Separation of Variables and Darboux Transformations

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Abstract

A classification of maximal Abelian subalgebras (MASAs) of the pseudoeuclidean Lie algebras is presented. It is then used to generate coordinate systems in the pseudo-Euclidean space M(p,q) in which the Laplace-Beltrami equation allows the separation of variables. The MASAs induce ignorable variables; i.e. variables that do not figure explicitly in the metric tensor. Darboux transformations are then used to introduce solvable potentials depending on these ignorable variables.

Résumé

La classification des sous-algèbres maximales abéliennes (SAMAs) d'algèbre pseudo-euclidiennes est présentée. On l'utilise pour construire des systèmes de coordonnées dans l'espace pseudo-euclidien M(p,q) das lesquels l'équation de Laplace-Beltrami admet une séparation des variables. Les SAMA induisent des variables ignorables, i.e. des variables qui n'apparaissent pas explicitement dans le tenseur métrique. Les transformations de Darboux sont utilisées pour introduire des potentiels solubles qui dépendent de ces variables ignorables.

1 Introduction

The purpose of this article is to show how classical Darboux transformations [1] can be applied to partial differential equations. The context will be that of the separation of variables in Laplace and wave equations, or more generally, in Laplace-Beltrami equations on homogeneous spaces. We shall use separable coordinates with a maximal number of ignorable variables, i.e. coordinates that do not figure in the metric tensor and hence in the Laplace-Beltrami operator. The ignorable variables are induced by the action of maximal Abelian subgroups of the isometry group of the corresponding space. A classification of the maximal Abelian subalgebras (MASAs) of the isometry algebra will provide a classification of coordinate systems with a maximal number of ignorable variables.

In Euclidean spaces only two types of ignorable variables exist. Cartesian ones correspond to translation generators P_i in a MASA, polar ones to rotation generators M_{jk}. In pseudoeuclidean spaces $E(p,q)$, $p \geq q \geq 1$ the situation is much richer, since MASAs of the isometry algebra can involve nilpotent elements in $o(p,q)$. This is already true for Minkowski space ($q=1$), as we shall see below.

Darboux transformations have gained new importance, new applications and new geometric applications with the advent of soliton theory (see Ref. [2], [3] and the references therein).

They were first introduced by Darboux in 1882 in a study of linear ordinary differential equations [1], in particular the Sturm-Liouville problem

$$\psi''(x) - u(x)\psi(x) = -E\psi(x). \tag{1.1}$$

Indeed, let us assume that we know two distinct solutions ψ and ψ_0 of the above equation corresponding to the eigenvalues E and E_0, respectively. The Darboux transformation

$$\tilde{\psi} = (\frac{d}{dx} - \frac{\psi_0'}{\psi_0})\psi \tag{1.2}$$

provides us with a solution of a related equation, namely

$$\tilde{\psi}''(x) - \tilde{u}(x)\tilde{\psi}(x) = -E\tilde{\psi}(x) \tag{1.3}$$

$$\tilde{u}(x) = u - 2(\frac{\psi_0'}{\psi_0})'. \tag{1.4}$$

Eq. (1.2) gives us solutions the new Sturm-Liouville problems in terms of known ones. Notice that general eigenvalue E is the same in the original equation (1.1) and in the new equation (1.2). The "specific" eigenvalue E_0 only figures via the eigenfunction ψ_0 in the new potential $\tilde{u}(x)$ of eq. (1.4) and the new solution $\tilde{\psi}(x)$ in eq. (1.2).

In Section 2 we sum up some previously obtained results [4, 5] on the MASAs of the pseudoeuclidean algebras $e(p,q)$ with an emphasis on $e(p,0)$ and $e(p,1)$. Section 3 is devoted to separable coordinate systems in Euclidean and Minkowski spaces related to these MASAs. Finally, in Section 4 we use Darboux transformations to introduce multidimensional solvable potentials into Laplace and wave equations.

