Some extensions of the Markov inequality for polynomials

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Abstract

Let $\mathbb{D}$ denote the unit disc of the complex plane and $\mathcal{P}_n$ the class of polynomials of degree at most n with complex coefficients. We prove that

$$\max_{z\,\in\,\partial\mathbb{D}}\left|\frac{p_k(z)-p_k(\bar{z})}{z-\bar{z}}\right|\leq n^{1+k}\max_{0\leq j\leq n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|\,,$$

where $p_0:=p$ belongs to $\mathcal{P}_n$ and for $k\geq 0$, $p_{k+1}(z):=zp_k'(z)$. We also show how this result contains or sharpens certain classical inequalities for polynomials due to Bernstein, Markov, Duffin and Schaeffer and others.

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Introduction

Let $\mathcal{P}_n$ be the class of polynomials $p(z) = \sum_{k=0}^{n} a_k(p)z^k$ of degree at most n with complex coefficients. We define, together with $\mathbb{D} := \{z \mid |z| < 1\}$,

$$\|p\|_{\mathbb{D}} := \max_{z\in\partial\mathbb{D}} |p(z)| \qquad \text{and} \qquad \|p\|_{[-1,1]} := \max_{-1\le x\le 1} |p(x)|.$$

The famous inequalities of, respectively, Bernstein and Markov state that for any $p \in \mathcal{P}_n$

$$\|p'\|_{\mathbb{D}} \le n\|p\|_{\mathbb{D}} \tag{1}$$

and

$$\|p'\|_{[-1,1]} \le n^2\|p\|_{[-1,1]}, \tag{2}$$

while

$$\|p'\|_{[-1,1]} \le n^2 \max_{0\le j\le n} \left|p\big(\cos(j\pi/n)\big)\right| \tag{3}$$

is a far reaching extension of (2) obtained by Duffin and Schaeffer [3] in 1941. We refer the reader to the recent book by Rahman and Schmeisser [4] or to the survey paper by Bojanov [1] for historical remarks and generalizations of these inequalities.

Let us consider a polynomial $p(z) := \sum_{k=0}^{n} a_k(p)z^k$ in $\mathcal{P}_n$ and an associated polynomial $P(z) := \sum_{k=0}^{n} a_k(p)T_k(z)$ where T_k denotes, for each integer $k \ge 0$, the k^{th} Chebyshev polynomial, i.e., $T_k(\cos\theta) = \cos(k\theta)$ for any real number θ. We have

$$P(\cos\theta) = \frac{p(e^{i\theta}) + p(e^{-i\theta})}{2}$$

and applying (3) to P we obtain the inequality

$$\left|\frac{e^{i\theta}p'(e^{i\theta}) - e^{-i\theta}p'(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}\right| \le n^2 \max_{0\le k\le n} \left|\frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2}\right| \tag{4}$$

valid for any real θ and equivalent to the Duffin and Schaeffer inequality.

Given a non-negative number t and a polynomial $p(z) := \sum_{k=0}^{n} a_k(p)z^k \in \mathcal{P}_n$ we define

$$p_t(z) := \sum_{k=0}^{n} k^t a_k(p)z^k.$$

Clearly, $p_t \in \mathcal{P}_n$, $p_0 = p$ and $p_{t+1}(z) = zp_t'(z)$ for $t \ge 0$. Our main result is the following

Theorem 1 *For any integer $j \ge 0$ and polynomial $p \in \mathcal{P}_n$,*

$$\left|\frac{p_j(e^{i\theta}) - p_j(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}\right| \le n^{1+j} \max_{0\le k\le n} \left|\frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2}\right| \tag{5}$$

for all real θ.

Our proof of Theorem 1 is completely independent of the known proofs of (3). This Theorem 1 therefore contains (3) as a special case ($j = 1$, compare with (4)). It also follows easily from (5) that

$$|p'_{j-1}(e^{i\theta})| \le n^j \max_{0\le k\le n} \left|\frac{p(e^{i(\theta+k\pi/n)}) + p(e^{i(\theta-k\pi/n)})}{2}\right|, \quad \theta \text{ real}, \tag{6}$$

for all $p \in \mathcal{P}_n$ and integer $j \ge 1$. It is therefore also clear that our Theorem 1 contains an improvement of Bernstein's inequality (1).

