Modulus of analytic classification for the generic unfolding of a codimension one resonant diffeomorphism or resonant saddle

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Abstract

In this paper we consider germs of one-parameter generic families of resonant analytic diffeomorphims and we give a complete modulus of analytic classification by means of the unfolding of the Ecalle modulus. We describe the parametric resurgence phenomenon. We apply this to give a complete modulus of orbital analytic classification for the unfolding of a generic resonant saddle of a two-dimensional vector field by means of the unfolding of its holonomy map. Here again the modulus is an unfolding of the Martinet-Ramis modulus of the resonant saddle. When the saddle passes through the resonance we observe a "transcritical bifurcation": the dynamics in the neighborhood of the saddle is governed by different parts of the unfolding of the modulus on each side of the bifurcation. We then include the time dependence and give a complete modulus of analytic conjugacy for the unfolding of a generic resonant saddle.

Résumé

Dans cet article nous considérons des germes de familles gén ériques à un paramètre de difféomorphismes analytiques résonants et nous montrons que le déploiement du module d'Écalle donne un module complet de classification analytique pour une famille. On décrit le phénomène de résurgence paramétrique. On applique ceci pour donner un module complet de classification analytique orbitale pour le déploiement d'un col rśonant générique d'un champ de vecteurs en dimension 2, via le déploiement d'un de ses difféomorphismes d'holonomie. Dans ce cas encore le module est donné par le déploiement du module de Martinet-Ramis pour un point de selle résonant. Quand le point de selle passe par la résonance on observe une "bifurcation transcritique" : de chaque côté de la bifurcation la dynamique au voisinage du point de selle est gouvernée par une partie différente du déploiement du module . On inclut ensuite la dépendance du temps et on donne un module complet de conjugaison analytique pour le déploiement d'un point de selle générique.

1   Introduction

This paper is part of a general program to study the dynamics of a germ of analytic diffeomorphism with a fixed point at the origin:

$$f(z) = \lambda_0 z + o(z), \qquad \lambda_0 \neq 0. \tag{1.1}$$

The behaviour is known for the different values of λ_0 and we want to "glue" these different behaviours in a global picture with λ_0 as a parameter. Indeed it is known that:

1. For $|\lambda_0| \neq 1$ then f is linearizable in the neighborhood of the origin, i.e. there exists a change of coordinate $Z = z + o(z) = h(z)$ such that

$$h \circ f \circ h^{-1}(Z) = \lambda_0 Z. \tag{1.2}$$

2. For $\lambda_0 = \exp(2\pi i \alpha)$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then the diffeomorphism is formally linearizable, i.e. there exists a formal change of coordinate

$$Z = h(z) = z + \sum_{n \geq 2} a_n z^n \tag{1.3}$$

such that (1.2) is satisfied at the formal level. We distinguish the two cases:

   - If α is diophantian (badly approximated by rational numbers) then the change of coordinate (1.3) is indeed analytic.
   - If α is Liouvillian (well approximated by rational numbers) then the change of coordinate (1.3) is generically divergent [11] (in the measure theoretic sense).

3. If $\lambda_0 = \exp(2\pi i \frac{p}{q})$ and the diffeomorphism is formally linearizable then it is analytically linearizable [2].

4. In the general case of $\lambda_0 = \exp(2\pi i \frac{p}{q})$ there are obstructions to linearizability at the level of the finite jet of $f(z)$. The geometric meaning of these obstructions correspond to the birth of periodic orbits of period q as λ_0 bifurcates from $\exp(2\pi i \frac{p}{q})$. These orbits are called "materializations of resonances" by Arnold [1], Ilyashenko and Pyartli [12]. It was conjectured that their presence would be the obstruction to linearization when perturbing λ_0 to λ of the form $\exp(2\pi i \alpha)$ with α Liouvillian irrational number. The works of Yoccoz [27] and Pérez-Marco [19] have shown that, although this happens quite often and in particular for the quadratic map, there are other types of nonlinearities which obstruct linearizability. In this paper we focus on the case $\lambda_0 = \exp(2\pi i \frac{p}{q})$ and f non linearizable with nonzero first resonant monomial.

As two non linearizable resonant diffeomorphisms of the form (1.1) are conjugate if and only if their q-th iterates are conjugate [10], we limit ourselves to the study of their q-th iterates which have, in suitable coordinates, the form

$$g(z) = f^q(z) = z + z^{kq+1} + o(z^{kq+1}). \tag{1.4}$$

The integer k is a first formal invariant. In this paper we limit ourselves to the generic case $k = 1$. As the diffeomorphism g is not linearizable we look for a nonlinear (more complex) "normal form". There exists a formal change of coordinates (1.3) which conjugates g with the time-one map of the flow of the vector field

$$\frac{z^{q+1}}{1 + az^q} \frac{\partial}{\partial z}, \tag{1.5}$$

where $a \in \mathbb{C}$ is an analytic invariant.

   - If the series (1.3) is convergent and h is analytic then we say that the diffeomorphism (1.1) is *normalizable*. If (1.1) with fixed λ_0 depends on a finite number of analytic parameters then, for each fixed k, this is satisfied for an analytic subset of the parameter space [7]. Again it is a very exceptional phenomenon.

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- In the generic case the normalizing series (1.3) is divergent and the modulus of analytic classification is a functional one. **Why?** Let us limit ourselves to the case $k = 1$. Then, when we perturb λ_0 from its initial value, the diffeomorphism has a fixed point and a periodic orbit of period q. If we perturb λ_0 to λ in the Poincaré domain (i.e. $|\lambda| \neq 1$) then both the fixed point and the periodic orbit are linearizable. In the neighborhoods of the fixed and periodic points it is possible to find changes of coordinates transforming the family g_ϵ to the "model family" unfolding (1.5) namely the time-one map of the flow of the vector field

$$\frac{z(z^q - \epsilon)}{1 + a(\epsilon)z^q} \frac{\partial}{\partial z} \qquad (1.6)$$

but the changes of coordinates to the model family in the neighborhood of the origin and in the neighborhood of the periodic points are not compatible. This was conjectured by Arnold and proved by Martinet [15] and Glutsyuk [8] in the case $\lambda_0 = 1$ and we extend it here to the case $\lambda_0 = \exp(2\pi i \frac{p}{q})$. What happens if we perturb λ_0 to λ in the Siegel direction ($|\lambda| = 1$)? The case $\lambda_0 = 1$ is treated in [14]. There it was shown that the dynamics near the two fixed points can be obtained from the unfolding of the Ecalle-Voronin functional modulus (ψ^0, ψ^∞), where ψ^0 and ψ^∞ are germs of analytic diffeomorphisms at the origin and at ∞. The non normalizability of (1.1) is expressed by the nonlinearity of at least one of ψ^0 and ψ^∞. Moreover the paper shows the existence of the *parametric resurgence phenomenon*: the nonlinearities of ψ^0 or ψ^∞ control the nonlinearizability of one of the fixed points for sequences of resonant parameter values converging to $\lambda_0 = 1$.

In this paper we extend to the case $\lambda_0 = \exp(2\pi i \frac{p}{q})$ the results of [14] for $\lambda_0 = 1$. The modulus of (1.1) is described in the literature by Écalle [7], Martinet-Ramis [17] and Ilyashenko [10] in slightly different ways. We work here with the description given by Ilyashenko [10] and we make the link with the presentation of Martinet-Ramis in Section 4.5. The modulus is given by a 2-tuple

$$(\psi_1^0, \psi_1^\infty) \qquad (1.7)$$

of germs of analytic functions at the origin and at infinity respectively, which we extend to a $2q$-tuple of germs of analytic functions

$$(\psi_1^0, \ldots, \psi_q^0, \psi_1^\infty, \ldots, \psi_q^\infty), \qquad (1.8)$$

where σ is the permutation of $\{1, \ldots, q\}$ generated by $j \mapsto j + p \pmod{q}$ and

$$\psi_{\sigma(j)}^{0,\infty}(w) = \exp\left(\frac{2\pi i}{q}\right) \psi_j^{0,\infty}\left(\exp\left(-\frac{2\pi i}{q}\right) w\right). \qquad (1.9)$$

We show that a complete modulus of analytic classification for a generic 1-parameter unfolding of (1.1) is given by an unfolding of (1.7), which we can identify with an unfolding of (1.8) which still satisfies (1.9). This unfolding can be taken continuous in the parameter over appropriate sectors of parameter space. From this unfolding one can study the dynamics near the bifurcating fixed and periodic points. In particular the parametric resurgence phenomenon again occurs.

We then apply these results to the generic 1-parameter unfolding of a vector field in the neighborhood of a generic resonant hyperbolic saddle. (A hyperbolic saddle is generic if the first resonant monomial of the orbital normal form is nonzero and a 1-parameter unfolding is generic if the derivative with respect to the parameter of the quotient of eigenvalues is nonzero.) We consider both orbital equivalence and conjugacy. We first show that a complete modulus for orbital analytic classification is given by the modulus of analytic classification of the unfolding of any of its holonomy maps (there is one for each separatrix). A complete modulus of classification of vector fields with a resonant hyperbolic saddle has been given by Voronin and Grintchy [26]. This modulus is composed of two parts: the first part is the Ecalle-Voronin modulus of the holonomy and the second part, the time-part, is composed by a pair of germs of analytic functions at the origin and at infinity respectively. We show that a complete modulus of analytic classification under

conjugacy between two germs of generic families of vector fields unfolding germs of vector fields with a generic resonant saddle is given by the unfolding of the Voronin-Grintchy modulus.

The point of view taken throughout the paper is to compare the family of diffeomorphisms (vector fields) with the model family which would be the family expected if we had convergence of the normalizing changes of coordinates.

The paper is organized in the following way. In Section 2 we "prepare the family" of diffeomorphisms so as to change the initial parameter to the "canonical parameter" which is in particular an analytic invariant. In Section 3 we construct Fatou coordinates bringing the family of diffeomorphisms to the model family. In Section 4 we describe the modulus of analytic classification for a generic family of analytic diffeomorphisms. In Section 5 we describe the parametric resurgence phenomenon. In Section 6 we describe the modulus of orbital analytic classification of a generic family unfolding a generic resonant saddle. In Section 7 we describe the "transcritical bifurcation" phenomenon which is natural in this context. Finally in Section 8 we give the modulus of analytic conjugacy of a generic family unfolding a generic resonant saddle. We end up with questions for future research.

2 Preparation of the family

We consider a germ of resonant generic diffeomorphism of the form

$$f_0(z) = e^{\frac{2i\pi p}{q}} z + \frac{e^{\frac{2i\pi p}{q}}}{q} z^{q+1} + o(z^{q+1}). \tag{2.1}$$

Then f_0^q has a fixed point at the origin of multiplicity $q+1$, which corresponds for f_0 to the coalescence of a fixed point with a periodic orbit of period q: the fixed point and periodic orbit bifurcate in a generic unfolding. Because we can always localize the fixed point at the origin, bring the family in normal form up to order $q+1$ and rescale, then a germ of generic unfolding can be taken of the form

$$f_\epsilon(z) = (e^{\frac{2i\pi p}{q}} - \alpha)z + \frac{e^{\frac{2i\pi p}{q}}}{q} z^{q+1} + o(z^{q+1}) \tag{2.2}$$

with α a small parameter. In the particular case $q = 1$ this means that we study the transcritical bifurcation since the generic unfolding of a double fixed point has been studied in [14] (the transcritical bifurcation means that we limit ourselves to unfoldings preserving the origin). We can limit ourselves to consider the conjugacy problem for the q-th iterate $g_\epsilon = f_\epsilon^q$ of f_ϵ. This will be proved in Lemma 4.13 below.

It is easier to work with the q-th iterate $g_0 = f_0^q$ (resp. $g_\epsilon = f_\epsilon^q$) of f_0 (resp. f_ϵ) because g_0 has multiplier equal to 1 at the origin. If we replace α by ϵ given by

$$(1 - \epsilon) = (e^{\frac{2\pi i p}{q}} - \alpha)^q, \tag{2.3}$$

which yields $\epsilon = q e^{\frac{2\pi i p}{q}} \alpha + o(\alpha)$ we can write g_ϵ as

$$g_\epsilon(z) = z(1 - \epsilon) + (1 + O(\epsilon))z^{q+1} + o(z^{q+1}). \tag{2.4}$$

Proposition 2.1 *There exists an analytic change of variable $(z, \epsilon) \mapsto (\tilde{z}, \tilde{\epsilon})$ tangent to the identity and fibered over the parameter space bringing the family (2.4) to the prepared form*

$$\tilde{g}_{\tilde{\epsilon}}(\tilde{z}) = \tilde{z} + \tilde{z}(\tilde{z}^q - \tilde{\epsilon})[1 + B(\tilde{\epsilon}) + A(\tilde{\epsilon})\tilde{z}^q + \tilde{z}(\tilde{z}^q - \tilde{\epsilon})(1 + h(\tilde{\epsilon}, \tilde{z}))], \tag{2.5}$$

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with fixed points $\tilde{z}_0 = 0$ and $\tilde{z}_j$, $j = 1, \ldots, q$ with $\tilde{z}_j^q = 1$, in which the multiplier $\lambda_0 = 1 - \tilde{\epsilon}(1 + B(\tilde{\epsilon}))$ of the fixed point $\tilde{z}_0 = 0$ satisfies

$$\lambda_0 = \exp(-\tilde{\epsilon}). \tag{2.6}$$

In particular the parameter $\tilde{\epsilon}$ is an analytic invariant for $\tilde{g}_{\tilde{\epsilon}}$. We call it the canonical parameter. Let $\lambda_1, \ldots \lambda_q$ be the multipliers of the fixed points $\tilde{z}_j$ where $\tilde{z}_j^q = \tilde{\epsilon}$. The formal parameter

$$a(\tilde{\epsilon}) = \frac{1}{\ln \lambda_0} + \sum_{j=1}^{q} \frac{1}{\ln \lambda_j} \tag{2.7}$$

depends analytically on $\tilde{\epsilon}$ and hence on ϵ. It is an analytic invariant of g_ϵ. (As $\lambda_0, \ldots \lambda_q$ are all close to 1 there is no problem in choosing the $\ln \lambda_j$ close to zero in a continuous way.)

PROOF. The equation of fixed points for g_ϵ yields $z = 0$ and $(1 + O(\epsilon))z^q - \epsilon + o(z^q) = 0$. We make the change of coordinate $\hat{z} = z(1 + O(\epsilon)) + o(z)$ so that the fixed points become $\hat{z}^q = \epsilon$. In the $\hat{z}$-coordinate this yields for g_ϵ:

$$\hat{g}_\epsilon(\hat{z}) = \hat{z} + \hat{z}(\hat{z}^q - \epsilon)h(\hat{z}, \epsilon) \tag{2.8}$$

with $h(\hat{z}, \epsilon) = 1 + c_0(\epsilon) + O(\hat{z})$. Using a rescaling in $(\hat{z}, \epsilon) \mapsto (1 + c_0(\epsilon))^{-1/q}\hat{z}, (1 + c_0(\epsilon)^{-1})\epsilon)$ we can of course suppose that $h(\hat{z}, \epsilon) = 1 + O(\hat{z})$. Using the Weierstrass division theorem we write

$$h(\hat{z}, \epsilon) - 1 = \sum_{j=1}^{q} c_j(\epsilon)\hat{z}^j + \hat{z}(\hat{z}^q - \epsilon)k(\hat{z}, \epsilon) \tag{2.9}$$

We will show in Lemma 2.2 below that $c_1(\epsilon) = \cdots = c_{q-1}(\epsilon) \equiv 0$, which yields that the multipliers of the fixed points $\hat{z}^q = \epsilon$ are exactly $1 + q\epsilon(1 + c_q(\epsilon)\epsilon)$.