2 MASAs of e(p,q)

2.1 General formulation

The pseudoeuclidean Lie algebra $e(p,q)$ is the semidirect sum of the pseudoorthogonal Lie algebra $o(p,q)$ and an abelian algebra $T(n)$ of translations

$$e(p,q) = o(p,q) \niplus T(n), \qquad n = p+q. \tag{2.1}$$

We will make use of the following matrix representation of the Lie algebra $e(p,q)$ and the corresponding Lie group $E(p,q)$. We introduce an "extended metric"

$$K_e = \begin{pmatrix} K & 0 \\ 0 & 0_1 \end{pmatrix}, \tag{2.2}$$

where K satisfies

$$K = K^T \in \mathbb{R}^{n\times n}, \quad n = p+q, \quad detK \neq 0, \tag{2.3}$$

$$sgnK = (p,q), \qquad p \geq q \geq 0. \tag{2.4}$$

Here $sgnK$ denotes the signature of K, where p and q are the numbers of positive and negative eigenvalues, respectively. Then $X_e \in e(p,q)$ and $H \in E(p,q)$ are represented as

$$X_e(X,\alpha) \equiv X_e = \begin{pmatrix} X & \alpha^T \\ 0 & 0 \end{pmatrix}, \qquad X \in \mathbb{R}^{n\times n}, \qquad \alpha \in \mathbb{R}^{1\times n}, \tag{2.5}$$

$$H = \begin{pmatrix} G & a^T \\ 0 & 1 \end{pmatrix}, \qquad G \in \mathbb{R}^{n\times n}, \qquad a \in \mathbb{R}^{1\times n}, \tag{2.6}$$

$$XK + KX^T = 0, \quad GKG^T = K, \qquad X_eK_e + K_eX_e^T = 0. \tag{2.7}$$

The vector $\alpha \in \mathbb{R}^{1\times n}$ represents the translations. We say that the translations are of positive, negative or zero length if

$$\alpha K \alpha^T > 0, \quad \alpha K \alpha^T < 0, \quad \alpha K \alpha^T = 0, \tag{2.8}$$

respectively. Zero length vectors are called isotropic.

We will be classifying maximal Abelian subalgebras of the pseudoeuclidean Lie algebra $e(p,q)$ into conjugacy classes under the action of the pseudoeuclidean Lie group $E(p,q)$.

2.2 Classification strategy

The classification of MASAs of $e(p,q)$ is based on the fact that $e(p,q)$ is the semidirect sum of the Lie algebra $o(p,q)$ and an abelian ideal $T(n)$ (the translations). We proceed in five steps.

1. Classify subalgebras $T(k_+,k_-,k_0)$ of $T(n)$. They are characterized by a triplet (k_+,k_-,k_0), where k_+, k_- and k_0 are the number of positive length, negative length and isotropic vectors, respectively. They are represented by matrices of type (2.5) with $X = 0$.

2. Find the centralizer $C(k_+,k_-,k_0)$ of $T(k_+,k_-,k_0)$ in $o(p,q)$

$$C(k_+,k_-,k_0) = \{X \in o(p,q) | [X, T(k_+,k_-,k_0)] = 0\}. \tag{2.9}$$

3. Construct all MASAs $M(k_+,k_-,k_0)$ of $C(k_+,k_-,k_0)$ and classify them under the action of the normalizer $N \equiv Nor[T(k_+,k_-,k_0), G]$ of $T(k_+,k_-,k_0)$ in the group $G \sim E(p,q)$.

4. Obtain a representative list of all splitting MASAs of $e(p,q)$ as direct sums

$$M(k_+,k_-,k_0) \oplus T(k_+,k_-,k_0) \tag{2.10}$$

and keep only those amongst them that are indeed maximal (and mutually inequivalent).

5. Construct all nonsplitting MASAs from the splitting ones and classify them under the action of the group $N \ni T(n)$. A nonsplitting MASA is any MASA of $e(p,q)$ not conjugate to one of the form (2.10). This involves some elementary cohomology theory and the procedure is described elsewhere [4].

2.3 MASAs of e(p,0) and e(p,1)

In this section we shall list all splitting MASAs of $e(p,0)$ and $e(p,1)$. The results are given as theorems. For the proofs we refer to the original articles [4, 5].

Let us first consider the real Euclidean algebra $e(n,0)$ with a basis $\{P_i, L_{jk}, i = 1,\dots,n,\ 1 \leq k \leq n\}$ consisting of infinitesimal translations and rotations. The structure of the MASAs is very simple and we have:

Theorem 2.1 *Every MASA of $e(p,0)$ splits into the direct sum $M(k) = F(k) \oplus T(k)$ and is $E(p,0)$ conjugate to precisely one subalgebra with*

$$F(k) = \{L_{12}, L_{34}, \ldots, L_{2l-1,2l}\}, \quad T(k) = \{P_{2l+1}, \ldots, P_p\}$$

where k is such that $p-k$ is even ($p-k$=2l).