We shall also obtain the following extensions of (3):

Theorem 2 *Let $x \in [-1, 1]$ and $n \ge 1$ such that $|T_n(x)| \le \frac{T_n'(x)}{n^2}$ or else $|T_n(x)| \le \frac{-T_n'(x)}{n^2}$. Then*

$$|p'(x)| \le |T_n'(x)| \max_{0\le k\le n} \left|p\left(\cos\left(\frac{k\pi}{n}\right)\right)\right|.$$

Theorem 3 *Let $p \in \mathcal{P}_{n+1}$; then for any $x \in [-1,1]$,*

$$\left|p'(x) - \frac{a_{n+1}(p)}{2^{n-1}}\left(\frac{1}{n+1}T'_{n+1}(x) + (n-1)T_n(x)\right)\right| \le n^2 \max_{0\le k\le n}\left|p\left(\cos\left(\frac{k\pi}{n}\right)\right)\right|.$$

Theorem 2 refines (3) because

$$|T'_n(x)| = \left|n\frac{\sin(n\theta)}{\sin(\theta)}\right| \le n^2, \quad x = \cos\theta.$$

Clearly Theorem 3 reduces to (3) when $p \in \mathcal{P}_n \subset \mathcal{P}_{n+1}$.

Some Lemmas

Using the notation

$$\sum_{j=0}^{n}{}'' a_j := \frac{a_0}{2} + \sum_{j=1}^{n-1} a_j + \frac{a_n}{2},$$

our auxiliary results are as follows:

Lemma 1 *For any real φ, $n \ge 2$ and $z \in \mathbb{D}$,*

$$\frac{z}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} = \sum_{j=0}^{n}{}'' \frac{c_n(j,\varphi)}{2}\left(\frac{1}{1-ze^{ij\pi/n}} + \frac{1}{1-ze^{-ij\pi/n}}\right) - \frac{2(z^n - \cos(n\varphi))z^{n+1}}{(1-ze^{i\varphi})(1-ze^{-i\varphi})(1-z^{2n})}$$

where $c_n(j,\varphi) = \frac{(-1)^j}{n}\frac{\cos(j\pi)-\cos(n\varphi)}{\cos(j\pi/n)-\cos(\varphi)}$ and $\sum_{j=0}^{n}{}'' |c_n(j,\varphi)| \le n$.

Lemma 2 *For any real φ, $n \ge 2$ and $z \in \mathbb{D}$,*

$$\frac{2(1-z\cos\varphi)}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} = \sum_{j=0}^{n}{}'' \frac{d_n(j,\varphi)}{2}\left(\frac{1}{1-ze^{ij\pi/n}} + \frac{1}{1-ze^{-ij\pi/n}}\right) + \frac{2z^{n+1}(\cos(n+1)\varphi - \cos(n-1)\varphi)}{(1+ze^{i\varphi})(1-ze^{-i\varphi})(1-z^{2n})}$$

where $d_n(j,\varphi) = \frac{(-1)^{j-1}}{n}\frac{\cos(n+1)\varphi-\cos(n-1)\varphi}{\cos(j\pi/n)-\cos(\varphi)}$ and $\sum_{j=0}^{n}{}'' |d_n(j,\varphi)| \le 2\left|\frac{n\sin\varphi}{\sin(n\varphi)}\right|$.

We only prove Lemma 2 in details. Let us fix $\varphi \in \mathbb{R}$ and consider

$$L_\varphi(z) := \frac{(1-\cos(\varphi)z)(1-z^{2n}) - z^{n+1}(\cos(n+1)\varphi - \cos(n-1)\varphi)}{(1-ze^{i\varphi})(1-ze^{-i\varphi})}$$

together with

$$R_\varphi(z) := \frac{1}{4n}\sum_{j=n+1}^{n}(-1)^{j-1}\frac{\cos(n+1)\varphi - \cos(n-1)\varphi}{\cos(j\pi/n)-\cos(\varphi)}\frac{1-z^{2n}}{1-ze^{ij\pi/n}}.$$