Unfortunately in general ϵ is not an analytic invariant in (2.8). To achieve this we need to make a scaling $(\tilde{z}, \tilde{\epsilon}) = (\hat{z}(1 + b(\epsilon))^{1/q}, \epsilon(1 + b(\epsilon)))$ in (2.8) transforming $\hat{g}_\epsilon$ into $\tilde{g}_{\tilde{\epsilon}}$ so that

$$\tilde{g}_{\tilde{\epsilon}}'(0) = 1 - \frac{\tilde{\epsilon}}{1 + b(\epsilon)} = \exp(-\tilde{\epsilon}) \tag{2.10}$$

The condition (2.10) (and (2.6)) will be satisfied for $b(\epsilon)$ satisfying $b(0) = 0$ by the implicit theorem.

The formal parameter $a(\tilde{\epsilon})$ defined in (2.7) is analytic for $\epsilon \neq 0$. As it is bounded at $\epsilon = 0$ it is analytic. □

Lemma 2.2 *In* (2.9) *we have*

$$c_1(\epsilon) = \cdots = c_{q-1}(\epsilon) \equiv 0. \tag{2.11}$$

PROOF. The proof uses that $\hat{g}_\epsilon(\hat{z})$ is the q-th iterate of $\hat{f}_\epsilon$ (where $\hat{f}_\epsilon$ is f_ϵ in $\hat{z}$-coordinate) so the multipliers λ_j of the fixed points $\hat{z}_j = e^{2\pi ij/q}\epsilon^{1/q}$ are all the same. This yields the system of equations $\sum_{k=1}^{q-1} \epsilon^{k/q}c_k = \sum_{k=1}^{q-1} \epsilon^{k/q}c_k e^{\frac{2\pi ijk}{q}}$. Hence the $\epsilon^{k/q}c_k$ are solutions of a homogeneous system of linear equations with matrix

$$M = \begin{pmatrix} e^{\frac{2\pi i}{q}} - 1 & e^{\frac{4\pi i}{q}} - 1 & \ldots & e^{\frac{2(q-1)\pi i}{q}} - 1 \\ e^{\frac{4\pi i}{q}} - 1 & e^{\frac{8\pi i}{q}} - 1 & \ldots & e^{\frac{4(q-1)\pi i}{q}} - 1 \\ \vdots & \vdots & \ldots & \vdots \\ e^{\frac{2(q-1)\pi i}{q}} - 1 & e^{\frac{4(q-1)\pi i}{q}} - 1 & \ldots & e^{\frac{2(q-1)^2\pi i}{q}} - 1 \end{pmatrix}. \tag{2.12}$$

We divide each column by its first coefficient. Then, starting from the last row, we subtract from each row the row immediately above. This yields a Vandermonde determinant which is nonzero. □

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As in [14] we want to compare the prepared family $\tilde{g}_{\tilde{\epsilon}}$ in (2.5) with the "model family" which is the time-one map of the vector field

$$\frac{\tilde{z}(\tilde{z}^q - \tilde{\epsilon})}{1 + a(\tilde{\epsilon})\tilde{z}^q} \frac{\partial}{\partial \tilde{z}}, \tag{2.13}$$

where $a(\tilde{\epsilon})$ is defined as in (2.7). The vector field (2.13) has singular points $\tilde{z}_0 = 0$, $\tilde{z}_1, \dots \tilde{z}_q$ with respective eigenvalues

$$\mu_0 = -\tilde{\epsilon}, \qquad \mu_j = \frac{q\tilde{\epsilon}}{1 + a(\tilde{\epsilon})\tilde{\epsilon}}, \quad j = 1, \dots, q. \tag{2.14}$$

As μ_0 and $\mu_0^{-1} + \sum_{j=1}^q \mu_j^{-1} = a(\tilde{\epsilon})$ are analytic invariants of (2.13) which also depend analytically on $\tilde{\epsilon}$ it follows that $\tilde{\epsilon}$ and $a(\tilde{\epsilon})$ are analytic invariants of (2.13). The multipliers of the time-one map of (2.13) are $\lambda_j = e^{\mu_j}$, i.e. there are precisely the multipliers of the fixed points of $\tilde{g}_{\tilde{\epsilon}}$.

Definition 2.3 A family (2.2) to which we have applied the change of coordinate and parameter of Proposition 2.1 is called *prepared*. We use the same term for the corresponding $g_\epsilon = f_\epsilon^q$.

Remark 2.4 This gives the geometric interpretation of the formal invariant $a(0)$: indeed $a(\epsilon)$ measures a shift between the fixed points and the periodic points through (2.7) and $a(0)$ is the limit shift.

3 The construction of Fatou coordinates

>From now on we will limit ourselves to a prepared family $f_\epsilon(z)$ such that

$$\begin{aligned} g_\epsilon(z) = f_\epsilon^q(z) &= z + z(z^q - \epsilon)[1 + B(\epsilon) + A(\epsilon)z^q + z(z^q - \epsilon)(1 + h(\epsilon, z))] \\ &= z + z(z^q - \epsilon)(1 + k_\epsilon(z)), \end{aligned} \tag{3.1}$$

with $k_\epsilon(z) = O(|\epsilon, z|)$ and $B(\epsilon) = \frac{(1 - \exp(-\epsilon))}{\epsilon} - 1$.

Fatou coordinates are changes of coordinates which transform the family (3.1) to the associated "model family" which is the time-one map of the vector field

$$\frac{z(z^q - \epsilon)}{1 + a(\epsilon)z^q} \frac{\partial}{\partial z}, \tag{3.2}$$

where $a(\epsilon)$ is chosen so that the multipliers of the periodic points $z_1, \dots, z_q$ of g_ϵ are given by $\lambda_j = \exp(\mu_j) = \exp(\frac{q\epsilon}{1 + \epsilon a(\epsilon)})$.

We give the construction with little details as it is quite standard and very similar to [14].

3.1 The two charts

We want to study the dynamics of the germ of family $g_\epsilon(z)$. So we will study its dynamics on any sufficiently small neighborhood of the origin in z-coordinate which we can choose of the form $U = \{z, |z| < r\}$ with $r \in (0, 1)$ for all sufficiently small values of the parameter ϵ in a small ball $V = \{\epsilon; |\epsilon| < \rho\}$. We limit ourselves to values of ϵ sufficiently small so that the fixed points of g_ϵ remain inside U. For this it suffices to take

$$\rho < \frac{r^q}{2}, \tag{3.3}$$

a condition which will be assumed throughout the paper.

We will not be able to give a uniform treatment for all $\epsilon \in V$ and we will need to cover V with two sectors, each of opening $2\pi - 2\delta$ with $\delta \in (0, \frac{\pi}{2})$, but we are essentially interested to δ arbitrarily small. The parameter $\delta \in (0, \frac{\pi}{2})$ is chosen at the beginning and kept fixed for all the treatment. The size of the neighborhoods U and V defined above (for V we will need a stronger condition than (3.3)) depends on δ. We will be able to give a uniform treatment of g_ϵ over the two following two sectors of V:

$$\begin{aligned} V_{\delta,+} &= \{\epsilon \in V \,|\, \arg \epsilon \in (-\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta)\} \\ V_{\delta,-} &= \{\epsilon \in V \,|\, \arg \epsilon \in (\frac{\pi}{2} + \delta, \frac{5\pi}{2} - \delta)\}. \end{aligned} \tag{3.4}$$

3.2 The lifted diffeomorphism

We first introduce a change of coordinate which nearly rectifies the family g_ϵ to the translation by 1 and sends the fixed points to infinity. Let

$$T_\beta(Z) = Z + \beta. \tag{3.5}$$

We will in particular consider $T_{\alpha(\epsilon)}(Z)$ with

$$\alpha(\epsilon) = \begin{cases} \frac{2\pi i}{q\epsilon} & \epsilon \neq 0 \\ 0 & \epsilon = 0. \end{cases} \tag{3.6}$$

We introduce the change of coordinate $p_\epsilon : S_\epsilon \to \mathbb{CP}^1 \setminus \{0, z_1, \ldots, z_q\}$ given by

$$z = p_\epsilon(Z) = \begin{cases} \left(\frac{\epsilon}{1 - e^{q\epsilon Z}}\right)^{1/q} & \epsilon \neq 0 \\ \left(-\frac{1}{qZ}\right)^{1/q} & \epsilon = 0, \end{cases} \tag{3.7}$$

where S_ϵ is the Riemann surface of the function

$$\begin{cases} \left(\frac{e^{q\epsilon Z} - 1}{\epsilon}\right)^{1/q} & \epsilon \neq 0 \\ Z^{1/q} & \epsilon = 0. \end{cases} \tag{3.8}$$
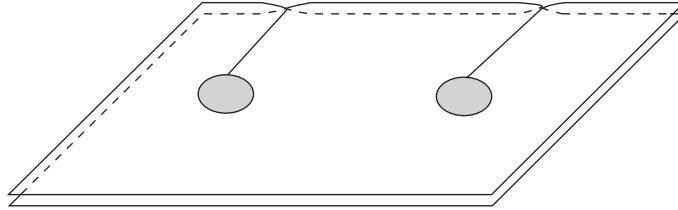

Figure 1: The domain of Z in the case $q = 2$

It is the composition of the map

$$\zeta = \tilde{p}_\epsilon(Z) = \begin{cases} \frac{\epsilon}{1 - e^{q\epsilon Z}} & \epsilon \neq 0 \\ -\frac{1}{qZ} & \epsilon = 0, \end{cases} \tag{3.9}$$

which is periodic of period $\alpha(\epsilon)$ and which we may consider as defined over S_ϵ with the map $z = \zeta^{1/q}$. On a strip of width $\alpha(\epsilon)$ if $\epsilon \neq 0$ the image of each sheet of S_ϵ by $\tilde{p}_\epsilon$ covers $\mathbb{CP} \setminus \{0, \epsilon\}$ once. Hence p_ϵ covers $\mathbb{CP} \setminus \{0, \epsilon\}$ once. Its (multivalued) inverse is given by:

$$Z = p_\epsilon^{-1}(z) = \begin{cases} \frac{1}{q\epsilon} \ln \frac{z^q - \epsilon}{z^q} & \epsilon \neq 0 \\ -\frac{1}{qz^q} & \epsilon = 0, \end{cases} \tag{3.10}$$

which is the composition of $z \mapsto \zeta = z^q$ with the multivalued map

$$Z = \tilde{p}_\epsilon^{-1}(\zeta) = \begin{cases} \frac{1}{q\epsilon} \ln \frac{\zeta - \epsilon}{\zeta} & \epsilon \neq 0 \\ -\frac{1}{q\zeta} & \epsilon = 0. \end{cases} \tag{3.11}$$

For $\epsilon \neq 0$ it is univalued when the image is restricted to a strip of width $\alpha(\epsilon)$. We can lift the map $T_{\alpha(\epsilon)}$ to S_ϵ.

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The image of $U \setminus \{0, z_1, \ldots, z_q\}$ under p_ϵ^{-1} is

$$\hat{U}_\epsilon = S_\epsilon \setminus \cup_{j \in \mathbb{Z}} B_j \tag{3.12}$$

where B_0 is the component of $p_\epsilon^{-1}(\mathbb{C} \setminus U)$ which contains the origin and $B_i = T_{\alpha(\epsilon)}^i(B_0) = T_{i\alpha(\epsilon)}(B_0)$. B_0 is called the *fundamental hole*. It is a q-covering of a neighborhood of the origin.

We lift the function $g_\epsilon(z)$ to a function $G_\epsilon(Z)$ commuting with $T_{q\alpha(\epsilon)}$.

Proposition 3.1 *The function G_ϵ is a small perturbation of the translation $Z \mapsto Z + 1$ in the C^1-topology. More precisely, there exists $K > 0$ such that, for $r > 0$ sufficiently small and condition (3.3),*

$$|G_\epsilon(Z) - Z - 1| < Kr. \tag{3.13}$$

$$|G_\epsilon'(Z) - 1| < Kr^{q+1}. \tag{3.14}$$

PROOF. The function $G_\epsilon(Z)$ can be written

$$
\begin{aligned}
G_\epsilon(Z) &= \tfrac{1}{q\epsilon} \ln\left(\tfrac{g(z)^q - \epsilon}{g(z)^q} \right) \\
&= \tfrac{1}{q\epsilon} \ln\left(\tfrac{(z^q - \epsilon)[1 + \sum_{j=1}^q \binom{q}{j} z^q (z^q - \epsilon)^{j-1}(1+k_\epsilon)^j]}{z^q [1 + (z^q - \epsilon)(1 + k_\epsilon)]^q} \right) \\
&= Z + \tfrac{1}{q\epsilon} \ln\left(\tfrac{1 + \sum_{j=1}^q \binom{q}{j} z^q (z^q - \epsilon)^{j-1}(1+k_\epsilon)^j}{1 + \sum_{j=1}^q \binom{q}{j}(z^q - \epsilon)^j (1+k_\epsilon)^j} \right) \\
&= Z + \tfrac{1}{q\epsilon} \ln\left(1 + \epsilon \tfrac{\sum_{j=1}^q \binom{q}{j}(z^q - \epsilon)^{j-1}(1+k_\epsilon)^j}{[1+(z^q - \epsilon)(1+k_\epsilon)]^q} \right) \\
&= Z + 1 + O(z, \epsilon),
\end{aligned}
\tag{3.15}
$$

since $k_\epsilon(z) = O(z, \epsilon)$.

Let $R_\epsilon(Z) = \frac{\sum_{j=1}^q \binom{q}{j}(z^q - \epsilon)^{j-1}(1+k_\epsilon)^j}{(1+(z^q - \epsilon)(1+k_\epsilon))^q}$. Then

$$\frac{dG_\epsilon}{dZ} = 1 + \frac{1}{q(1 + \epsilon R_\epsilon)} \frac{dR_\epsilon}{dz}(z) \frac{dz}{dZ}. \tag{3.16}$$

The result follows as $\frac{dR_\epsilon}{dz}$ is bounded for small (z, ϵ) and

$$\frac{dz}{dZ} = \frac{1}{\frac{dZ}{dz}} = z(z^q - \epsilon) = O(|z|^{q+1} + |\epsilon z|). \tag{3.17}$$

$\square$

>From now on we suppose

- Condition (3.3);

- r is sufficiently small so that if we define

$$M(r) := Kr, \tag{3.18}$$

then

$$M(r) < \frac{\delta}{4}; \tag{3.19}$$

- r is sufficiently small so that the estimates of Proposition 3.1 are satisfied.

3.3 Translation domains

The Fatou coordinates are defined on maximal domains in Z-space called translation domains.

Definition 3.2 A line $\ell \subset \hat{U}_\epsilon$ is called an *admissible line* if ℓ and $G_\epsilon(\ell)$ are disjoint and the strip $\hat{C}_\epsilon(\ell)$ between ℓ and $G(\ell)$ is included in $\hat{U}_\epsilon$. The strip $\hat{C}_\epsilon(\ell)$ is called an *admissible strip*.