Now let us consider the Lie algebra $e(n,1)$. Every MASA of $e(n,1)$ can be represented by a matrix pair $\{M, K_e\}$, satisfying

$$M(k_+, k_-, k_0) \equiv M = \begin{pmatrix} M_0 & & & & & \gamma^T \\ & M_1 & & & & 0 \\ & & \ddots & & & \vdots \\ & & & M_l & & 0 \\ & & & & 0_{k_+} & x^T \\ & & & & & 0_1 \end{pmatrix}, \tag{2.11}$$

$$K_e = \begin{pmatrix} K_0 & & & \\ & I_{2l} & & \\ & & I_{k_+} & \\ & & & 0_1 \end{pmatrix}, \qquad sgnK_0 = (n - k_+ - 2l, 1),$$

where $M_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$, $a_i \in \mathrm{R}$, $x \in \mathrm{R}^{1\times k_+}$, $M_0 \in \mathrm{R}^{(n-k_+-2l+1)\times(n-k_+-2l+1)}$, $M_0K_0 + K_0M_0^T = 0$. The conditions on M_0, γ and x are given in two theorems.

Theorem 2.2 *Three different kinds of splitting MASAs of $e(p,1)$ exist. They are characterized by the triplet (k_+, k_-, k_0):*

A) $M(k_+,1,0)$, $0 \le k_+ \le p$

$$M_0 = 0 \in \mathrm{R}, \quad \gamma^T = z \in \mathrm{R} \quad \text{and} \quad K_0 = -1 \tag{2.12}$$

$p-k_+$ is even, $0 \le l \le \frac{p-k_+}{2}$, $d = dim M(k_+,1,0) = 1 + l + k_+$, $\left[\frac{p+3}{2}\right] \le d \le p+1$

B) $M(k_+,0,0), 0 \le k_+ \le p-1$

$$M_0 = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}, \qquad \gamma^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad K_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.13}$$

where $p-k_+$ is odd, $0 \le l \le \frac{p-k_+-1}{2}$, $d = dim M(k_+,0,0) = 1 + l + k_+$, $\left[\frac{p+2}{2}\right] \le d \le p$

C) $M(k_+,0,1)$, $0 \le k_+ \le p-2$

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^T \\ 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^T = \begin{pmatrix} z \\ 0_\mu \\ 0 \end{pmatrix}, \qquad K_0 = \begin{pmatrix} & & 1 \\ & I_\mu & \\ 1 & & \end{pmatrix} \tag{2.14}$$

where $1 \le \mu \le p-1$ and $0 \le l \le \frac{p-k_+-2}{2}$, $z \in \mathrm{R}$, $\alpha \in \mathrm{R}^{1\times\mu}$, $d = dim M(k_+,0,1) = \mu + l + k_+ + 1$, $\left[\frac{p+3}{2}\right] \le d \le p$.

All entries a_i, x, z, α and c are free.

Theorem 2.3 *Nonsplitting MASA's of $e(p,1)$ are obtained from splitting ones of type C in Theorem 2.2 and are conjugate to precisely one MASA of the form*
i) for $\mu \geq 2$

$$M_0 = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\alpha^T \\ 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^T = \begin{pmatrix} z \\ A\alpha^T \\ 0 \end{pmatrix} \tag{2.15}$$

where A is a diagonal matrix with $a_1 = 1 \geq |a_2| \geq \ldots \geq |a_\mu| \geq 0$ and $TrA = 0$, K_0 is as in (2.14)
ii) for $\mu = 1$ we have a special case for which the nonsplitting MASA has the following form

$$M_0 = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & -a \\ 0 & 0 & 0 \end{pmatrix}, \quad \gamma^T = \begin{pmatrix} z \\ 0 \\ a \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.16}$$

No other nonsplitting MASAs of $e(p,1)$ exist.