It is readily seen that L_φ and R_φ are polynomials in $\mathcal{P}_{2n-1}$. A simple computation gives

$$R_\varphi(e^{-ij\pi/n}) = L_\varphi(e^{-ij\pi/n}) = (-1)^j\frac{\sin(n\varphi)\sin(\varphi)}{\cos(j\pi/n)-\cos(\varphi)}$$

at the $2n$ distinct points $e^{-ij\pi/n}$, $j=-n+1,\ldots,n$. Clearly then the polynomials L_φ and R_φ must coincide on the whole complex plane and the identity of Lemma 2 follows. We further have with $Z=e^{i\varphi}$

$$\begin{aligned}
\sum_{j=0}^{n}{}'' \frac{1}{|\cos j\pi/n-\cos\varphi|} &= \sum_{j=0}^{n}{}'' \frac{2}{|1-Ze^{ij\pi/n}||1-Ze^{-ij\pi/n}|}\\
&= \frac{1}{|1-Z|^2}+\sum_{j=1}^{n-1}\frac{2}{|1-Ze^{ij\pi/n}||1-Ze^{-ij\pi/n}|}+\frac{1}{|1+Z|^2}\\
&\le \frac{1}{|1-Z|^2}+\sum_{j=1}^{n-1}\frac{1}{|1-Ze^{ij\pi/n}|^2}+\frac{1}{|1-Ze^{-ij\pi/n}|^2}+\frac{1}{|1+Z|^2}\\
&= \sum_{j=0}^{2n-1}\frac{1}{|1-w_jZ|^2}\\
&= \sum_{j=0}^{2n-1}\frac{-w_jZ}{(1-w_jZ)^2}\\
&= \frac{n^2}{\sin^2(n\varphi)}
\end{aligned}$$

where $\{w_j\}_{j=0}^{2n-1}$ is the set of distinct $2n^{\text{th}}$ roots of unity. It follows that

$$\begin{aligned}
\sum_{j=0}^{n}{}'' |d_n(j,\varphi)| &= \sum_{j=0}^{n}{}'' \frac{2|\sin(n\varphi)||\sin(\varphi)|}{n|\cos j\pi/n-\cos\varphi|}\\
&\le \frac{2n|\sin\varphi|}{|\sin n\varphi|}.
\end{aligned}$$

This completes the proof of Lemma 2.

The proof of Lemma 1 is very similar and is based on the fact that the polynomials (in $\mathcal{P}_{2n-1}$)

$$\ell_\varphi(z) := z\frac{1-e^{in\varphi}z^n}{1-e^{i\varphi}z}\,\frac{1-e^{-in\varphi}z^n}{1-e^{-i\varphi}z}$$

and

$$r_\varphi(z) := \frac{1}{2n}\sum_{j=-n+1}^{n}(-1)^j\frac{\cos(j\pi)-\cos(n\varphi)}{\cos(j\pi/n)-\cos(\varphi)}\,\frac{1-z^{2n}}{1-e^{ij\pi/n}z}$$

also satisfy $r_\varphi(e^{-ij\pi/n})=\ell_\varphi(e^{-ij\pi/n})$, $j=-n+1,\ldots,n$. A proof that

$$\sum_{j=0}^{n}{}'' |c_n(j,\varphi)| \le n$$

can be found in [2].

We end this section by an application of Lemma 2.

Corollary 1 *Let $p\in\mathcal{P}_n$ and $\varphi\in\mathbb{R}$. Then*

$$|p(e^{i\varphi})+p(e^{-i\varphi})| \le \begin{cases} n\left|\frac{\sin(\varphi)}{\sin(n\varphi)}\right| \max\limits_{0\le j\le n}|p(e^{ij\pi/n})+p(e^{-ij\pi/n})|, & \textit{if } e^{2in\varphi}\ne 1,\\ \max\limits_{0\le j\le n}|p(e^{ij\pi/n})+p(e^{-ij\pi/n})|, & \textit{if } e^{2in\varphi}=1,\end{cases}$$