Lemma 3.3 *There exists $\theta_0(r)$ defined by*

$$\tan\theta_0(r) = 2M(r) \leq \frac{\delta}{2} \tag{3.20}$$

such that if the angle $\theta(\ell)$ of the line with the horizontal axis satisfies

$$\theta_0(r) \leq \theta(\ell) \leq \pi - \theta_0(r) \tag{3.21}$$

then $G_\epsilon(\ell)$ is located to the right of ℓ.

PROOF. Let θ_1 be the angle of the horizontal line through Z with the line through Z and tangent to the circle centered in $Z+1$ of radius $M(r) = Kr$. Then $|\sin\theta_1| = M(r) < |\tan\theta_1|$. We need $\sin\theta_0 > |\sin\theta_1|$. We let $\tan\theta_0 = 2M(r) \leq \frac{\delta}{2}$. As $\delta < \frac{\pi}{2} < 2$ then $\tan\theta_0 < 1$ yielding that $\theta_0 < \frac{\pi}{4}$. Hence $\cos\theta_0 > \frac{1}{\sqrt{2}}$ yielding that $\sin\theta_0 = 2M(r)\cos\theta_0 > \sqrt{2}M(r) > \sqrt{2}|\sin\theta_1|$.

The condition $\tan\theta_0 = \frac{\delta}{2}$ is easier to manipulate than a condition on $\sin\theta_0$. $\qquad\square$

The translation domains are the saturation of admissible strips under iterations of G_ϵ.

Definition 3.4 Let ℓ be an admissible line for G_ϵ. The *translation domain* associated with ℓ is the set

$$Q_\epsilon(\ell) = \{Z \in \hat{U}_\epsilon | \exists n \in \mathbb{Z} \; G_\epsilon^n(Z) \in \hat{C}_\epsilon(\ell) \quad \text{and} \quad \forall j \in [0,n] \subset \mathbb{Z}, G_\epsilon^j(Z) \in \hat{U}_\epsilon\} \tag{3.22}$$

(For $n < 0$, $[0,n] = \{j \in \mathbb{Z} | n \leq j \leq 0.\}$)

Proposition 3.5   *1. The domain $Q_\epsilon(\ell)$ is a simply connected open subset of $\hat{U}_\epsilon$.*

   2. $\hat{C}_\epsilon(\ell) \setminus \ell$ is a fundamental domain for G_ϵ restricted to $Q_\epsilon(\ell)$: each G_ϵ-orbit in $Q_\epsilon(\ell)$ has one and only one point in this subset.

   3. If ℓ' is another admissible line, then $\ell' \subset Q_\epsilon(\ell)$ if and only if $\ell \subset Q_\epsilon(\ell')$. This defines an equivalence relation among the admissible lines for G_ϵ, each equivalence class corresponding to a different translation domain.

Definition 3.6 A *Lavaurs translation domain* (Figure 2) is a domain associated with an admissible line passing between the fundamental hole and one of its two adjacent holes (notation Q_ϵ^L).
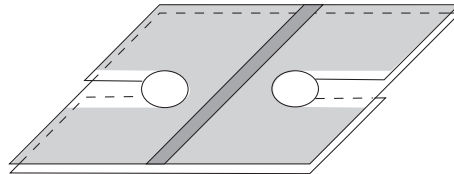

Figure 2: A fundamental domain $\hat{C}_\epsilon(\ell)$ associated to an admissible line ℓ and the Lavaurs translation domain it generates (the figure is drawn for $q = 2$)

Remark 3.7 It is also possible to define *Glutsyuk translation domains* associated with admissible lines parallel to the line of holes. The projection of the corresponding admissible strips are fundamental domains having the shape of annuli (tori once quotiented by g_ϵ). But Glutsyuk translations domains do not exist for all values of ϵ and this is why we have prefered the other approach.

Proposition 3.8 *Let us suppose that* r, ρ, δ *and* θ_0 *satisfy the following conditions*

$$\rho < r^{2q}, \qquad \tan\theta_0(r,\rho) = 2M(r) < \frac{\delta}{2}, \tag{3.23}$$

and $r < \frac{1}{2}$ *is sufficiently small so that*

$$r^q < \frac{\delta\pi\sqrt{2}}{16}. \tag{3.24}$$

Then for $V_{\delta,+}$ *(resp.* $V_{\delta,-}$*) to each admissible line for some* ϵ_0 *with* $\arg\epsilon_0 = \pi/2$ *(resp.* $\arg\epsilon_0 = -\pi/2$*) is associated a unique family of Lavaurs translation domains associated with admissible lines depending continuously on* ϵ *(see Figure* 3 *for the case of* $V_{\delta,+}$*).*
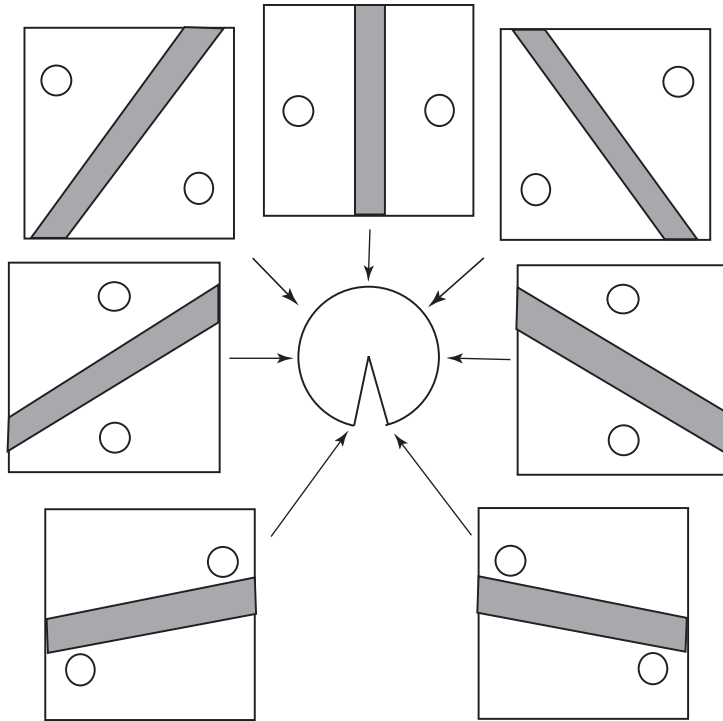

Figure 3: A continuous family of admissible lines and strips for $\epsilon \in V_{\delta,+}$ (for the sake of simplicity we have not drawn the ramification of S_ϵ at the holes)

PROOF. Let us take the case of $V_{\delta,+}$. The worst cases are the extreme cases when $\arg(\epsilon) = -\frac{\pi}{2} + \delta$ and $\arg(\epsilon) = \frac{3\pi}{2} - \delta$. Let us discuss the second case. The slope of the line joining the holes is $\tan\delta$. The radius of the holes is bounded by

$$-\frac{1}{q\epsilon}\ln\left(1 - \frac{\epsilon}{r^q}\right) < \frac{1}{q\epsilon}\frac{\epsilon/r^q}{1 - \epsilon/r^q} < \frac{2}{qr^q} \tag{3.25}$$

since $-\ln(1-k) < \frac{k}{1-k}$ and $\rho < \frac{r^q}{2}$. Hence the vertical distance between the top of one ball and the bottom of the next one is greater than $|\alpha|\sin\delta - \frac{4}{qr^q}$. We want to pass a strip admissible line with slope

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$|\tan \theta_0(r)| = 2M(r) < \frac{\delta}{2}$. Its maximum vertical size is less than $(1 + M(r)) \tan \theta_0(r) < \frac{1}{2}(1 + M(r))\delta < \delta$ since $M(r) < \frac{\delta}{4} < 1$. The horizontal distance between the center of two balls is $|\alpha| \cos \delta$, and the largest horizontal distance between two points of the different balls is less than $|\alpha| \cos \delta + \frac{4}{qr^q}$. Hence it suffices to have

$$\frac{\delta}{2} < \frac{|\alpha| \sin \delta - \frac{4}{qr^q} - \delta}{|\alpha| \cos \delta + \frac{4}{qr^q}}. \tag{3.26}$$

As $\sin \delta - \frac{\delta}{2} \cos \delta > \frac{\delta}{2\sqrt{2}}$ (see comment below) it suffices to have

$$|\alpha| \frac{\delta}{2\sqrt{2}} > \frac{4}{qr^q} + \delta \left(1 + \frac{2}{qr^q}\right), \tag{3.27}$$

which is satisfied if we have

$$\begin{aligned} |\alpha| \frac{\delta}{4\sqrt{2}} &> \frac{4}{qr^q} \\ |\alpha| \frac{\delta}{4\sqrt{2}} &> \frac{4\delta}{qr^q} > \delta \left(1 + \frac{2}{qr^q}\right). \end{aligned} \tag{3.28}$$

As $|\alpha| = \frac{2\pi}{q\rho}$, the first inequality is satisfied under condition (3.24). The second is satisfied for $r^{2q} < \frac{\pi}{2}$ which is automatically satisfied for $r < \frac{1}{2}$.

Let $m(\delta) = \sin \delta - \frac{\delta}{2} \cos \delta - \frac{\delta}{2\sqrt{2}}$. Then $m(0) = 0$, $m'(0) > 0$ and $m''(\delta) > 0$ yielding that $m'(\delta) > 0$ for all $\delta \in [0, \pi/2]$. Hence $m(\delta) > 0$ for $\delta \in (0, \pi/2]$. $\qquad\square$

3.4 Existence of Fatou coordinates

Theorem 3.9 *Let $Q_\epsilon = Q_\epsilon(\ell)$ be any translation domain and $Z_0(\epsilon) \in Q_\epsilon$.*

1. *There exists a holomorphic diffeomorphism $\Phi_\epsilon : Q_\epsilon \to \mathbb{C}$, such that*

$$\Phi_\epsilon(G_\epsilon(Z)) = \Phi_\epsilon(Z) + 1, \tag{3.29}$$

   for $Z \in Q_\epsilon \cap G_\epsilon^{-1}(Q_\epsilon)$. Moreover

$$\lim_{Im(Z) \to \pm\infty} Im(\Phi_\epsilon(Z)) = \pm\infty \tag{3.30}$$

2. *If $\Phi_{1,\epsilon}$ and $\Phi_{2,\epsilon}$ are two solutions of (3.29), then there exists $A \in \mathbb{C}$ such that $\Phi_{2,\epsilon}(Z) = A + \Phi_{1,\epsilon}(Z)$. In particular if $Z_0(\epsilon) \in Q_\epsilon(\ell)$ there exists a unique holomorphic diffeomorphism Φ_ϵ satisfying (3.29) together with $\Phi_\epsilon(Z_0(\epsilon)) = 0$.*

PROOF. The proof is exactly the same as in [14] as it relies only on (3.13), (3.14) and (3.19). We put it here for the sake of completeness. The technique we use is identical to that of Shishikura [23], as adapted in [14]. It consists in constructing a quasi-conformal conjugacy of G_ϵ to the translation by 1 and then using Ahlfors-Bers theorem to transform it into a conformal conjugacy.

All along the proof we do not mention the ϵ-dependence. Let ℓ be an admissible line in the translation domain Q, $\hat{C}(\ell)$ the corresponding strip and let Z_1 be any point of ℓ. Points of ℓ can be written as $Z_1 + Y e^{i\theta}$, $Y \in \mathbb{R}$, where $\theta = \theta(\ell) \in (\theta_0, \pi - \theta_0)$ is the angle of ℓ with $\mathbb{R}$. We recall that $\theta_0(r) \leq \theta(\ell) \leq \pi - \theta_0(r)$. We define $h_1 : C_0 = \{(X, Y) \in \mathbb{R}^2 \mid 0 \leq X \leq 1\} \to \hat{C}(\ell)$ by:

$$h_1(X, Y) = (1 - X)(Z_1 + Y e^{i\theta}) + X G_\epsilon(Z_1 + Y e^{i\theta}). \tag{3.31}$$

Then

$$\begin{aligned} \frac{\partial h_1}{\partial X} &= G_\epsilon(Z_1 + Y e^{i\theta}) - (Z_1 + Y e^{i\theta}) \\ \frac{\partial h_1}{\partial Y} &= X e^{i\theta} G_\epsilon'(Z_1 + Y e^{i\theta}) + e^{i\theta}(1 - X). \end{aligned} \tag{3.32}$$

Using the estimates (3.13) and (3.14), these formulas imply that

$$\frac{\partial h_1}{\partial X} = 1 + u(X,Y), \quad \frac{\partial h_1}{\partial Y} = e^{i\theta} + v(X,Y), \qquad \text{with } |u|,|v| \le M(r) \le \frac{\delta}{4}. \tag{3.33}$$

Let $\mu_{h_1} = \frac{\partial h_1}{\partial \bar{Z}} / \frac{\partial h_1}{\partial Z}$ be the dilatation coefficient field of h_1. One has

$$\frac{\partial h_1}{\partial \bar{Z}} = \frac{1}{2}[1 + u + i(e^{i\theta} + v)] \text{ and } \frac{\partial h_1}{\partial Z} = \frac{1}{2}[1 + u - i(e^{i\theta} + v)]. \tag{3.34}$$

When $u, v \equiv 0$, i.e. when $G_\epsilon \equiv T_1$, μ_{h_1} reduces to $\mu^0 = \frac{1+ie^{i\theta}}{1-ie^{i\theta}} = \frac{i\cos\theta}{1+\sin\theta}$ and

$$\left|\mu^0\right| = \frac{1}{\sqrt{1 + \tan^2\theta} + |\tan\theta|} \le \frac{1}{1 + 4M(r)} \tag{3.35}$$

as $|\tan\theta| > \tan\theta_0 = 2M(r)$. From (3.34) one can write

$$\mu_{h_1} = \mu^0 \left(1 + \frac{u - iv}{1 - ie^{i\theta}}\right)^{-1} + \frac{u + iv}{1 - ie^{i\theta} + u - iv}. \tag{3.36}$$

Let us remark that $\left|1 - ie^{i\theta}\right| \ge \sqrt{2}$. Then, from (3.36) one deduces $(M = M(r))$

$$\begin{aligned}
||\mu_{h_1}||_\infty &= Sup\{|\mu_{h_1}(z)| \mid z \in C_0\} \\
&\le (1 + 4M)^{-1}(1 - M)^{-1} + M(1 - M)^{-1} \\
&= 1 - \frac{2M - 8M^2}{(1+4M)(1-M)} < 1,
\end{aligned} \tag{3.37}$$

for $1 - 4M > 0$ which is satisfied for $\delta < 1$. So h_1 is a quasi-conformal mapping on the strip C_0 and satisfies $h_1^{-1}(G_\epsilon(Z)) = h_1^{-1}(Z) + 1$ for $Z \in \ell$ when $M(r, \rho)$ is small enough. Moreover, $\mu = \mu_{h_1}$ is a Beltrami field on C_0. (This just means that μ is defined by a L^∞-function with a norm strictly less than 1). One can also write that $\mu = h_1^* \mu_0$, where μ_0 is the standard Beltrami field on $\mathbb{C}$ (defined by the function 0).

