+(y-xy_{\bar{x}})^2)}{(1+x_-^2+y^2)(1+x^2+y^2)}\right)$	$y'' = C\left(\frac{1+y'^2+(y-xy')^2}{1+x^2+y^2}\right)^{3/2}$

Table 1: (continued)

Group	Difference Equation	Mesh	Differential Equation
$D_{3,15}$	—	$g(x_-, x, x_+) = 0$	$y' = 0$
$D_{3,16}$	—	$g(x_-, x, x_+) = 0$	—
$D_{4,1}$	$y_x - y_{\bar{x}} = C_1(1 + y_x y_{\bar{x}})$	$h_+ = C_2 h_- \sqrt{\dfrac{1+y_{\bar{x}}^2}{1+y_x^2}}$	$y'' = 0$
$D_{4,2}$	$y_{x\bar{x}} = 0$	$h_+ = h_- \dfrac{1+xx_+}{1+xx_-}\, g\left(\dfrac{h_-}{1+xx_-}\right)$	$y'' = 0$
$D_{4,3}$	$y_{x\bar{x}} = C_1 h_+^{a-2}$	$h_+ = C_2 h_-$	$y'' = \begin{cases} 0 & \text{if } a \neq 2, \\ C & \text{if } a = 2 \end{cases}$
$D_{4,4}$	$y_{x\bar{x}} = \ln(h_- h_+) + C_1$	$h_+ = C_2 h_-$	—
$D_{4,5}$	$y_{x\bar{x}} = 0$	$h_+ = h_- g(h_-)$	$y'' = 0$
$D_{4,6}$	$y_{x\bar{x}} = C_1 \dfrac{y_{\bar{x}}}{h_-}$	$h_+ = C_2 h_-$	$y'' = 0$
$D_{4,7}$	$y_{x\bar{x}} = 0$	$h_+ = h_- g\left(\dfrac{h_-}{x}\right)$	$y'' = 0$
$D_{4,8}$	$y_{x\bar{x}} = \dfrac{C_1}{h_+ h_-} y$	$h_+ y_- = C_2 h_- y_+$	$y'' = 0$
$D_{5,1}$	$y_{x\bar{x}} = 0$	$h_+ = C h_-$	$y'' = 0$
$D_{5,2}$	$y_{x\bar{x}} = 0$	$h_+ = C h_-$	$y'' = 0$
$D_{6,1}$	$y_{x\bar{x}} = 0$	$h_+ = C h_-$	$y'' = 0$
$D_{8,1}$	$y_{x\bar{x}} = 0$	—	$y'' = 0$

Table 2: Invariant discretizations of the equation $y'' = 0$. The discrete equation is $y_{x\bar{x}} = 0$, the invariant mesh is given in the table. We only list the maximal invariance algebras for a given mesh. Throughout g is an arbitrary function of its arguments, nonsingular for $h_- \to 0$.

dim L	Algebra	Mesh
1	$D_{1,1}$: $\quad X_1 = \frac{\partial}{\partial y}$	$h_+ = h_- g\left(x, \dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$
2	$D_{2,1}$: $\quad X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y}$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, h_-\right)$
2	$D_{2,3}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}, \dfrac{h_-}{x}\right)$
3	$D_{3,4}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial y},\ X_3 = y\frac{\partial}{\partial y}$	$h_+ = h_- g(x, h_-)$
3	$D_{3,7}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2}\right)$
3	$D_{3,8}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$	$h_+ = h_- g\left(\sqrt{h_- h_+} \exp\left(-\dfrac{y_x + y_{\bar{x}}}{2}\right)\right)$
3	$D_{3,9}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y},\ k \neq 0, 1$	$h_+ = h_- g\left(\dfrac{y_x + y_{\bar{x}}}{2} h_-^{(1-k)}\right)$
3	$D_{3,10}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = (kx+y)\frac{\partial}{\partial x} + (ky - x)\frac{\partial}{\partial y}$	$h_+ \sqrt{1+y_x^2}\exp(k\arctan(y_x)) = h_- \sqrt{1+y_{\bar{x}}^2}\exp(k\arctan(y_{\bar{x}})) + g(h_- \sqrt{1+y_{\bar{x}}^2}\exp(k\arctan(y_{\bar{x}})))$
3	$D_{3,11}$: $\quad X_1 = \frac{\partial}{\partial x},\ X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\ X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$\dfrac{1}{y}\left(\dfrac{h_+}{y_+} + \dfrac{h_-}{y_-}\right) = 4\dfrac{1}{y^2}\dfrac{h_+ h_-}{h_+ + h_-} g\left(\dfrac{1}{y^2}\dfrac{h_+ h_-}{h_+ + h_-}\right)$
3	$D_{3,14}$: $\quad X_1 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\ X_2 = xy\frac{\partial}{\partial x} + (1+y^2)\frac{\partial}{\partial y},\ X_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$\dfrac{h_+^2(1+y_x^2) + (y - xy_x)^2}{(1+x^2+y^2)(1+x_+^2+y_+^2)} = g\left(\dfrac{h_-^2(1+y_{\bar{x}}^2) + (y - xy_{\bar{x}})^2}{(1+x_-^2+y_-^2)(1+x^2+y^2)}\right)$
4	$D_{4,2}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial y},\ X_3 = y\frac{\partial}{\partial y},\ X_4 = (1+x^2)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$h_+ = h_- \dfrac{1+xx_+}{1+xx_-} g\left(\dfrac{h_-}{1+xx_-}\right)$
4	$D_{4,5}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial y},\ X_4 = y\frac{\partial}{\partial y}$	$h_+ = h_- g(h_-)$
4	$D_{4,7}$: $\quad X_1 = \frac{\partial}{\partial y},\ X_2 = x\frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = y\frac{\partial}{\partial y}$	$h_+ = h_- g\left(\dfrac{h_-}{x}\right)$

Table 2: (continued)

dim L	Algebra	Mesh
4	$D_{4,8}$: $\quad X_1 = \frac{\partial}{\partial x},\ X_2 = x\frac{\partial}{\partial x},\ X_3 = y\frac{\partial}{\partial y},\ X_4 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$	$h_+ = Ch_- \frac{y_+}{y_-}$
6	$D_{6,1}$: $\quad X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial y},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = y\frac{\partial}{\partial x},\ X_5 = x\frac{\partial}{\partial y},\ X_6 = y\frac{\partial}{\partial y}$	$h_+ = Ch_-$