We first define the Hadamard product of two analytic functions $f(z) := \sum\limits_{n=0}^{\infty} a_n(f)z^n$ and $g(z) := \sum\limits_{n=0}^{\infty} a_n(f)z^n$ by

$$f\star g(z) := \sum_{n=0}^{\infty} a_n(f)b_n(f)z^n.$$

Then for any $p \in \mathcal{P}_n$ and $\varphi \in \mathbb{R}$, we obtain from Lemma 2,

$$\begin{aligned} p(ze^{i\varphi}) + p(ze^{-i\varphi}) &= \left(\frac{1}{1-ze^{i\varphi}} + \frac{1}{1-ze^{-i\varphi}}\right) \star p(z) \\ &= \frac{2(1-z\cos(\varphi))}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} \star p(z) \\ &= \sum_{j=0}^{n}{}'' d_n(j,\varphi) \frac{p(ze^{ij\pi/n}) + p(ze^{-ij\pi/n})}{2}. \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} |p(e^{i\varphi}) + p(e^{-i\varphi}) &\leq \frac{1}{2} \sum_{j=0}^{n}{}'' |d_n(j,\varphi)| \max_{0\leq j\leq n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| \\ &\leq \left|\frac{n\sin(\varphi)}{\sin(n\varphi)}\right| \max_{0\leq j\leq n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| \end{aligned}$$

and the result follows. It can also be checked that the above inequality is strict when $p \in \mathcal{P}_n$, $n \geq 2$, $p \not\equiv 0$, $e^{i\varphi} \notin \{w_j\}_{j=0}^{2n-1}$.

The translation of Corollary 1 in terms of polynomials over $[-1,1]$ makes more explicit its relation to the Duffin and Schaeffer inequality (3). We have

Corollary 1′ *Let $p \in \mathcal{P}_n$ and $-1 \leq x \leq 1$. Then*

$$|T_n'(x)||p(x)| \leq n^2 \max_{0\leq j\leq n} \big|p\big(\cos(j\pi/n)\big)\big|.$$

For $p \not\equiv 0$, equality holds if and only if $x = +1$ and

$$|p(1)|\big(\text{or } |p(-1)|\big) = \max_{0\leq j\leq n} \big|p\big(\cos(j\pi/n)\big)\big|.$$

Proof of Theorem 1

Let $q \in \mathcal{P}_n$ and $\varphi \in [0,\pi]$. By Lemma 1, we have

$$\begin{aligned} \frac{q(ze^{i\varphi}) - q(ze^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} &= \frac{z}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} \star q(z) \\ &= \sum_{j=0}^{n}{}'' \frac{c_n(j,\varphi)}{2} \big(q(ze^{ij\pi/n}) + q(ze^{-ij\pi/n})\big) \end{aligned}$$

and in particular for $z = 1$,

$$\frac{q(e^{i\varphi}) - q(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = \sum_{j=0}^{n}{}'' c_n(j,\varphi) \frac{q(e^{ij\pi/n}) + q(e^{-ij\pi/n})}{2}. \tag{8}$$

Letting now $\varphi = 0$ in (8) we obtain

$$q'(1) = \sum_{j=0}^{n}{}'' c_n(j,0) \frac{q(e^{ij\pi/n}) + q(e^{-ij\pi/n})}{2}.$$

and more generally

$$e^{i\varphi} q'(e^{i\varphi}) = \sum_{j=0}^{n}{}'' c_n(j,0) \frac{q(e^{i(\varphi+j\pi/n)}) + q(e^{i(\varphi-j\pi/n)})}{2}. \tag{9}$$

We shall prove the following statement by induction on $k \geq 0$: there exist real numbers $\alpha_{j,k}(\theta)$ $(j = 0,1,\ldots,n)$ such that for any $p \in \mathcal{P}_n$ and $n \geq 1$

$$\frac{p_k(e^{i\theta}) - p_k(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{n} \alpha_{j,k}(\theta) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}, \quad \theta \in [0,\pi] \tag{10}$$

and $\sum_{j=0}^{n} |\alpha_{j,k}(\theta)| \le n^{1+k}$, $\theta \in [0,\pi]$. The truth of Theorem 1 is clearly a consequence of (10). A proof of (10) for $k = 0, 1$ has been given in [2]; clearly such a proof also follows from (8) and Lemma 1. Let us now assume that (10) is valid for a certain integer k and any polynomial $q \in \mathcal{P}_n$. By (9), we obtain

$$e^{\pm i\theta} p_k'(e^{\pm i\theta}) = \sum_{j=0}^{n} c_n(j,0) \frac{p_k(e^{i(\pm\theta + j\pi/n)}) + p_k(e^{i(\pm\theta - j\pi/n)})}{2}. \tag{11}$$