3 Separation of variables in the Laplace Beltrami operator

Specifically for the spaces $M(p,q)$ of this article, we generate the coordinates as follows. We use the realization (2.6) of the group $E(p,q)$, but restrict H to be a maximal Abelian subgroup of $E(p,q)$. We have $G =< \exp X >$, where X is one of the MASAs we have constructed. We then write

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = e^X \begin{pmatrix} s \\ 1 \end{pmatrix}, \qquad s \in \mathbb{R}^{p+q} \tag{3.1}$$

where s represents a vector in a subspace of $M(p,q)$ parameterized by nonignorable variables $(s_1, \ldots, s_k)$, and X is a MASA of $e(p,q)$, parametrized by a set of ignorable variables.

Euclidean space M(p)

In cartesian coordinates we have

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \tag{3.2}$$

Space $M(p)$ is split into a direct sum of one and two-dimensional spaces. In each $M(1)$ we have a Cartesian coordinate x_i, corresponding to the translation P_i. In each subspace $M(2)$ we have polar coordinates, e.g. $M_{12} = \frac{\partial}{\partial \alpha_1}$ corresponds to

$$\begin{aligned} x_1 &= s_1 \cos\alpha_1 \\ x_2 &= s_1 \sin\alpha_1 \end{aligned} \tag{3.3}$$

with α_1 ignorable.

Minkowski space M(p,1)

The Laplace-Beltrami operator in Cartesian coordinates is written as

$$\Box_{p,1}\Psi = E\Psi$$

$$\Delta_{LB} \equiv \Box_{p,1} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_0^2}. \tag{3.4}$$

We introduce a separable system of coordinates in each indecomposable subspace of $M(p,1)$. Space $M(1,0)$ corresponds to a Cartesian coordinate, $M(2,0)$ to polar coordinate as in eq. (3.3). Now let us consider the coordinates corresponding to $M(k,1)$.

$$\begin{array}{lll} M(0,1) & : & x_0 \ \ \text{(cartesian coordinate)} \\ M(1,1) & : & x_0 = s\cosh\alpha, \quad x_1 = s\sinh\alpha \ \ \text{(hyperbolic coordinates)} \\ & & x_0 = s\sinh\alpha, \quad x_2 = s\cosh\alpha \\ & & \text{(for } x_0^2 - x_1^2 = \pm s^2, \text{ respectively)} \end{array}$$

M(2,1) : There are two ignorable variables, z and a and we have

$$\begin{array}{rcl} x_0 + x_1 & = & r\sqrt{2} + 2a \\ x_0 - x_1 & = & ra^2\sqrt{2} + \frac{2}{3}a^3 - z\sqrt{2} \\ x_2 & = & -a^2 - ar\sqrt{2}. \end{array} \tag{3.5}$$

And the operator in this $M(2,1)$ subspace of $M(p,1)$ is:

$$\begin{aligned} \square_{2,1} = \sqrt{2}\frac{\partial^2}{\partial r \partial z} + \frac{1}{2}\frac{1}{r^2}\frac{\partial^2}{\partial a^2} + \frac{1}{r^2}\frac{\partial}{\partial r^2} - \frac{\sqrt{2}}{r^2}\frac{\partial^2}{\partial r \partial a} \\ + \frac{1}{\sqrt{2}}\frac{1}{r}\frac{\partial}{\partial z} - \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{1}{\sqrt{2}}\frac{1}{r^3}\frac{\partial}{\partial a} \end{aligned} \tag{3.6}$$

Let us consider the space $M(k,1)$ with $k \geq 2$ and the splitting MASA (2.14) Applying eq. (3.1) with

$$X = \begin{pmatrix} 0 & \alpha & 0 & z \\ 0 & 0 & -\alpha^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad s = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r \end{pmatrix}, \qquad r \in \mathbb{R} \tag{3.7}$$

we obtain the coordinates

$$\begin{array}{rcl} x_k + x_0 & = & r\sqrt{2} \\ x_k - x_0 & = & -r\alpha\alpha^T\frac{1}{\sqrt{2}} + z\sqrt{2} \\ x_1 & = & -r\alpha_1 \\ & \vdots & \\ x_{k-1} & = & -r\alpha_{k-1}. \end{array} \tag{3.8}$$

The wave operator in these coordinates is

$$\square_{k,1} = 2\frac{\partial^2}{\partial z \partial r} + \frac{k-1}{r}\frac{\partial}{\partial z} + \frac{1}{r^2}\sum_{i=1}^{k-1}\frac{\partial^2}{\partial \alpha_i^2}. \tag{3.9}$$