We extend μ to all of $\mathbb{C}$ by means of the translation T_1: the extended μ is periodic of period 1, is in $L^\infty(\mathbb{C})$ and has a L^∞-norm: $||\mu||_\infty = ||\mu_{h_1}||_\infty < 1$ (μ may have discontinuities along the lines $\{Re\,Z = c \mid c \in \mathbb{Z}\}$). Then this extended μ is a Beltrami field on $\mathbb{C}$.

The universal covering

$$w = E(W) = \exp(-2\pi i W) \tag{3.38}$$

from $\mathbb{C}$ to $\mathbb{C}^*$ induces a holomorphic diffeomorphism from $\mathbb{C}/T_1$ to $\mathbb{C}^*$. As μ is invariant by T_1 the map E induces a Beltrami field $\tilde{\mu}$ on $\mathbb{C}^*$ with the same norm : $\mu = E^*(\tilde{\mu})$. Considering the Riemann sphere S^2 as $\mathbb{C}^* \cup \{0, \infty\}$, one can extend $\tilde{\mu}$ on S^2 by, for instance, $\tilde{\mu}(0) = \tilde{\mu}(\infty) = 0$. Then $\tilde{\mu}$ defines a Beltrami field on the Riemann sphere.

By Ahlfors-Bers measurable mapping theorem there exists a unique quasi-conformal mapping $\tilde{h}_2 : S^2 \to S^2$ such that $\tilde{h}_2^* \mu_0 = \tilde{\mu}$, and $\tilde{h}_2(0) = 0, \tilde{h}_2(\infty) = \infty, \tilde{h}_2(1) = 1$. As $0, 1 \in E^{-1}(1)$, this map lifts to a quasi-conformal map $h_2 : \mathbb{C} \to \mathbb{C}$ sending 0 to 0 and 1 to 1. Indeed, one can lift $\tilde{h}_2$ into a map h_2 such that $h_2(0) = 0$. The circle in S^2 which turns one time around 0 or ∞ lifts into the line segment $[0, 1]$ in $\mathbb{C}$. This means that $h_2(1) = 1$. We have also that $Im(h_2(X + iY)) \to \pm\infty$ when $Y \to \pm\infty$.

The most important property of h_2 is that it commutes with T_1. To see this, consider the homeomorphism $H_2 = h_2 \circ T_1 \circ h_2^{-1}$. It induces the identity on S^2 and must then be a power of the deck transformation T_1 of the universal covering map E : i.e. $H_2 = T_1^n$ for some $n \in \mathbb{Z}$. Now $H_2(0) = h_2 \circ T_1(0) = h_2(1) = 1$. This forces $n = 1$ and then $H_2 = T_1$, i.e $h_2 \circ T_1 = T_1 \circ h_2$.

We define $\phi : \hat{C}(\ell) \to \mathbb{C}$ by $\phi = h_2 \circ h_1^{-1}$ and extend it by T_1 to a mapping $\phi : \mathbb{C} \to \mathbb{C}$ which is quasi-conformal and preserves the standard conformal structure. Hence it is conformal. For $Z \in \ell$ one has $T_1 \circ \phi(Z) = \phi \circ G_\epsilon(Z)$. Then ϕ extends to a map Φ of Q into $\mathbb{C}$ by $\Phi(Z) = \phi \circ G_\epsilon^n(Z) - n$ where $n \in \mathbb{Z}$ is such that $G_\epsilon^n(Z) \in \hat{C}(\ell)$. This map Φ is a holomorphic diffeomorphism which verifies: $\Phi \circ G_\epsilon = T_1 \circ \Phi$.

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The property (3.30) follows from the definition of h_1 and the fact that $Im(h_2(X + iY)) \to \pm\infty$ when $Y \to \pm\infty$.

If $\Phi_{i,\epsilon}$, $i = 1, 2$, satisfy (3.29), let $\chi_{1,\epsilon} = \Phi_{2,\epsilon} \circ (\Phi_{1,\epsilon})^{-1}$ and $\chi_{2,\epsilon} = \Phi_{1,\epsilon} \circ (\Phi_{2,\epsilon})^{-1}$. Both $\chi_{j,\epsilon}$ satisfy $\chi_{j,\epsilon}(W + 1) = \chi_{j,\epsilon}(W) + 1$ and this relation allows to extend them to global functions on $\mathbb{C}$. Moreover $\chi_{2,\epsilon} \circ \chi_{1,\epsilon}$ is the identity over a strip of width 1, hence everywhere on $\mathbb{C}$. It follows that $\Phi_{2,\epsilon} \circ \Phi_{1,\epsilon}^{-1}$ is a translation. $\qquad\square$

Definition 3.10 A function Φ_ϵ constructed in Theorem 3.9 is called a *Fatou coordinate associated with the translation domain Q_ϵ*. The base point of a Fatou coordinate is the point $Z_0(\epsilon) = \Phi_\epsilon^{-1}(0)$.

3.5 Dependence on the parameters of Fatou coordinates

Theorem 3.11 *Let g_ϵ be a prepared family (3.1) and G_ϵ the lifted unfolding. Let $\delta > 0$ and r_0, ρ_0 be given sufficiently small and let r, ρ be chosen so that the conclusions of Proposition 3.1 hold. Let Q_ϵ be a family of translation domains for ϵ in one of the sectors $V_{\delta,\pm}$ constructed by means of an admissible line depending continuously on ϵ.*

1. *The family $(Q_\epsilon)_{\epsilon \in V_{\delta,\pm}}$ is continuous in the following sense. Let us consider*

$$Q_\pm = \cup_{\epsilon \in V_{\delta,\pm}} (\{\epsilon\} \times Q_\epsilon) \subset \mathbb{C}^2 \qquad (3.39)$$

   Then $Q_\pm$ is an open subset of $V_{\delta,\pm} \times \mathbb{C}$. Moreover $\cap_{\epsilon \in V_{\delta,\pm}} Q_\epsilon \neq \emptyset$.

2. *Let $Z_0(\epsilon) \in Q_\epsilon$ depend holomorphically on ϵ (including at $\epsilon = 0$) and let Φ_ϵ be the Fatou coordinate defined on Q_ϵ for $\epsilon \in V_{\delta,\pm}$ and normalized by $\Phi_\epsilon(Z_0(\epsilon)) = 0$.*

   Let $\Phi_\pm : Q_\pm \to \mathbb{C}$ defined by $\Phi_\pm(\epsilon, Z) = \Phi_\epsilon(Z)$. The function $\Phi_\pm$ is holomorphic in $Int(Q)$ (i.e. for $\epsilon \neq 0$), and continuous in Q.

PROOF. The proof is as in [14]. $\qquad\square$

Definition 3.12 The function $\Phi_\pm : Q_\pm \to \mathbb{C}$ of Theorem 3.11 is called a *global Fatou coordinate associated to the sector $V_{\delta,\pm}$*.

4 The modulus of analytic classification

Before defining the modulus it is necessary to understand better the geometry of the domain of definition of (3.7) and (3.10). When $\epsilon = 0$, S_0 is the standard Riemann surface of the function $Z^{1/q}$ obtained by glueing together q sheets along cuts from 0 to ∞. Fatou coordinates are defined on translation domains which belong to the complement of a q-sheeted neighborhood of 0. If we consider an admissible line located in a sheet on one side of the hole and the translation domain it generates, then for $q \geq 2$ this domain intersects exactly two translation domains associated to admissible lines located on the other side of the hole B_0 (see Figure 4). Moreover each of the two intersections is simply connected yielding that a comparison of the two Fatou coordinates is possible only in a domain containing a half-plane. When $\epsilon \neq 0$ we have a similar picture but repeated at each of the holes. Remember that the whole surface looks like Figure 1.

So, for the sector $V_{\delta,+}$ (resp. $V_{\delta,-}$), we consider $2q$ global Fatou coordinates $\Phi_{j,+}^\pm$ (resp. $\Phi_{j,-}^\pm$) generated by admissible lines $\ell_{j,+}^\pm(\epsilon)$ (resp. $\ell_{j,-}^\pm(\epsilon)$), $j = 1, \ldots q$, located respectively between B_0 and either B_1 or B_{-1} on the different sheets and generating admissible strips $\hat{C}_{j,\epsilon,+}^\pm$ (resp. $\hat{C}_{j,\epsilon,-}^\pm$). The lines are chosen so that no positive iterates of points of a fundamental domain generated by a line $\ell_{j,\pm}^-$ ever enters a fundamental domain generated by a line $\ell_{j,\pm}^+$. So lines $\ell_{j,-}^-$ and $\ell_{j,+}^+$ (resp. $\ell_{j,+}^-$ and $\ell_{j,-}^+$) pass through B_0 and B_{-1} (resp. B_0 and B_1). (For the index j we work (mod q).) They generate translation domains $Q_{j,\epsilon,\pm}^\pm$. Their indices are chosen so that the translation domains of $\ell_{j,\pm}^+(\epsilon)$ and $\ell_{j,\pm}^-(\epsilon)$ (resp. $\ell_{j+1,\pm}^+(\epsilon)$ and $\ell_{j,\pm}^-(\epsilon)$) intersect and
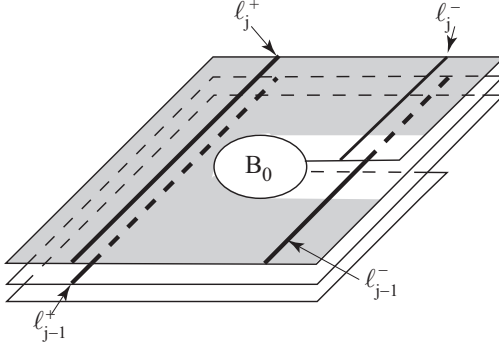
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Figure 4: Four admissible lines and one translation domain (here $q = 3$)

contain an "upper domain" (resp. "lower domain"), i.e. a domain whose intersection with $\hat{C}^{\pm}_{j,\epsilon,\pm}$ contains an upper end (resp. lower end) of the cylinder $\hat{C}^{\pm}_{j,\epsilon,\pm}/G_\epsilon$. We give ourselves $2q$ base points $Z^{\pm}_{j,\pm}(\epsilon)$ located in the different translation domains and depending analytically on ϵ. This gives us, for each sector $V_{\delta,\pm}$, $2q$ global Fatou coordinates $\Phi^{\pm}_{j,\epsilon,\pm}$ associated to each of the $2q$ admissible lines $\ell^{\pm}_{j,\pm}(\epsilon)$ and base points $Z^{\pm}_{j,\pm}(\epsilon)$, $j = 1, \ldots, q$.

We define

$$\begin{cases} \Psi^{\infty}_{j,\epsilon,\pm} = \Phi^{-}_{j,\epsilon,\pm} \circ (\Phi^{+}_{j,\epsilon,\pm})^{-1} \\ \Psi^{0}_{j,\epsilon,\pm} = \Phi^{-}_{j,\epsilon,\pm} \circ (\Phi^{+}_{j+1,\epsilon,\pm})^{-1}, \end{cases} \tag{4.1}$$

$j = 1, \ldots, q$, where we identify $\Phi^{+}_{q+1,\epsilon,\pm} = \Phi^{+}_{1,\epsilon,\pm}$.

Whenever possible we will drop the lower indices $\pm$ referring to the sectors.

Proposition 4.1 *Here we drop the lower indices $\pm$ in the $\Psi^{0,\infty}_{j,\epsilon,\pm}$.*

1. *Each map $\Psi^{0,\infty}_{j,\epsilon}$ commutes with the translation by 1: $\Psi^{0,\infty}_{j,\epsilon} \circ T_1 = T_1 \circ \Psi^{0,\infty}_{j,\epsilon}$. Hence $\Psi^{\infty}_{j,\epsilon}$ (resp. $\Psi^{0}_{j,\epsilon}$) induces a mapping $\hat{\Psi}^{\infty}_{j,\epsilon}$ (resp. $\hat{\Psi}^{0}_{j,\epsilon}$) defined on an open set of $\Phi^{+}_{j,\epsilon}(Q^{+}_{j,\epsilon} \cap Q^{-}_{j,\epsilon})/\mathbb{Z}$ (resp. $\Phi^{+}_{j+1,\epsilon}(Q^{-}_{j,\epsilon} \cap Q^{+}_{j+1,\epsilon})/\mathbb{Z}$) of the cylinder $\mathbb{C}/\mathbb{Z}$ with values in $\mathbb{C}/\mathbb{Z}$.*

2. *Using the exponential function $W \mapsto w = E(W) = \exp(-2i\pi W)$, we can identify $\mathbb{C}/\mathbb{Z}$ with the sphere minus two points : $\mathbb{CP}^1 \setminus \{0, \infty\}$. The upper end of the cylinder $\mathbb{C}/\mathbb{Z}$, corresponds to $\infty \in \mathbb{CP}^1$ and the lower end to 0. Conjugating $\Psi^{0}_{j,\epsilon}$ (resp. $\Psi^{\infty}_{j,\epsilon}$) with this map yields an analytic map $\psi^{0}_{j,\epsilon}$ (resp. $\psi^{\infty}_{j,\epsilon}$) defined in the neighborhood of 0 (resp. ∞) on $\mathbb{CP}^1$:*

$$\psi^{0,\infty}_{j,\epsilon}(w) = \exp\left(-2i\pi \Psi^{0,\infty}_{j,\epsilon}\left(-\frac{1}{2i\pi}\log(w)\right)\right), \tag{4.2}$$

where

$$\psi^{0}_{j,\epsilon}(0) = 0, \quad \psi^{\infty}_{j,\epsilon}(\infty) = \infty. \tag{4.3}$$

3. *The functions $\psi^{0,\infty}_{j,\epsilon,\pm}$ depend analytically on $\epsilon \neq 0$ in $V_{\delta,\pm}$ and are continuous in ϵ at $\epsilon = 0$.*

PROOF. The proof is identical to that made in [14]. □

We now need to exploit that $g_\epsilon = f^n_\epsilon$. This will yield that only $\Psi^{0,\infty}_{1,\epsilon}$ are independent and the other $\Psi^{0,\infty}_{j,\epsilon}$, $j > 1$ are conjugate to them by translations.