Now since

$$\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{e^{i\theta}p_k'(e^{i\theta}) - e^{-i\theta}p_k'(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}},$$

it follows from (11) that

$$\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = {\sum_{j=0}^{n}}'' c_n(j,0) \frac{p_k(e^{ij\pi/n}e^{i\theta}) - p_k(e^{ij\pi/n}e^{-i\theta})}{2(e^{i\theta} - e^{-i\theta})} + {\sum_{j=0}^{n}}'' c_n(j,0) \frac{p_k(e^{-ij\pi/n}e^{i\theta}) - p_k(e^{-ij\pi/n}e^{-i\theta})}{2(e^{i\theta} - e^{-i\theta})}.$$

Applying the induction hypothesis we get

$$\frac{p_k(e^{\pm ij\pi/n}e^{i\theta}) - p_k(e^{\pm ij\pi/n}e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{\varphi=0}^{n} \alpha_{\ell,k}(\theta) \frac{p(e^{i(\pm j\pi/n + \ell\pi/n)}) + p(e^{i(\pm j\pi/n - \ell\pi/n)})}{2}$$

with $\sum_{\ell=0}^{n} |\alpha_{\ell,k}(\theta)| \le n^{1+k}$. Finally we have

$$\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{1}{2} {\sum_{j=0}^{n}}'' c_n(j,0) \sum_{\ell=0}^{n} \alpha_{\ell,k}(\theta) \frac{p(e^{i(j+\ell)\pi/n}) + p(e^{-i(j+\ell)\pi/n})}{2} + \frac{1}{2} {\sum_{j=0}^{n}}'' c_n(j,0) \sum_{\ell=0}^{n} \alpha_{\ell,k}(\theta) \frac{p(e^{i(j-\ell)\pi/n}) + p(e^{-i(j-\ell)\pi/n})}{2}.$$

Clearly, the right hand-side of the above is a sum of the type

$$\sum_{j=0}^{n} \alpha_{j,k+1}(\theta) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}, \quad \alpha_{\ell,k+1}(\theta) \text{ real},$$

and obviously since

$$\sum_{j=0}^{n} |a_{j,k+1}(\theta)| \le {\sum_{j=0}^{n}}'' |c_n(j,\theta)| \sum_{\ell=0}^{n} |\alpha_{\ell,k}(\theta)| \le n \cdot n^{k+1} = n^{k+2},$$

the final result follows.

We shall end this section by some remarks concerning the sharpness of Theorem 1. Let us first point out that the inequality (5) becomes an equality for certain choices of polynomials p; indeed

$$\left| \frac{p_j(e^{i\theta}) - p_j(e^{-i\theta})}{e^{i\theta} - e^{i\theta}} \right| = n^{1+j} \max_{0 \le k \le n} \left| \frac{p(e^{ik\pi/n}) + p(e^{ik\pi/n})}{2} \right|$$

for any $j = 0, 1, 2, \ldots$, $\theta = 0$ or $\theta = \pi$ and $p(z) \equiv Kz^n$ for some complex constant K. As shown in [2], there are no other cases of equality if $j = 1$ but there are many other cases of equality if $j = 0$. To discuss the cases of equality for $j > 1$ seems to be beyond the scope of our method. It is not, however, difficult to establish that (compare with (10)) for $k = 0, 1, 2, \ldots$

$$\sum_{j=0}^{n} |\alpha_{j,k}(\theta)| = n^{1+k} \iff \theta = 0 \quad \text{or} \quad \theta = \pi$$

i.e., the inequality (5) is always strict if $\theta \ne 0$, π and the polynomial p does not vanish identically. We also remark that the statement

$$\left| \frac{p_j(z) - p_j(\bar{z})}{z - \bar{z}} \right| \le n \left| \frac{p_{j-1}(z) - p_{j-1}(\bar{z})}{z - \bar{z}} \right|, \quad z \in \partial\mathbb{D},\ j \ge 1,\ p \in \mathcal{P}_n,$$

can be seen numerically to be false and therefore cannot yield a simpler inductive proof of Theorem 1.