The variables z and α_i are ignorable (only r figures in eq.(3.9)) and solution of the wave equation then separates

$$\psi = R(r)e^{mz}\prod_{i=1}^{k-1} e^{b_i\alpha_i}, \tag{3.10}$$

with $R(r)$

$$R(r) = r^{\frac{1-k}{2}} \exp\left(\frac{1}{r}\frac{\sum_{i=1}^{k-1} b_i^2}{2m}\right)\exp\left(\frac{Er}{2m}\right). \tag{3.11}$$

Now consider the space $M(k,1)$ for $k \geq 3$ and the nonsplitting MASA (2.15). The coordinates we obtain are

$$\begin{array}{rcl} x_k + x_0 & = & r\sqrt{2} \\ x_k - x_0 & = & \frac{1}{\sqrt{2}}(2z - r\alpha\alpha^T + \alpha A\alpha^T) \\ x_1 & = & (q_1 - r)\alpha_1 \\ & \vdots & \\ x_{k-1} & = & (q_{k-1} - r)\alpha_{k-1}. \end{array} \tag{3.12}$$

The wave operator is

$$\Box_{k,1} = 2\frac{\partial^2}{\partial z \partial r} - \left(\sum_{i=1}^{k-1} \frac{1}{(q_i - r)}\right) \frac{\partial}{\partial z} + \sum_{i=1}^{k-1} \frac{1}{(q_i - r)^2} \left(\frac{\partial^2}{\partial \alpha_i^2}\right). \tag{3.13}$$

We see that α_k and z are ignorable variables. The solution of the wave equation then separates and we have

$$\Psi = R(r) e^{mz} \prod_{i=1}^{k-1} e^{b_i \alpha_i} \tag{3.14}$$

with $R(r)$ equal to

$$R(r) = \prod_{i=1}^{k-1} (q_i - r)^{-\frac{1}{2}} \exp\left(-\frac{1}{2m} \sum_{i=1}^{k-1} \frac{b_i^2}{q_i - r}\right) \exp\left(\frac{Er}{2m}\right). \tag{3.15}$$

4 Darboux transformations

4.1 Examples of Darboux transformations in 1 dimension

As stated in the introduction the Darboux transformations (1.2) relates solutions of two different one-dimensional Schrödinger equation (1.1) and (1.3) with potentials $u(x)$ and $\tilde{u}(x)$, respectively.

Let us give several elementary examples to be used below. We always start from a solvable system for which all solutions are known.

1. The free particle: $u(x) = 0$
   For $E_0 = 0$ in eq. (1.1) we have $\psi_0 = ax + b$. With no loss of generality we can shift b to $b = 0$, since we assume $a \neq 0$. Eq. (1.4) then gives an inverse square potential

$$\tilde{u}(x) = \frac{2}{x^2}. \tag{4.1}$$

   For $E_0 = \omega^2 > 0$ the solution of eq. (1.1) is $\psi_0 = A \sin(\omega x + \alpha)$. Again with no loss of generality we can shift α to $\alpha = 0$ and obtain the Poeschl-Teller potential [6]

$$\tilde{u}(x) = \frac{2\omega^2}{\sin^2 \omega x}. \tag{4.2}$$

   For $E_0 = -\omega^2 < 0$ we take $\psi_0 = A \sinh \omega x$ and obtain a hyperbolic version of the potential (4.2), namely

$$\tilde{u}(x) = \frac{2\omega^2}{\sinh^2 \omega x}. \tag{4.3}$$

2. The harmonic oscillator: $u(x) = \omega^2 x^2$
   Let us choose the solution $\psi_0 = x e^{\frac{\epsilon\omega}{2} x^2}$, $\epsilon = \pm 1$, corresponding to $E_0 = -3\epsilon\omega$. We obtain

$$\tilde{u}(x) = \omega^2 x^2 + \frac{2}{x^2} - 2\epsilon\omega. \tag{4.4}$$

3. The Coulomb potential: $u(x) = \frac{2\alpha}{x}$
   Let us choose the solution $\psi_0 = x e^{\alpha x}$ corresponding to the energy $E_0 = -\alpha^2$. The new potential is

$$\tilde{u}(x) = \frac{2\alpha}{x} + \frac{2}{x^2}. \tag{4.5}$$

4.2 Darboux transformations in Laplace-Beltrami equations

One interpretation of the separation variables in a Laplace-Beltrami equation is that we require the separated solutions $\psi(z_1, \ldots z_n)$ to be the eigenfunctions of a complete set of n second order commuting operators in the enveloping algebra of the isometry algebra:

$$\triangle_{LB}\, \psi = -E\psi \tag{4.6}$$

$$Y_i\psi = \lambda_i\psi, \quad i = 1, \ldots, n-1 \tag{4.7}$$

$$[Y_i, Y_k] = 0, \quad [\triangle_{LB}, Y_i] = 0$$

$$\psi = \prod_{i=1}^{n} \psi_i(z_i) \tag{4.8}$$

The specific feature of ignorable variables α_i is that for them the corresponding operator Y_i is actually the square of an element of the isometry algebra $Y_i = X^2$. The function $\psi_i(\alpha_i)$ then satisfies

$$X\psi_i = \frac{\partial}{\partial \alpha_i} = \lambda_i\psi_i \tag{4.9}$$

and hence we have $\psi_i = e^{\lambda_i \alpha_i}$. The only way that ignorable variables figure in the Laplace-Beltrami operator is via the derivative operator $\partial/\partial\alpha_i$. If the coordinates are orthogonal, only second derivatives are present. Such is always the case in real Euclidean spaces and often also in Minkowski ones. We can then replace eq. (4.9) by

$$\frac{\partial^2}{\partial \alpha_i^2}\psi_i = \mu_i\psi_i. \tag{4.10}$$

A Darboux transformation can then be applied to eq. (4.10) to introduce a solvable potential into eq. (4.10) and also into the original eq. (4.6). If the separable coordinates are not orthogonal, then terms of the type $\partial^2/\partial z \partial \alpha$ may occur, where α is ignorable, z not. Then the potential introduced into (4.10) will destroy separability.

Let us now look at specific coordinate systems in Minkowski space and consider the Laplace-Beltrami equation with potential

$$(\Box_{k,1} - u)\psi = -E\psi. \tag{4.11}$$

We start with the coordinate system (3.8). To preserve separation of variables the potential u must have the form

$$u = f(r) + \frac{1}{r^2}\sum_{1}^{k-1} v_i(\alpha_i) \tag{4.12}$$

The ignorable variable z is not orthogonal (see eq. (3.9)), however the variables α_i do enter in the correct form. Writing $\psi(x) = R(r)Z(z)\prod_{i=1}^{k-1} A_i(\alpha_i)$, we obtain the following equations for each A_i

$$(\frac{\partial^2}{\partial \alpha_i^2} + v_i(\alpha_i))A_i = -\mu_i A_i, \;\; i = 1, \ldots, k-1 \tag{4.13}$$

and, for each α_i separately, we can perform a Darboux transformation. Starting with the potential $v_i(\alpha_i) = 0$ we obtain a potential of the form (4.1), (4.2), or (4.3) with x replaced by α_i. Similarly, starting with a harmonic oscillator, or Coulomb potential we obtain (4.4), or (4.5), respectively, again with $x = \alpha_i$. Returning to Cartesian coordinates, we have

$$\alpha_i = -\frac{x_i\sqrt{2}}{x_k + x_0}, \quad i = 1, \ldots k-1 \tag{4.14}$$

in all potentials.

The situation in the coordinates (3.12) is quite similar. A separable potential has the form (4.12), the equations for $A_i(\alpha_i)$ have the form (4.13). Darboux transformations can be used to induce all the potentials (4.1), ..., (4.5) with

$$x = \alpha_i = \frac{x_i\sqrt{2}}{\sqrt{2}q_i - x_k - x_0}, \quad i = 1, \ldots, k-1 \tag{4.15}$$

The Darboux transformation can be made for each variable α_i separately and a different one can be chosen in each case.

5 Conclusions

The examples considered in this contribution are typical for a more general situation. Instead of pseudo-euclidean spaces we could consider the separation of variables in other homogeneous spaces. The corresponding isometry or conformal groups will provide us with tools for separating variables in the Laplace-Beltrami operator. Darboux transformations can then be used to generate integrable systems with non-trivial interactions in these spaces. This can be done for ignorable variables that are orthogonal. In the examples of Section 4 these are the ignorable variables α_i, not hoverer the variable z, that is ignorable but not orthogonal to the variable r. A crucial step in this procedure and in the construction of coordinate systems with ignorable variables is classification of MASAs of the corresponding isometry or conformal Lie algebra.

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