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(a) $\epsilon = 0$ (b) $\epsilon \in V_{\delta,+}$, $\arg \epsilon = \pi/2$ (c) $\epsilon \in V_{\delta,-}$, $\arg \epsilon = -\pi/2$
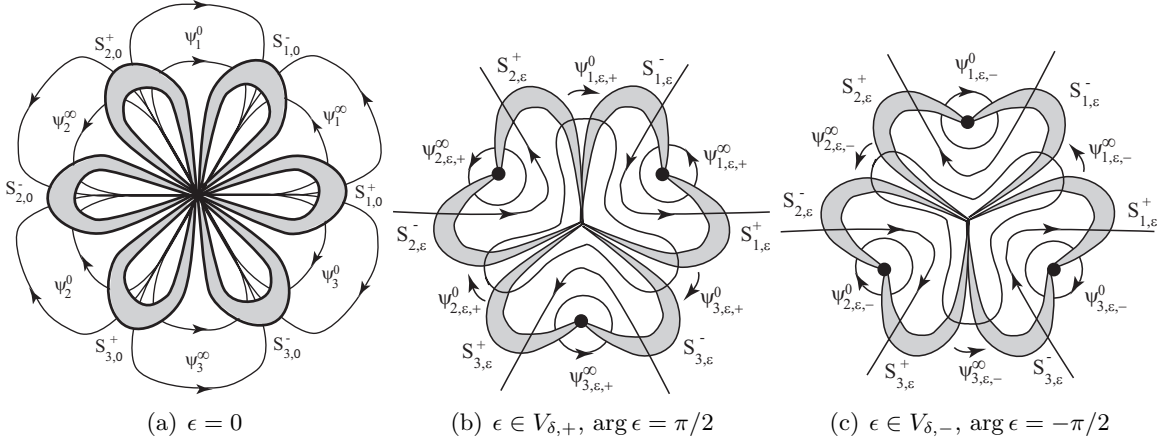
Figure 5: The maps $\psi_{j,\epsilon}$ for different values of ϵ

Normalized set of Fatou coordinates. Fatou coordinates are uniquely determined by the base points. By Proposition 4.1 (1) the maps $\Psi_{j,\epsilon,\pm}^{0,\infty}(W) - W$ can be expanded as Fourier series with constant terms $A_{j,\epsilon,\pm}^{0,\infty}$. It is possible to choose the base points of the Fatou coordinates in Theorem 3.9 (2) so that all $A_{j,\epsilon,\pm}^{0,\infty} = A_{\epsilon}^{0,\infty}$ for some constants $A_{\epsilon}^{0,\infty}$ independent of j and of the sector $V_{\delta,+}$ or $V_{\delta,-}$ and such that $A_{\epsilon}^0 = -A_{\epsilon}^\infty$ (see for instance details in [10] for the case $\epsilon = 0$ and also Lemma 4.2 below). A set of Fatou coordinates $\Phi_{j,\epsilon}^{\pm}$, $j = 1, \ldots q$, such that the corresponding transition maps $\Psi_{j,\epsilon}^{0,\infty}$, $j = 1, \ldots, q$, have this property, is called a *normalized set of Fatou coordinates*.

Lemma 4.2 *We consider a normalized set of Fatou coordinates. Then $A_{\epsilon}^0 = -A_{\epsilon}^\infty = \pi i a/q$ and the derivatives of $\psi_{j,\epsilon}^{0,\infty}$ satisfy*

$$\begin{cases} (\psi_{j,\epsilon}^0)'(0) = \exp(2\pi^2 a/q) \\ (\psi_{j,\epsilon}^\infty)'(\infty) = \exp(2\pi^2 a/q). \end{cases} \tag{4.4}$$

PROOF. The proof of the lemma will follow from the proof of (2) in Proposition 5.2. $\square$

Proposition 4.3 *We consider a map g_ϵ as in (3.1), being the q-th iterate of a map f_ϵ as in (2.2), the corresponding lifted diffeomorphism G_ϵ and a normalized set of Fatou coordinates on either $V_{\delta,+}$ or $V_{\delta,-}$.*

1. *Let σ defined by $\sigma(j) = j + p \pmod{q}$ be the shift which represents the iterates of $\exp(2\pi i/q)$ under multiplication by $\exp(2\pi i p/q)$. Then*

$$\Psi_{\sigma(j),\epsilon}^{0,\infty}(W) = \Psi_{j,\epsilon}^{0,\infty}(W - 1/q) + 1/q. \tag{4.5}$$

2.

$$\psi_{\sigma(j),\epsilon}^{0,\infty}(w) = \exp\left(-\frac{2\pi i}{q}\right) \psi_{j,\epsilon}^{0,\infty}\left(\exp\left(\frac{2\pi i}{q}\right) w\right). \tag{4.6}$$

3. *Once $\Phi_{1,\epsilon}^{\pm}$ is chosen the other Fatou coordinates can be taken such that*

$$\Phi_{\sigma(j),\epsilon}^{\pm} \circ F_\epsilon = T_{\frac{1}{q}} \circ \Phi_{j,\epsilon}^{\pm}. \tag{4.7}$$

PROOF. The map f_ϵ commutes with $g_\epsilon = f_\epsilon^q$. Hence $F_\epsilon = p_\epsilon^{-1} \circ f_\epsilon \circ p_\epsilon$ commutes with G_ϵ. We deduce that:

$$\Phi_{\sigma(j),\epsilon}^{\pm}(G_\epsilon(F_\epsilon(Z))) = \Phi_{\sigma(j),\epsilon}^{\pm}(F_\epsilon(Z)) + 1 = (\Phi_{\sigma(j),\epsilon}^{\pm} \circ F_\epsilon)(G_\epsilon(Z)). \tag{4.8}$$
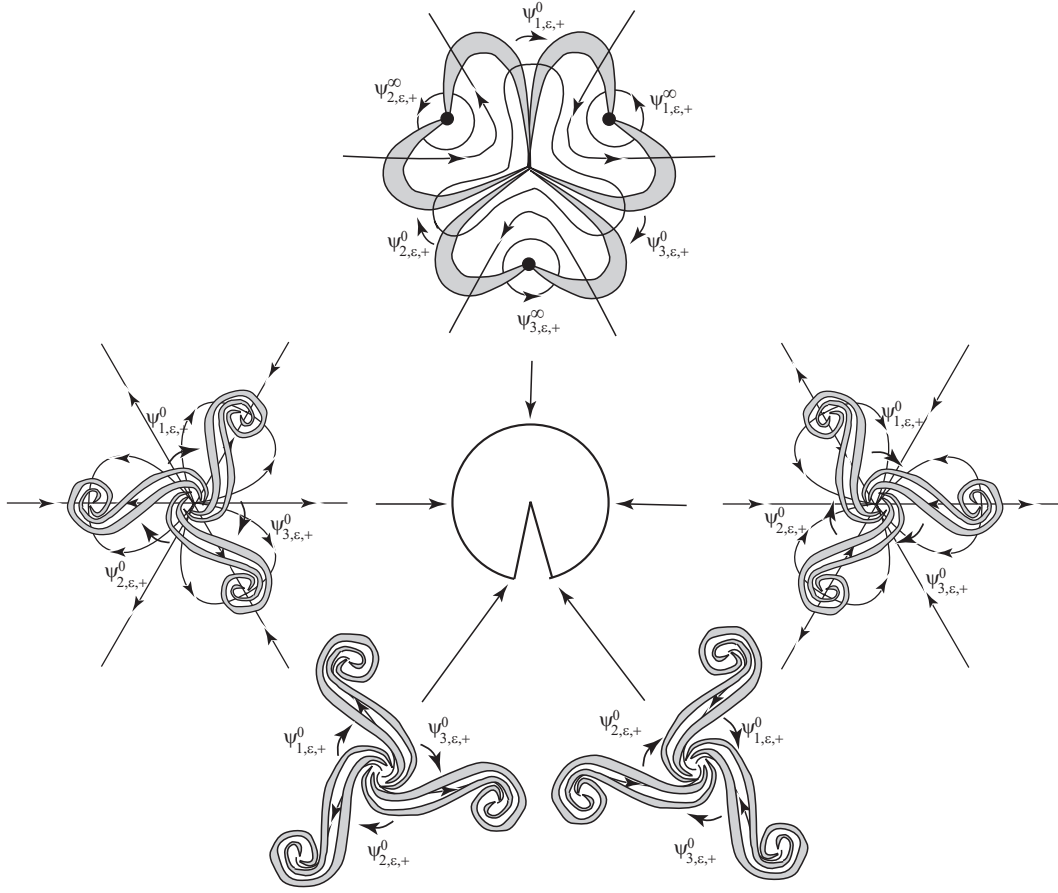
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Figure 6: The crescents and maps $\psi_{j,\epsilon,+}^{0,\infty}$ for $\epsilon \in V_{\delta,+}$

Then $\Phi_{\sigma(j),\epsilon}^{\pm} \circ F_\epsilon$ is a Fatou coordinate on a translation domain containing $\ell_j^{\pm}(\epsilon)$. Hence there exists $C_{\epsilon,j}^{\pm}$ such that

$$\Phi_{\sigma(j),\epsilon}^{\pm} \circ F_\epsilon = T_{C_{\epsilon,j}^{\pm}} \circ \Phi_{j,\epsilon}^{\pm}. \tag{4.9}$$

Using (4.1) we get

$$\begin{cases} \Psi_{\sigma(j),\epsilon}^{\infty}(W) = \Psi_{j,\epsilon}^{\infty}(W - C_{\epsilon,j}^{+}) + C_{\epsilon,j}^{-} \\ \Psi_{\sigma(j),\epsilon}^{0}(W) = \Psi_{j,\epsilon}^{0}(W - C_{\epsilon,j+1}^{+}) + C_{\epsilon,j}^{-}. \end{cases} \tag{4.10}$$

As the set of Fatou coordinates are normalized we get $C_{j,\epsilon}^{\pm} = C_\epsilon$.

Moreover, using $F_\epsilon^q = G_\epsilon$ we get $\Phi_{\sigma^q(j),\epsilon}^{\pm} \circ G_\epsilon(W) = \Phi_{j,\epsilon}^{\pm}(W) + qC_\epsilon = \Phi_{\sigma^q(j),\epsilon}^{\pm}(W) + 1$. As $\sigma^q(j) = j$ we get $C_\epsilon = 1/q$. □

Definition 4.4 Let Diff^0 (resp. Diff^∞) be the set of germs of diffeomorphisms of $\mathbb{CP}^1$ defined in the neighborhood of 0 (resp. ∞) and fixing 0 (resp. ∞).

1. We consider the set of 2-tuples of diffeomorphisms $(\psi_1^\infty, \psi_1^0) \in (\mathrm{Diff}^\infty \times \mathrm{Diff}^0)$ having equal first derivatives at their distinguished point:

$$(\psi_1^0)'(0) = (\psi_1^\infty)'(\infty).$$

We define an equivalence relation on it by

$$(\psi_1^\infty, \psi_1^0) \sim (\overline{\psi}_1^\infty, \overline{\psi}_1^0) \iff \exists c \in \mathbb{C}^* \quad \overline{\psi}_1^{0,\infty}(w) = c^{-1}\psi_1^{0,\infty}(cw). \tag{4.11}$$

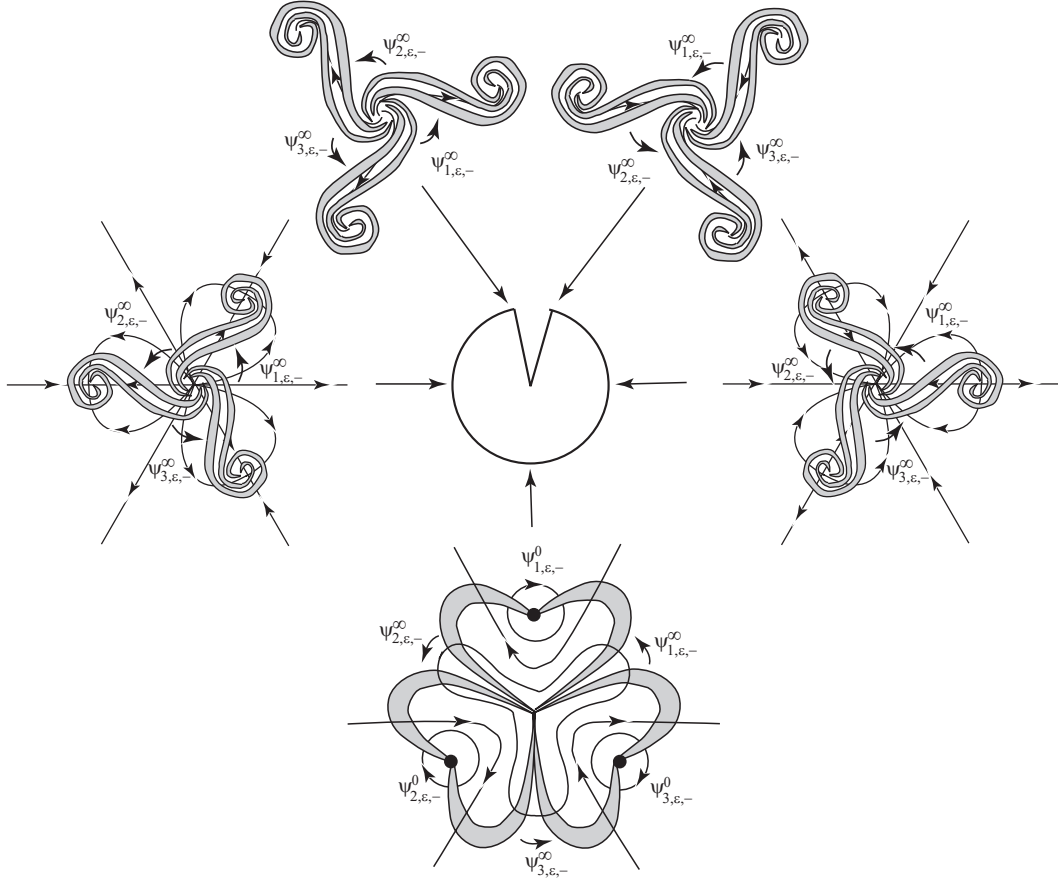Let $\mathcal{M}$ be the quotient space.

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Figure 7: The crescents and maps $\psi_{j,\epsilon,-}^{0,\infty}$ for $\epsilon \in V_{\delta,-}$

2. We identify $\mathcal{M}$ with the set of $2q$-tuples of diffeomorphims $\psi = (\psi_1^\infty, \psi_1^0, \psi_2^\infty, \psi_2^0, \dots \psi_q^\infty, \psi_q^0)$ where the $\psi_j^{\infty,0}$ satisfy

$$\psi_{\sigma(j)}^{\infty,0}(w) = \exp\left(-\frac{2\pi i}{q}\right)\psi_j^{\infty,0}\left(\exp\left(\frac{2\pi i}{q}\right)w\right). \tag{4.12}$$

In particular all $\psi_j^{\infty,0}$ have the same derivative at their respective distinguished point.

Corollary 4.5 *For a prepared family $\mathcal{G} = \{g_\epsilon\}_{\epsilon \in V}$ with $V = V_{\delta,+} \cup V_{\delta,-}$ of the form* (3.1) *we have two applications*

$$m_{\mathcal{G},\pm} : V_{\delta,\pm} \to \mathcal{M}, \qquad \epsilon \mapsto m_{\mathcal{G},\pm}(\epsilon), \tag{4.13}$$

where $m_{\mathcal{G},\pm}(\epsilon)$ is the equivalence class of $(\psi_{1,\epsilon,\pm}^\infty, \psi_{1,\epsilon,\pm}^0)$ (which is the same as that of $\psi_{\epsilon,\pm} = (\psi_{1,\epsilon,\pm}^\infty, \psi_{1,\epsilon,\pm}^0, \dots, \psi_{q,\epsilon,\pm}^\infty, \psi_{q,\epsilon,\pm}^0)$. This equivalence class depends only on $\mathcal{G}$ and not of the choice of the base points.

Remark 4.6 The two maps $m_{\mathcal{G},\pm}$ do not coincide on the intersection $V_{\delta,+} \cap V_{\delta,-}$. Indeed for $\epsilon \in V_{\delta,+}$ (resp. $\epsilon \in V_{\delta,+}$) the point 0 and ∞ of the spheres correspond respectively to the fixed point $z_0 = 0$ of g_ϵ and the fixed points $z_1, \dots, z_q$ of g_ϵ (resp. the fixed points $z_1, \dots, z_q$ of g_ϵ and the fixed point $z_0 = 0$ of g_ϵ). A neighborhood of them on the spheres corresponds to a sectorial neighborhood of the corresponding fixed points of g_ϵ.