Let us notice that the definition of $p_j(z) := \sum_{k=0}^{\infty} k^j a_k(p) z^k$ extends to positive but non necessarily integer values of j and it is therefore a legitimate (but apparently hard) question to ask whether or not Theorem 1 holds for these values of j. In this context, let us mention that the slightly weaker inequality

$$|p_t|_{\mathbb{D}} \leq n^t |p|_{\mathbb{D}}, \quad p \in \mathcal{P}_n$$

holds for all real $t \geq 1$ but does not hold in general for $0 < t < 1$. This unpublished result is due to Mohopatra, Qazi and Rahman ([4, Section 14.5]).

We finally wish to mention the following two technical results which are a consequence of our proof of Theorem 1:

Corollary 2 $\sum_{j=0}^{n}{}'' \left|\left(c_n(j,\varphi)\sin(\varphi)\right)^{(2k)}\right| \leq n^{2k+1}\sin(\varphi), \quad k \geq 0,\ \varphi \in [0,\pi].$

Corollary 3 $\sum_{j=0}^{n}{}'' |d_n(j,\varphi)^{(2k+1)}| \leq n^{2k+2}\sin(\varphi), \quad k \geq 0,\ \varphi \in [0,\pi].$

We only prove the first statement. By applying Lemma 1 we obtain after differentiation, for any $p \in \mathcal{P}_n$,

$$\frac{p_{2k}(e^{i\varphi}) - p_{2k}(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = (-1)^k i \sum_{j=0}^{n}{}'' \frac{(c_n(j,\varphi)\sin(\varphi))^{(2k)}}{\sin\varphi} \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}$$

and by (10)

$$\frac{p_{2k}(e^{i\varphi}) - p_{2k}(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = \sum_{j=0}^{n} \alpha_{j,2k}(\varphi) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}.$$

Applying successively the last two identities to the polynomials $p(z) := z^\ell$, $0 \leq \ell \leq n$, we obtain

$$\sum_{j=0}^{n}{}'' \frac{|(c_n(j,\varphi)\sin(\varphi))^{(2k)}|}{\sin(\varphi)} = \sum_{j=0}^{n} |\alpha_{j,2k}(\varphi)|$$

because the matrix $\left[\cos\left(\frac{j\ell\pi}{n}\right)\right]_{\substack{0\leq j\leq n\\ 0\leq \ell\leq n}}$ is invertible and a further application of (10) yields the result. Corollary 2 can be obtained similarly, using this time Lemma 2.

Proof of Theorem 2

Theorem 2 is not new and is indeed a special case of a result due to Shadrin [5]. We include our simple proof here because it is a straightforward consequence of our method. We have by (7) for any $p \in \mathcal{P}_n$,

$$\frac{e^{i\varphi}p'(e^{i\varphi}) - e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = \sum_{j=0}^{n}{}'' \frac{\delta_n'(j,\varphi)}{\sin(\varphi)} \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}$$

where $\delta_n(j,\varphi) := \frac{(-1)^{j-1}}{n} \frac{\sin(n\varphi)\sin(\varphi)}{\cos(j\pi/n) - \cos(\theta)}$ and all we need to prove is that under the hypothesis $|T_n(x)| \leq \frac{1}{n^2} T_n'(x)$ and $x = \cos(\varphi)$, $0 \leq \varphi \leq \pi$, then

$$\sum_{j=0}^{n}{}'' \frac{|\delta_n'(j,\varphi)|}{\sin(\varphi)} \leq T_n'(x).$$

Easy computations show that

$$\sum_{j=0}^{n}{}'' \frac{|\delta_n'(j,\varphi)|}{\sin(\varphi)} = \left| \sum_{j=0}^{n}{}'' \frac{(1+x_j)(1-x)^2}{2(x_j - x)^2} \frac{(1/n^2)T_n'(x) - T_n(x)}{1-x} + \frac{(1-x_j)(1-x)^2}{2(x_j-x)^2} \frac{(1/n^2)T_n'(x) + T_n(x)}{1+x} \right|$$

with $x_j = \cos\left(\frac{j\pi}{n}\right)$. Because of our hypothesis we have

$$\frac{(1/n^2)T_n'(x) \pm T_n(x)}{1 \pm x} \geq 0$$

and therefore

$$\begin{aligned}
\sum_{j=0}^{n}{}'' \frac{|\delta_n'(j,\varphi)|}{\sin(\varphi)} &= \sum_{j=0}^{n}{}'' \left(\frac{(1+x_j)(1-x)^2}{2(x_j-x)^2} \frac{(1/n^2)T_n'(x) - T_n(x)}{1-x} + \frac{(1-x_j)(1-x)^2}{2(x_j-x)^2} \frac{(1/n^2)T_n'(x) + T_n(x)}{1+x} \right) \\
&= \frac{T_n'(x)}{n^2} \left(\frac{1}{1-x^2} - \frac{d}{dx} \left(\sum_{j=1}^{n-1} \frac{1-x^2}{x-x_j} \right) - \sum_{j=1}^{n-1} \frac{x}{x-x_j} \right) + T_n(x) \left(\frac{-x}{1-x^2} + \sum_{j=1}^{n-1} \frac{1}{x-x_j} \right) \\
&= \frac{T_n'(x)}{n^2} \left(\frac{1}{1-x^2} - \frac{d}{dx} \left((1-x^2) \frac{T_n''(x)}{T_n'(x)} \right) - x \frac{T_n''(x)}{T_n'(x)} \right) + T_n(x) \left(\frac{-x}{1-x^2} + \frac{T_n''(x)}{T_n'(x)} \right) \\
&= \frac{T_n'(x)}{n^2} \left(n^2 + \frac{n^4}{1-x^2} \frac{T_n^2(x)}{T_n'(x)^2} \right) + T_n(x) \left(-n^2 \frac{T_n(x)}{(1-x^2)T_n'(x)} \right) \\
&= T_n'(x).
\end{aligned}$$

In these computations we have used the fact that the set of zeros of $T_n'(x)$ is $\{x_j\}_{j=1}^{n-1}$ and the standard Chebyshev differential equation. The proof in the case where $|T_n(x)| < -\frac{1}{n^2}T_n'(x)$ runs along the same lines.

We clearly have proved that, under one of the hypothesis $|T_n(x)| \leq \pm\frac{1}{n^2}T_n'(x)$,

$$\max_{p \in \mathcal{P}_n} \frac{\left| \frac{e^{i\theta}p'(e^{i\theta}) - e^{-i\theta}p'(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \right|}{\max\limits_{0 \leq j \leq n} \left| \frac{p(e^{-ij\pi/n}) + p(e^{-ij\pi/n})}{2} \right|} = |T_n'(x)|, \quad x = \cos\theta. \tag{12}$$

We can also establish for which polynomials p the maximum in (12) is attained. Our proof and the triangle inequality clearly show that for any such p, there exists a real number φ such that

$$(-1)^j \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} = e^{i\varphi} \max_{0 \leq k \leq n} \left| \frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2} \right| := Me^{i\varphi}, \quad 0 \leq j \leq n$$

and for the polynomial $P(z) := \sum\limits_{k=0}^{n} a_k(p)T_k(z)$ this simply means that

$$(-1)^j P\left(\cos\frac{j\pi}{n}\right) = Me^{i\varphi}, \quad 0 \leq j \leq n.$$

By the Lagrange interpolation formula (with nodes at the zeros of $(1-z^2)T_n'(z)$) we finally obtain

$$\begin{aligned}
P(z) &= \sum_{j=0}^{n} P(x_j)(-1)^{j-1} \frac{(1-z^2)T_n'(z)}{n^2(z-x_j)} \\
&= Me^{i\varphi} \sum_{j=0}^{n}{}'' - \frac{(1-z^2)T_n'(z)}{n^2(z-x_j)} \\
&= Me^{i\varphi} T_n(z),
\end{aligned}$$

i.e., P is a multiple of T_n or equivalently p is a monomial of degree n. In other words

$$\max_{p \in \mathcal{P}_n} \frac{|p'(x)|}{\max\limits_{0 \leq j \leq n} |p(\cos j\pi/n)|} = |T_n'(x)|$$

and the maximum can be attained only for multiples of the n^{th} Chebyshev polynomial.