Definition 4.7 Two germs of analytic families f_ϵ and $\overline{f}_{\overline{\epsilon}}$ of diffeomorphisms with a fixed point at the origin are *conjugate* if there exists a germ of analytic diffeomorphism $H(\epsilon, z) = (k(\epsilon), h(\epsilon, z))$ fibered over the parameter space such that

$$h_\epsilon \circ f_\epsilon = \overline{f}_{k(\epsilon)} \circ h_\epsilon \tag{4.14}$$

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where $h_\epsilon(z) = h(\epsilon, z)$.

Theorem 4.8 *We consider two prepared families $\mathcal{F} = \{f_\epsilon\}$ and $\overline{\mathcal{F}} = \{\overline{f}_\epsilon\}$ of the form (2.2) and the families of their q-th iterates $\mathcal{G} = \{g_\epsilon\}$ and $\overline{\mathcal{G}} = \{\overline{g}_\epsilon\}$ of the form (3.1). We choose common sectors $V_{\delta,\pm}$ on which the previous analysis applies. Then the two families are conjugate if and only if $m_{\mathcal{G},\pm} = m_{\overline{\mathcal{G}},\pm}$.*

PROOF. It is clear that two anlytically conjugate families g_ϵ and $\overline{g}_\epsilon$ have the same invariant. Indeed in Proposition 2.1 we showed that the canonical parameter is an analytic invariant, so the conjugacy is over the identity and it suffices to compare the two families for a given $\hat{\epsilon} \in V^\pm$ corresponding to some $\epsilon \in V$. >From a conjugacy between g_ϵ and $\overline{g}_\epsilon$ we construct an equivalence between the Fatou coordinates, etc., which will yield equality of the moduli. We postpone the proof of the converse since it uses the notion of Lavaurs phase which will be discussed in the next section.

Theorem 4.8 can be generalized to families not in prepared form.

Definition 4.9 The 2-tuple $m_{\mathcal{G}} = (m_{\mathcal{G},+}, m_{\mathcal{G},-})$ of Corollary 4.5 is called the *modulus* of the prepared family $\mathcal{F}$ (and of the prepared family $\mathcal{G}$).

Theorem 4.10 *To any 1-parameter analytic family $\mathcal{H} = \{h_\eta\}$ which is a generic unfolding of a generic resonant fixed point (i.e. $\frac{\partial^2 h_\eta}{\partial z \partial \eta} \neq 0$) we associate a prepared analytic family $\mathcal{F} = \{f_\epsilon\}$ and its q-th iterate $\mathcal{G} = \{g_\epsilon\}$. We call $m_{\mathcal{H}}$ the modulus $m_{\mathcal{G}}$ of $\mathcal{G}$. Then*

1. *$m_{\mathcal{H}}$ is well defined.*

2. *Two families $\mathcal{H}$ and $\overline{\mathcal{H}}$ are analytically conjugate if and only if they have the same formal invariant $a(0)$ for $\epsilon = 0$ and $m_{\mathcal{H}} = m_{\overline{\mathcal{H}}}$. Hence $a(0)$ together with $m_{\mathcal{H}}$ is a complete invariant of analytic classification under conjugacy.*

PROOF. The passage from a family to its prepared form is analytic in the parameter. Moreover the parameter of the prepared family is uniquely defined and canonical as it is an analytic invariant. The conjugacy between two prepared families is constructed as in Theorem 4.8. Composing it with the changes of coordinate and parameter bringing the families to their prepared forms yields a conjugacy between the initial families. $\square$

4.1 The Lavaurs phase

Definition 4.11   1. For $V_{\delta,+}$, the q Lavaurs translations are the maps

$$L_{j,\epsilon,+} = \Phi_{j,\epsilon,+}^+ \circ T_{-q\alpha(\epsilon)} \circ (\Phi_{j,\epsilon,+}^-)^{-1} : Q_{j,+}^- \to Q_{j,+}^+. \tag{4.15}$$

2. For $V_{\delta,-}$, the q Lavaurs translations are the maps

$$L_{j,\epsilon,-} = \Phi_{j+1,\epsilon,-}^+ \circ T_{-q\alpha(\epsilon)} \circ (\Phi_{j,\epsilon,-}^-)^{-1} : Q_{j,-}^- \to Q_{j+1,-}^+. \tag{4.16}$$

Proposition 4.12 *The maps $L_{j,\epsilon,\pm}$ commute with $W \mapsto W + 1$. Hence they induce automorphisms of $\mathbb{C}/\mathbb{Z}$. By conjugating with $W \mapsto w = E(W) = \exp(-2i\pi W)$ they yield diffeomorphisms of $\mathbb{CP}^1$ preserving 0 and ∞, hence linear maps $l_{j,\epsilon,\pm}$ defined by*

$$l_{j,\epsilon,\pm}(w) = \nu_{j,\pm}(\epsilon)w. \tag{4.17}$$

So the map $L_{j,\epsilon,\pm}$ is a translation $W \mapsto W + \tau_{j,\pm}(\epsilon)$ where

$$\nu_{j,\pm}(\epsilon) = \exp(-2i\pi \, \tau_{j,\pm}(\epsilon)). \tag{4.18}$$

The $\nu_{j,\pm}$ depend only of a and ϵ when we have a normalized set of Fatou coordinates and are calculated below in (5.6).

PROOF. The proof is completely straightforward. See for instance [14]. $\square$

4.2 Proof of Theorems 4.8 and 4.10

END OF PROOF OF THEOREM 4.8. Here we prove the converse, i.e. two prepared families with same modulus are analytically conjugate. The proof is in three steps. Considering two prepared families $\mathcal{G} = \{g_\epsilon = f_\epsilon^q\}$ and $\overline{\mathcal{G}} = \{\overline{g}_\epsilon = \overline{f}_\epsilon^q\}$ which have the same modulus (and hence the same canonical parameter ϵ) we first construct for each ϵ a conjugacy between g_ϵ and $\overline{g}_\epsilon$. Using our global Fatou coordinates this will in fact yield conjugacies $h_{\epsilon,\pm}$ for $\epsilon \in V_{\delta,\pm}$, each depending analytically on $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$. We have shown that it yields a conjugacy between f_ϵ and $\overline{f}_\epsilon$. We finally show that it is possible to construct a conjugacy which depends analytically of ϵ. This will be shown in Theorem 4.15 below.

Because the families have the same modulus we can consider that on each sector $V_{\delta,\pm}$ they have a common set of translations domains $Q_{j,\epsilon,\pm}^\pm$ and the same transition functions $\Psi_{j,\pm}^{0,\infty} = \overline{\Psi}_{j,\pm}^{0,\infty}$ defined on the corresponding intersections. We first consider a conjugacy $H_{\epsilon,\pm}$ defined on the union of the translation domains $Q_\pm = \cup_{j=1}^q (Q_{j,\epsilon,\pm}^+ \cup Q_{j,\epsilon,\pm}^-)$. Then we check that the composition $h_{\epsilon,\pm} = p_\epsilon \circ H_{\epsilon,\pm} \circ p_\epsilon^{-1}$ yields a conjugacy over U. The map $H_{\epsilon,\pm}$ is defined as

$$H_{\epsilon,\pm}(Z) = \begin{cases} (\overline{\Phi}_{j,\epsilon,\pm}^+)^{-1} \circ \Phi_{j,\epsilon,\pm}^+ & Z \in Q_{j,\epsilon,\pm}^+ \\ (\overline{\Phi}_{j,\epsilon,\pm}^-)^{-1} \circ \Phi_{j,\epsilon,\pm}^- & Z \in Q_{j,\epsilon,\pm}^-. \end{cases} \tag{4.19}$$

The map $H_{\epsilon,\pm}$ is well defined because the two families have the same modulus. We need to show that $h_{\epsilon,\pm} = p_\epsilon \circ H_{\epsilon,\pm} \circ p_\epsilon^{-1}$ is well defined over U. For this we need to show that $H_{\epsilon,\pm}$ commutes with the translation $T_{q\alpha(\epsilon)}$, where α is given in (3.6). For the rest of the proof we do not write the dependence in ϵ. We write the proof for $V_{\delta,+}$ and do not write the index referring to it in the functions $\Phi_j^\pm$, $\overline{\Phi}_j^\pm$ and $\Psi_j^{0,\infty}$.

$$\begin{cases} \Psi_j^\infty \circ L_{j,+} = \Phi_j^- \circ T_{-q\alpha} \circ (\Phi_j^-)^{-1} \\ \overline{\Psi}_j^\infty \circ \overline{L}_{j,+} = \overline{\Phi}_j^- \circ T_{-q\alpha} \circ (\overline{\Phi}_j^-)^{-1}. \end{cases} \tag{4.20}$$

We will show in Proposition 5.2 below that $L_{j,\epsilon,\pm}$ depends only on the modulus, and hence that $L_{j,\epsilon,\pm} = \overline{L}_{j,\epsilon,\pm}$. So

$$\begin{aligned} (\overline{\Phi}_j^-)^{-1} \circ \Phi_j^- \circ T_{-q\alpha} &= (\overline{\Phi}_j^-)^{-1} \circ \Psi_j^\infty \circ L_{j,+} \circ \Phi_j^- \\ &= (\overline{\Phi}_j^-)^{-1} \circ \overline{\Psi}_j^\infty \circ \overline{L}_{j,+} \circ \Phi_j^- \\ &= (\overline{\Phi}_j^-)^{-1} \circ \overline{\Phi}_j^- \circ T_{-q\alpha} \circ (\overline{\Phi}_j^-)^{-1} \circ \Phi_j^- \\ &= T_{q-\alpha} \circ (\overline{\Phi}_j^-)^{-1} \circ \Phi_j^-. \end{aligned} \tag{4.21}$$

Similarly, using that

$$L_{j,+} \circ \Psi_j^0 = \Phi_j^+ \circ T_{-q\alpha} \circ (\Phi_{j+1}^+)^{-1}, \tag{4.22}$$

we get

$$\begin{aligned} (\overline{\Phi}_j^+)^{-1} \circ \Phi_j^+ \circ T_{-q\alpha} &= (\overline{\Phi}_j^+)^{-1} \circ L_{j,+} \circ \Psi_j^0 \circ \Phi_{j+1}^+ \\ &= (\overline{\Phi}_j^+)^{-1} \circ \overline{L}_{j,+} \circ \overline{\Psi}_j^0 \circ \Phi_{j+1}^+ \\ &= T_{-q\alpha} \circ (\overline{\Phi}_{j+1}^+)^{-1} \circ \Phi_{j+1}^+ \\ &= T_{-q\alpha} \circ (\overline{\Phi}_j^+)^{-1} \circ \Phi_j^+. \end{aligned} \tag{4.23}$$

The last line follows as $H_{\epsilon,\pm}$ is well defined. The diffeomorphism $H_{\epsilon,\pm}$ induces an analytic equivalence h_ϵ between the two diffeomorphims g_ϵ and $\overline{g}_\epsilon$ except at the fixed points. Since the equivalence is bounded it can be extended at the fixed points. The domain of h_ϵ contains a ball of radius r independent of ϵ.

The proof on $V_{\delta,-}$ is analogous.

The last step of the proof is to show that it is possible to choose h_ϵ depending analytically on ϵ. This will be done in Theorem 4.15 below. $\qquad\square$

Lemma 4.13 *Let $g_\epsilon = f_\epsilon^q$ and $\overline{g}_\epsilon = \overline{f}_\epsilon^q$. If g_ϵ and $\overline{g}_\epsilon$ are conjugate then f_ϵ and $\overline{f}_\epsilon$ are conjugate.*

PROOF. Here again we drop the lower indices referring to the sectors $V_{\delta,\pm}$. Let F_ϵ and $\overline{F}_\epsilon$ be the lifts of f_ϵ and $\overline{f}_\epsilon$. >From (4.7) we get

$$
\begin{aligned}
F_\epsilon &= (\Phi_{\sigma(j)}^{\pm})^{-1} \circ T_{1/q} \circ \Phi_j^{\pm} \\
\overline{F}_\epsilon &= (\overline{\Phi}_{\sigma(j)}^{\pm})^{-1} \circ T_{1/q} \circ \overline{\Phi}_j^{\pm}.
\end{aligned}
\tag{4.24}
$$

The map H_ϵ defined in (4.19) which is a conjugacy between G_ϵ and $\overline{G}_\epsilon$ clearly induces a conjugacy between F_ϵ and $\overline{F}_\epsilon$. H_ϵ induces a map h_ϵ which conjugates f_ϵ and $\overline{f}_\epsilon$. $\qquad\square$

Remark 4.14 Because of the analytic character of the maps $\psi_{j,\epsilon,\pm}^{\infty,0}$, in order to show that two families are analytically equivalent it suffices to show that they have the same $m_{\mathcal{G},+}(\epsilon)$ or the same $m_{\mathcal{G},-}(\epsilon)$, or even the same $m_{\mathcal{G},\pm}(\epsilon)$ for ϵ in an open subsector of $V_{\delta,\pm}$ with vertex at the origin.

4.3 Symmetries of families unfolding a resonant fixed point

In this section we discuss briefly the symmetries of prepared families g_ϵ of the form (3.1) as a tool to prove:

Theorem 4.15 *If there exist conjugacies $h_{\epsilon,\pm}$ for $\epsilon \in V_{\delta,\pm}$ between two prepared families g_ϵ and $\overline{g}_\epsilon$ as in (3.1), depending analytically on $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$, then the two families g_ϵ and $\overline{g}_\epsilon$ are analytically conjugate.*

Definition 4.16   1. The *group of symmetries* of g_0 is the commutator of g_0 inside the group of germs of analytic diffeomorphisms tangent to the identity at the origin.

   2. Similarly, given g_ϵ defined on a neighborhood containing its fixed points, we will call *symmetry* of g_ϵ any analytic diffeomorphism on the same neighborhood which commutes with it.

The continuous symmetries of g_ϵ unfold the symmetries of g_0. So we first recall these.