Proof of Theorem 3

It is possible to apply the identity of Lemma 1 to Hadamard products of polynomials greater than n. For example for $p \in \mathcal{P}_{n+1}$ we have

$$\frac{p(e^{i\varphi}) - p(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = \sum_{j=0}^{n}{}'' c_n(j,\varphi) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} + 2a_{n+1}(p)\cos(n\varphi) \tag{13}$$

and (9) gets transformed into

$$e^{i\varphi} p'(e^{i\varphi}) = \sum_{j=0}^{n}{}'' c_n(j,0) \frac{p(e^{i(\varphi+j\pi/n)}) + p(e^{i(\varphi-j\pi/n)})}{2} + 2a_{n+1}(p) e^{i(n+1)\varphi}. \tag{14}$$

Therefore

$$\begin{aligned}
&\frac{e^{i\varphi}p'(e^{i\varphi}) - e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} \\
&\quad = \frac{1}{2}\sum_{j=0}^{n}{}'' c_n(j,0)\left(\frac{p(e^{ij\pi/n}e^{i\varphi}) - p(e^{ij\pi/n}e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} + \frac{p(e^{-ij\pi/n}e^{i\varphi}) - p(e^{-ij\pi/n}e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}}\right) + 2a_{n+1}(p)\frac{\sin(n+1)\varphi}{\sin(\varphi)}
\end{aligned}$$

and applying (13) to the polynomials $p(e^{\pm ij\pi/n}z)$

$$\begin{aligned}
&\frac{e^{i\varphi}p'(e^{i\varphi}) - e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} \\
&\qquad = \frac{1}{2}\sum_{j,\ell=0}^{n}{}'' c_n(j,0)c_n(\ell,\varphi)\left(\frac{p(e^{i(j+\ell)\pi/n}) + p(e^{-i(j+\ell)\pi/n})}{2} + \frac{p(e^{i(j-\ell)\pi/n}) + p(e^{-i(j-\ell)\pi/n})}{2}\right) \\
&\qquad\qquad + 2a_{n+1}(p)\cos(n\varphi)\sum_{j=0}^{n} c_n(j,0)(-1)^j\cos(j\pi/n) + 2a_{n+1}(p)\frac{\sin(n+1)\varphi}{\sin(\varphi)}.
\end{aligned}$$

We now use (14) with $\varphi = 0$ and $p(z) \equiv z^{n+1}$ and get

$$(n-1) = \sum_{j=0}^{n} c_n(j,0)(-1)^j \cos(j\pi/n)$$

and finally, as in the proof of Theorem 1,

$$\left|\frac{e^{i\varphi}p'(e^{i\varphi}) - e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} - 2a_{n+1}(p)\left(\frac{\sin(n+1)\varphi}{\sin(\varphi)} + (n-1)\cos(n\varphi)\right)\right| \le n^2 \max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}\right|.$$

By the usual transformation, this is seen to be equivalent to the statement of Theorem 3.

Some more of our results can be similarly generalized. For example we have, given $p \in \mathcal{P}_{n+1}$, $\varphi \in [0,\pi]$,

$$\left|\frac{p(e^{i\varphi}) - p(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} - 2a_{n+1}(p)\cos(n\varphi)\right| \le n \max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}\right|.$$

A similar treatment of (7) yields (compare with Corollary $1'$):

$$|T_n'(x)|\left|p(x) + \frac{a_{n+1}(p)}{2^{n-1}}\frac{(1-x^2)T_n'(x)}{n}\right| \le n^2 \max_{0\le j\le n}\left|p(\cos\left(\frac{j\pi}{n}\right)\right|, \tag{15}$$

valid for $p \in \mathcal{P}_{n+1}$, $x \in [-1,1]$. Further, equality holds in (15) for certain $p \in \mathcal{P}_n$ but also for $p(z) := (1-z^2)T_n'(z) \in \mathcal{P}_{n+1}$.

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