Proposition 4.17 *[10] Depending on the modulus $(\psi_{1,0}^0, \psi_{1,0}^\infty)$ we get the following cases:*

(1) *If g_0 is generic, i.e. $\psi_{1,0}^0$ or $\psi_{1,0}^\infty$ does not commute with any linear map, then the symmetry group of g_0 is the group of iterates $\{g_0^n | n \in \mathbb{Z}\}$.*

(2) *If g_0 is not embedable and $m \in \mathbb{N}$ is maximum so that there exists k_0 satisfying $g_0 = k_0^m$ with k_0 tangent to the identity (i.e. $\psi_{1,0}^0(w) = w\xi_{1,0}^0(w^m)$ and $\psi_{1,0}^\infty(w) = w\xi_{1,0}^\infty(w^m)$ and one of them is nonlinear), then the symmetry group of g_0 is the group of iterates $\{k_0^n | n \in \mathbb{Z}\}$.*

(3) *If g_0 is embedable, i.e. $\psi_{1,0}^0$ and $\psi_{1,0}^\infty$ are linear and g_0 is conjugate by m_0 to the time-one map v^1 of the flow of the vector field v given in (1.5) then all symmetries of g_0 are conjugate by m_0 to the time-t maps v^t of the flow of v for $t \in \mathbb{C}$.*

Proposition 4.18 *We consider a prepared family g_ϵ unfolding g_0.*

(1) *If g_0 is generic i.e. $\psi_{1,0}^0$ or $\psi_{1,0}^\infty$ does not commute with any linear map, then, for sufficiently small ϵ, any symmetry of g_ϵ is of the form g_ϵ^n for $n \in \mathbb{Z}$. In particular if γ_ϵ is a symmetry of g_ϵ depending continuously on ϵ in a sector, and such that $\gamma_0 = id$, then $\gamma_\epsilon = id$.*

(2) *If g_0 is not embedable and $m \in \mathbb{N}$ is maximum so that there exists k_0 satisfying $g_0 = k_0^m$ with k_0 tangent to the identity (i.e. $\psi_{1,0}^0(w) = w\xi_{1,0}^0(w^m)$ and $\psi_{1,0}^\infty(w) = w\xi_{1,0}^\infty(w^m)$ and one of them is nonlinear) and if γ_ϵ is a symmetry of g_ϵ depending continuously on ϵ in a sector such that $\gamma_0 = id$, then $\gamma_\epsilon = id$.*

(3) *If g_0 is embedable, then one of the following cases occurs:*

   (a) *If γ_ϵ is a symmetry of g_ϵ depending continuously on ϵ in a sector, and such that $\gamma_0 = id$, then $\gamma_\epsilon = id$.*

(b) *For all ϵ the map g_ϵ is embedable, i.e. conjugate to the time-one map of the flow v_ϵ^1 of the vector field (1.6) under m_ϵ and its continuous symmetries are conjugate by m_ϵ to the time-$t(\epsilon)$ maps $v^{t(\epsilon)}$ of the flow of v for a continuous function $t(\epsilon)$ with values in $\mathbb{C}$. The map $t(\epsilon)$ is unique as soon as it unfolds the identity, in which case it makes sense to call the corresponding symmetry the $t(\epsilon)$-th iterate $g_\epsilon^{t(\epsilon)}$ of g_ϵ.*

PROOF. A symmetry sends orbits to orbits. For $\epsilon \neq 0$ the orbit structure is completely determined by the quotient of a sphere ($\mathbb{CP}^1$) by the return maps in the neighborhood of 0 and ∞. So a symmetry is given by a diffeomorphism of the sphere preserving 0 and ∞ (i.e. a linear map) which commutes with the return maps.

(1) This case occurs as soon as one of $\psi_{1,0}^0$ and $\psi_{1,0}^\infty$ is nonlinear and both are not of the form $\psi_{1,0}^0(w) = w\xi_{1,0}^0(w^m)$ and $\psi_{1,0}^\infty(w) = w\xi_{1,0}^\infty(w^m)$ for some $m > 1$. This can be seen on a finite jet. (Indeed if $\psi_{1,0}^0(w) = \sum_{i=1}^\infty a_i w^i$ and $\psi_{1,0}^\infty(w) = \sum_{i=1}^\infty b_i w^i$ this occurs as soon as there exists $m, n > 1$ with $(m, n) = 1$ such that $a_n \neq 0$ or $b_n \neq 0$ and simultaneously $a_m \neq 0$ or $b_m \neq 0$.) Then the same property is true for $\psi_{1,\epsilon}^0$ and $\psi_{1,\epsilon}^\infty$ for ϵ sufficiently small. So all symmetries γ_ϵ of g_ϵ are of the form g_ϵ^n with $n \in \mathbb{Z}$. If a family γ_ϵ depends continuously on ϵ then n needs to be constant and $n = 0$ is the only possibility if we add the condition that $\lim_{\epsilon \to 0} h_\epsilon = id$.

(2) is similar. Note that the discrete symmetries may or may not be preserved in the unfolded family. Continuous families of symmetries will be given by some κ_ϵ^n for a fixed $n \in \mathbb{Z}$ where $\kappa_\epsilon^d = g_\epsilon$, $d|m$ and κ_ϵ is continuous in ϵ.

(3)

(a) The first case occurs as soon as one of $\psi_{1,\epsilon}^0$ or $\psi_{1,\epsilon}^\infty$ is nonlinear. Indeed suppose that $\psi_{1,\epsilon}^0(w) = a_1(\epsilon)w + a_s(\epsilon)w^s + o(w^s)$ with $a_s \not\equiv 0$. As $a_s(\epsilon)$ depends analytically on $\epsilon \neq 0$ it is nonzero on an open dense subset on which we can apply the same argument as in (1) or (2) since the only possible symmetries are discrete.

(b) Let us look at an individual symmetry H_ϵ of v_ϵ, given by the time-$t(\epsilon)$ map of its flow. Then $H_\epsilon'(0) = \exp(-\epsilon t(\epsilon))$ and $H_\epsilon'(z_i) = \exp\left(\frac{q\epsilon t(\epsilon)}{1+a\epsilon}\right)$. Different times $t(\epsilon)$ and $\tau(\epsilon)$ yield the same symmetry H_ϵ if and only if there exists $k, k' \in \mathbb{Z}$ such that $T(\epsilon) = t(\epsilon) - \tau(\epsilon) = -\frac{2k\pi i}{\epsilon} = \frac{2k'\pi i(1+a\epsilon)}{q\epsilon}$. The only continuous solution $T(\epsilon)$ satisfying $T(0) = 0$ is $T \equiv 0$.

$\square$

PROOF OF THEOREM 4.15. The idea of the proof is the following: on each of the sectors $V_{\delta,\pm}$ we have constructed diffeomorphisms $h_{\epsilon,\pm}$ between g_ϵ and $\overline{g}_\epsilon$ which depend analytically on $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$. To get the conclusion it suffices to prove that it is possible to choose the $h_{\epsilon,\pm}$ so that they coincide on the intersections of $V_{\delta,\pm}$. Indeed $\gamma_\epsilon = (h_{\epsilon,-})^{-1} \circ h_{\epsilon,+}$ is a symmetry of g_ϵ. Moreover γ_ϵ depends analytically on $\epsilon \neq 0$ and has a continuous limit at $\epsilon = 0$. It is of course possible to adjust the $h_{\epsilon,\pm}$ so that $h_{0,+} = h_{0,-}$. Then $\gamma_0 = id$. In cases (1), (2) and (3)(a) of Proposition 4.18 it follows that $\gamma_\epsilon = id$.

So we only need to discuss case (3)(b). Let $\lambda_\pm(\epsilon) = h_{\epsilon,\pm}'(0)$. We define on $V_{\delta,\pm}$ the symmetries $\gamma_{\epsilon,\pm}$ of g_ϵ which have the property that $\gamma_{\epsilon,\pm}'(0) = (\lambda_\pm(\epsilon))^{-1}$. Indeed g_ϵ is conjugate by $k_{\epsilon,\pm}$ to the time one map of the flow of v_ϵ given in (1.6). Each symmetry $\gamma_{\epsilon,\pm}$ of g_ϵ is conjugate by $k_{\epsilon,\pm}$ to the time t map v_ϵ^t of the flow of v_ϵ. As $v_\epsilon^t(0) = \exp(-\epsilon t)$ we must choose $t_\pm(\epsilon) = \frac{\ln \lambda_\pm(\epsilon)}{\epsilon}$ in $\gamma_{\epsilon,\pm}$, i.e. $\gamma_{\epsilon,\pm} = g_\epsilon^{t_\pm(\epsilon)}$. We replace the maps $h_{\epsilon,\pm}$ by $\overline{h}_{\epsilon,\pm} = \gamma_{\epsilon,\pm} \circ h_{\epsilon,\pm}$. As $\overline{h}_{\epsilon,+}'(0) = \overline{h}_{\epsilon,-}'(0) = 1$ and the two maps $\overline{h}_{\epsilon,+}$ and $\overline{h}_{\epsilon,-}$ both conjugate g_ϵ with $\overline{g}_\epsilon$ then they coincide on $V_{\delta,+} \cap V_{\delta,-}$.

$\square$

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4.4 The Glutsyuk point of view

Instead of taking admissible lines as in Proposition 3.8 it is also possible to take admissible lines parallel to the lines of holes as in Figure 8 but only for values of ϵ such that $|\arg \epsilon - k\frac{\pi}{2}| > \delta$ with $k \in \mathbb{Z}$, which we call the *Glutsyuk domain*. Then the fundamental domains are tori as G_ϵ commutes with $T_{q\alpha}$ (details as in
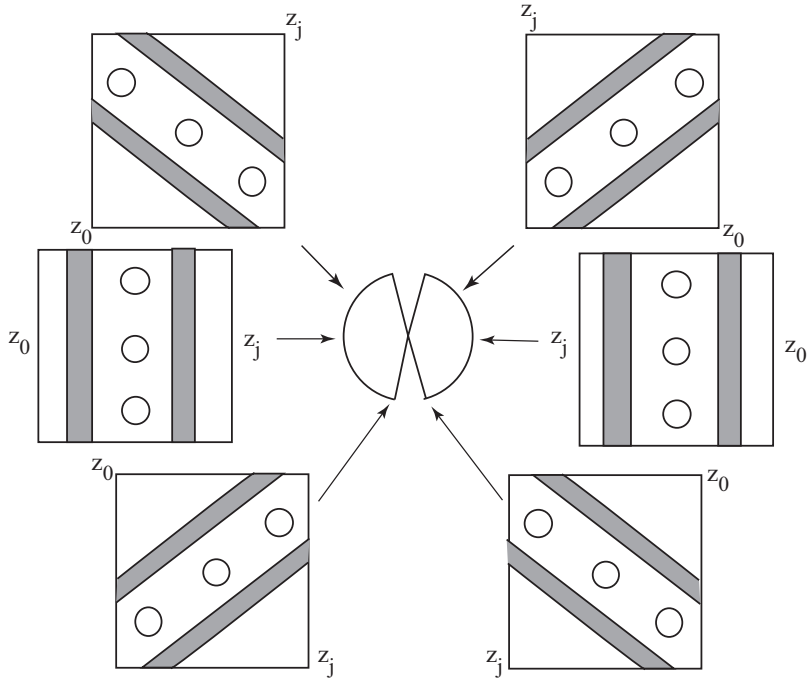

Figure 8: Continuous families of admissible lines and strips for ϵ in the Glutsyuk domain (for the sake of simplicity we have not drawn the ramification of S_ϵ at the holes)

[14]). The Fatou coordinates on the associated translation domains yield analytic changes of coordinates to the model family in the neighborhood of each of the fixed points of g_ϵ: these are named $\Phi^G_{j,\epsilon,\pm}$ for those covering a neighborhood of z_j and $\Phi^{G,j}_{0,\epsilon,\pm}$ for those covering a neighborhood of z_0 (there are q of these, one in each sheet of the covering). The lower index is $+$ (resp. $-$) if Re $\epsilon > 0$ (resp. Re $\epsilon < 0$). As in the proof of Proposition 4.3 we can show that they can be chosen so as to satisfy

$$\begin{array}{rcl}
\Phi^G_{\sigma(j),\epsilon,\pm}(F_\epsilon(Z)) & = & \Phi^G_{j,\epsilon,\pm}(Z) + \frac{1}{q} \\
\Phi^{G,\sigma(j)}_{0,\epsilon,\pm}(F_\epsilon(Z)) & = & \Phi^{G,j}_{0,\epsilon,\pm}(Z) + \frac{1}{q}.
\end{array} \tag{4.25}$$

>From the shape of the Riemann surface as in Figure 1 it is clear that the domain of any $\Phi^G_{j,\epsilon,\pm}$ intersects the domain of any $\Phi^{G,m}_{0,\epsilon,\pm}$.

The transitions between the Fatou coordinates are given by

$$\begin{array}{rcl}
\Psi^G_{j,\epsilon,+} & = & \Phi^{G,j}_{0,\epsilon,+} \circ (\Phi^G_{j,\epsilon,+})^{-1} \\
\Psi^G_{j,\epsilon,-} & = & \Phi^G_{j,\epsilon,-} \circ (\Phi^{G,j}_{0,\epsilon,-})^{-1}.
\end{array} \tag{4.26}$$

(There also exists other functions Ψ corresponding to the other intersections but they can be deduced from these ones by means of (4.25).) They depend continuously on ϵ as $\epsilon \to 0$. At the limit the domain becomes disconnected and the $\Psi^G_{j,\epsilon,\pm}$ tend to Ψ^0_j on one half of the domain and Ψ^∞_j on the other half. If one of the $\Psi^{0,\infty}_j$ is not a translation then so does the corresponding $\Psi^G_{j,\epsilon,\pm}$. So the changes of coordinates to the model in the unfolding in the Poincaré domain are not compatible. We write very few details as things are completely similar to [8] and [14] and as this is not needed for what follows.

Remark 4.19 The projection of a Glutsyuk translation domain on which we can bring the family to the model yields a neighborhood of one fixed point on which we can linearize the diffeomorphism.

4.5 The Martinet-Ramis point of view

In [17] Martinet-Ramis present the orbit space of f_0 as the union of two spheres identified in the neighborhoods of 0 and ∞ by two germs of diffeomorphisms (instead of our descriptions with $2q$-spheres and $2q$ germs of diffeomorphisms). Their description carries over to the unfolding. Indeed

Proposition 4.20 *Over each sector $V_{\delta,\pm}$ the orbit space of f_ϵ is described by the union of the two spheres $S_{1,\epsilon}^+ \cup S_{1,\epsilon}^-$ identified in the neighborhood of ∞ (resp. 0) by $\psi_{1,\epsilon}^\infty$ (resp. $\tilde{\psi}_{1,\epsilon}^0$) where*

$$\tilde{\psi}_{1,\epsilon}^0 = \psi_{1,\epsilon}^0 \circ L_m \tag{4.27}$$

with

$$L_m(w) = \exp\left(\frac{2\pi i m}{q}\right) w, \qquad where \qquad mp \equiv -1 \,(mod\; q). \tag{4.28}$$

PROOF. The map f_ϵ induces a global diffeomorphim between $S_{j,\epsilon}^\pm$ and $S_{\sigma(j),\epsilon}^\pm$ fixing 0 and ∞, hence a linear map. From (4.7) it is easily seen that $w \in S_{j,\epsilon}^\pm$ and $\overline{w} \in S_{\sigma(j),\epsilon}^\pm$ belong to the same orbit if $\overline{w} = \exp\left(-\frac{2\pi i}{q}\right) w$. Let m be such that $mp \equiv -1 \,(\text{mod } q)$. Then $w \in S_{2,\epsilon}^+$ belongs to the same orbit as $\overline{w} = \exp\left(-\frac{2\pi i m}{q}\right) w = L_m^{-1}(w) \in S_{1,\epsilon}^+$. Then identifying points with same orbits in $S_{2,\epsilon}^+$ and $S_{1,\epsilon}^-$ through $\psi_{1,\epsilon}^0$ is the same as identifying points with same orbits in $S_{1,\epsilon}^+$ and $S_{1,\epsilon}^-$ through $\tilde{\psi}_{1,\epsilon}^0 = \psi_{1,\epsilon}^0 \circ L_m$. $\square$

We do not discuss this point of view any longer as this is not needed in the rest of the paper.

5 The parametric resurgence phenomenon

For $\epsilon \neq 0$ the construction of Fatou coordinates allows to define renormalized return maps. These maps allow to study the dynamics of the fixed points of g_ϵ.

5.1 The renormalized maps as an alternative description to the modulus

We consider a normalized set of Fatou coordinates generated by admissible lines $l_j^\pm(\epsilon)$. These lines together with their images $G_\epsilon(l_j^\pm(\epsilon))$ determine strips $\hat{C}_{j,\epsilon}^\pm$. Their images by p_ϵ are crescents $C_{j,\epsilon}^\pm$. Their quotient under g_ϵ are conformally equivalent to $\mathbb{CP}^1 \setminus \{0, \infty\}$ by Proposition 4.1. We call these quotient spaces $S_{j,\epsilon}^\pm$.

Proposition 5.1 *1. For $\epsilon \in V_{\delta,+} \setminus \{0\}$ there exist for the map g_ϵ:*

> *i) a renormalized return map: $k_{j,\epsilon,+} : S_{j,\epsilon}^+ \to S_{j,\epsilon}^+$, $j = 1, \ldots, q$, defined by*

$$k_{j,\epsilon,+} = l_{j,\epsilon,+} \circ \psi_{j,\epsilon,+}^\infty, \tag{5.1}$$

> *where the map $k_{j,\epsilon,+}$ is conjugate to $\Phi_{j,\epsilon,+}^+ \circ T_{-q\alpha} \circ (\Phi_{j,\epsilon,+}^+)^{-1}$ by means of E given in (3.38);*

> *ii) a renormalized return map: $k_{0,\epsilon,+} : S_{1,\epsilon}^+ \to S_{1,\epsilon}^+$ defined by*

$$k_{0,\epsilon,+} = l_{1,\epsilon,+} \circ \psi_{1,\epsilon,+}^0 \circ \cdots \circ l_{q,\epsilon,+}^+ \circ \psi_{q,\epsilon,+}^0, \tag{5.2}$$

> *where the map $k_{0,\epsilon,+}$ is conjugate to $\Phi_{1,\epsilon,+}^+ \circ T_{-q^2\alpha} \circ (\Phi_{1,\epsilon,+}^+)^{-1}$ by means of E.*

2. For $\epsilon \in V_{\delta,-} \setminus \{0\}$ there exist for the map g_ϵ:

i) a renormalized return map: $k_{j,\epsilon,-} : S_{j,\epsilon}^- \to S_{j,\epsilon}^-$, $k = 1, \ldots, q$, *defined by*

$$k_{j,\epsilon,-} = \psi_{j,\epsilon,-}^0 \circ l_{j,\epsilon,-}^-, \qquad (5.3)$$

where the map $k_{j,\epsilon,-}$ *is conjugate to* $\Phi_{j,\epsilon,-}^- \circ T_{-q\alpha} \circ (\Phi_{j,\epsilon,-}^-)^{-1}$ *by means of* E;

ii) a renormalized return map: $k_{0,\epsilon,-} : S_{1,\epsilon}^+ \to S_{1,\epsilon}^+$ *defined by*

$$k_{0,\epsilon,-} = l_{q,\epsilon,-} \circ \psi_{q,\epsilon,-}^\infty \circ \cdots \circ l_{1,\epsilon,-} \circ \psi_{1,\epsilon,-}^\infty, \qquad (5.4)$$

where the map $k_{0,\epsilon,-}$ *is conjugate to* $\Phi_{1,\epsilon,-}^+ \circ T_{-q^2\alpha} \circ (\Phi_{1,\epsilon,-}^+)^{-1}$ *by means of* E.

PROOF. The proof is completely straightforward. ☐

Proposition 5.2 *1. The first derivatives* $k_{0,\epsilon,+}'(0)$, $k_{j,\epsilon,-}'(0)$, $j = 1, \ldots, q$, $k_{0,\epsilon,-}'(\infty)$, $k_{j,\epsilon,+}'(\infty)$, $j = 1, \ldots, q$, *are analytic invariants. Their values are:*

$$\begin{cases} k_{0,\epsilon,+}'(0) = e^{4\pi^2/\mu_0(\epsilon)} = e^{-4\pi^2/\epsilon} \\ k_{j,\epsilon,-}'(0) = e^{4\pi^2/\mu_j(\epsilon)} = e^{4\pi^2(1+a\epsilon)/(q\epsilon)} & j = 1, \ldots, q \\ k_{0,\epsilon,-}'(\infty) = e^{4\pi^2/\mu_0(\epsilon)} = e^{-4\pi^2/\epsilon} \\ k_{j,\epsilon,+}'(\infty) = e^{4\pi^2/\mu_j(\epsilon)} = e^{4\pi^2(1+a\epsilon)/(q\epsilon)} & j = 1, \ldots, q \end{cases} \qquad (5.5)$$

as $\mu_0(\epsilon)$ *and* $\mu_j(\epsilon)$ *are given in* (2.14).

2. *The maps* $l_{j,\epsilon,\pm}$ *are independent of* j *when we deal with a normalized set of Fatou coordinates. They are given by* $l_{j,\epsilon,\pm}(w) = \nu_\pm(\epsilon)$ *with*

$$\nu_\pm(\epsilon) = \exp\left(\mp \left(\frac{4\pi^2}{q\epsilon} + \frac{2\pi^2 a}{q} \right) \right). \qquad (5.6)$$

They depend on ϵ, $a(\epsilon)$ *and* q.

3. *The* $(q+1)$-*tuples* $(k_{0,\epsilon,\pm}, \ldots, k_{q,\epsilon,\pm})$ *defined in Proposition* 5.1 *are representatives of the modulus.*

4. *The maps* $L_{j,\epsilon,\pm}$ *depend only of the formal part of the modulus and of the parameter* ϵ.

PROOF.

1. It is classical that, if a diffeomorphism with a fixed point at the origin has a multiplier of the form $\exp(2\pi i\beta)$, then the renormalized return map has a multiplier $\exp(-\frac{2\pi i}{\beta})$. A proof in the case $\beta \in \mathbb{R}$ appears in [27]. A geometric proof for β non real appears in [14].

2. We have the two systems, each in two unknowns:

$$\begin{cases} k_{0,\epsilon,+}'(0) = \prod_{j=1}^q (\psi_{j,\epsilon,+}^0)'(0)(\nu_+(\epsilon))^q = e^{-4\pi^2/\epsilon} \\ k_{j,\epsilon,+}'(\infty) = (\psi_{j,\epsilon,+}^\infty)'(\infty)(\nu_+(\epsilon))^{-1} = e^{4\pi^2(1+a\epsilon)/(q\epsilon)} & j = 1, \ldots, q \end{cases} \qquad (5.7)$$

and

$$\begin{cases} k_{j,\epsilon,-}'(0) = (\psi_{j,\epsilon,-}^0)'(0)\nu_-(\epsilon) = e^{4\pi^2(1+a\epsilon)/(q\epsilon)} & j = 1, \ldots, q \\ k_{0,\epsilon,-}'(\infty) = \prod_{j=1}^q (\psi_{j,\epsilon,-}^\infty)'(\infty)(\nu_-(\epsilon))^{-q} = e^{-4\pi^2/\epsilon} \end{cases} \qquad (5.8)$$

from which we find (5.6) and also (4.4), hence proving Lemma 4.2.

3. It is clear that the $\psi_{j,\epsilon,\pm}^{0,\infty}$ can be recovered from the $(q+1)$-tuples $(k_{0,\epsilon,\pm}, \ldots, k_{q,\epsilon,\pm})$.

4. The formula (5.6) shows that the maps $l_{j,\epsilon,\pm}(w) = \nu_\pm(\epsilon)w$ depend only of the formal part of the modulus. Then so do the translations $L_{j,\epsilon,\pm}$. ☐

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5.2   The parametric resurgence

We call *parametric resurgence* the phenomenon in which the non triviality of the modulus for $\epsilon = 0$ allows to conclude to the non-linearizability of g_ϵ (or f_ϵ) on some sequences of parameter values converging to the origin.

Proposition 5.3 *We suppose that for all $j = 1, \ldots q$, $(\psi_{j,\epsilon,\pm}^{0,\infty})'(0) = \exp(2\pi^2 a/q)$. In the sequel we will speak of normal form of resonant maps at ∞. By this we mean the normal form at the origin of the conjugate under $\tilde{w} = 1/w$ of the corresponding map.*

(1) *Let us suppose that the coefficients of $\psi_{1,0}^0$ are such that $\xi_1 = \exp(2\pi i p/m - 2\pi^2 a/q)\psi_{1,0}^0$ is not linearizable at the origin because the s-th coefficient of the normal form does not vanish, then the periodic (resp. fixed) points $z_1, \ldots z_q$, of f_ϵ (resp. g_ϵ) will be nonlinearizable as soon as ϵ has the form $\epsilon = -\frac{2\pi i m}{q(p+nm)+2\pi i a m}$ with n sufficiently large. More precisely the s-th coefficient of the renormalized return map of f_ϵ^q is nonzero.*

(2) *Let us suppose that the coefficients of $\psi_{1,0}^\infty$ are such that $\xi_2 = \exp(2\pi i p/m + 2\pi^2 a/q)\psi_{1,0}^\infty$ is not linearizable at ∞ because the s-th coefficient of the normal form does not vanish, then the periodic (resp. fixed) points $z_1, \ldots z_q$, of f_ϵ (resp. g_ϵ) will be nonlinearizable as soon as ϵ has the form $\epsilon = \frac{2\pi i m}{q(p+nm)-2\pi i a m}$ with n sufficiently large. Indeed the s-th coefficient of the renormalized return map of f_ϵ^q is nonzero.*

(3) *Let us suppose that all $\psi_{1,0}^0, \ldots, \psi_{q,0}^0$ are such that the map*

$$\xi_3 = \psi_{1,0}^0 \circ l \circ \psi_{2,0}^0 \circ l \circ \cdots \circ \psi_{q,0}^0 \circ l, \tag{5.9}$$

where l is a linear map such that $\xi_3'(0) = \exp(2\pi i p/m)$, is not linearizable at the origin because the s-th coefficient of the normal form does not vanish, then the fixed point $z_0 = 0$ of f_ϵ (and g_ϵ) will be nonlinearizable as soon as ϵ has the form $\epsilon = \frac{2\pi i m}{p+nm}$ with n sufficiently large. Indeed the s-th coefficient of the renormalized return map of f_ϵ^q is nonzero.

(4) *Let us suppose that all $\psi_{1,0}^\infty, \ldots, \psi_{q,0}^\infty$ are such that*

$$\xi_4 = \psi_{q,0}^\infty \circ l \circ \cdots \circ \psi_{2,0}^\infty \circ l \circ \psi_{1,0}^\infty \circ l, \tag{5.10}$$

where l is a linear map such that $\xi_4'(\infty) = \exp(-2\pi i p/m)$, is not linearizable at ∞ because the s-th coefficient of the normal form does not vanish, then the fixed point $z_0 = 0$ of f_ϵ (and g_ϵ) will be nonlinearizable as soon as ϵ has the form $\epsilon = -\frac{2\pi i m}{p+nm}$ with n sufficiently large. Indeed the s-th coefficient of the renormalized return map of f_ϵ^q is nonzero.

PROOF. If a map has the form $g(z) = \exp(2\pi i \beta)z + o(z)$, then its renormalized return map has the form $h(z) = \exp(-2\pi i/\beta)z + o(z)$ (see Proposition 5.2).

(1) and (2) The renormalized return map around z_1 has the form $k_{1,\epsilon,-} = \psi_{1,\epsilon,-}^0 \circ l_{1,\epsilon,-}$, where $\psi_{1,\epsilon,-}^0$ depends continuously on ϵ. A priori $l_{1,\epsilon,-}$ is a wild map, but we limit ourselves to values of ϵ such that $k_{1,\epsilon,-}'(0) = \exp(2\pi i \frac{p}{m})$, i.e. $l_{1,\epsilon,-}(w) = \exp(2\pi i p/m - 2\pi^2 a/q)w$. The map $k_{1,\epsilon,-}$ is hence resonant. It is nonlinearizable as soon as one of the coefficients of its normal form is nonzero. As $\psi_{1,\epsilon,-}^0$ depends continuously on ϵ the non vanishing of the s-th coefficient of the normal form of ξ_1 implies the non vanishing of the s-th coefficient of the normal form of $k_{1,\epsilon,-}$. The situation is similar at the other singular points because of (4.6).

(3) and (4) This case is very similar to the previous one. Indeed we limit ourselves to values of ϵ for which the renormalized return map $k_{0,\epsilon,+}$ (resp. $k_{0,\epsilon,-}$) has same derivative at the origin(resp. at ∞) as ξ_3 (resp. ξ_4) and is very close to it. So if the s-th coefficient of the normal form of ξ_3 (resp. ξ_4) is non vanishing then the same is true for the s-th coefficient of $k_{0,\epsilon,+}$ (resp. $k_{0,\epsilon,-}$) for ϵ sufficiently small as described.

$\square$

6 The modulus of orbital analytic classification of a family unfolding a resonant saddle

In [22] it is shown that the modulus of orbital analytic classification of a generic family unfolding a generic saddle-node under weak equivalence is given by the modulus of the unfolded holomomy map of its strong separatrix. We show that the same holds for a generic family unfolding a generic resonant saddle.

6.1 Orbital preparation of the family

We consider a vector field with a generic resonant saddle at the origin, i.e. such that the first coefficient of its normal form is nonzero and a generic family unfolding it.

Proposition 6.1 *With a change of coordinates and scaling of time we can bring the family to the form* Θ_η *given by*

$$\Theta_\eta = \begin{cases} \dot{x} = x \\ \dot{y} = y[-\frac{p}{q}(1+\eta) + Au + h(x,y)], \end{cases} \tag{6.1}$$

with $A \neq 0$, $u = x^p y^q$ *and* $h(x,y) = o(u)$.

PROOF. As it is very standard we only give the main idea. A change of coordinates brings the analytic separatrices to the axis and scaling allows to transform the system to

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= y[-\frac{p}{q}(1+\eta) + O(|x,y|)]. \end{aligned} \tag{6.2}$$

The final form is achieved through a change of coordinates of the form

$$y = Y(1 + \sum_{j=0}^{p-1} a_j(Y)x^j + \sum_{k=0}^{q-1} b_k(x)Y^k). \tag{6.3}$$

$\square$

The holonomy map of its x-separatrix has the form

$$f_\eta(y) = \exp\left(-2\pi i \frac{p}{q}(1+\eta)\right) y + \left(2\pi i A \exp(-2\pi i \frac{p}{q}) + O(\eta)\right) y^{q+1} + o(y^{q+1}). \tag{6.4}$$

If we choose to scale x and y so that $A = \frac{1}{2\pi i q}$, then the map is almost of the form (2.2): only p is changed to $-p$ and the parameter is different.

An orbital preparation of the family must of course bring the holonomy map and its q-th iterate to a prepared form. It is possible to apply a scaling $(x,y) \mapsto (\alpha x, \beta y)$ with $\alpha^p \beta^q = 1$ so that the family is defined in the region $|x| < 2$ and the holonomy is defined as a map from $\{x = 1\}$ to itself. When we make further changes of coordinates we apply if necessary other scalings of this type so that the holonomy is always defined as a map from $\{x = 1\}$ to itself.

Theorem 6.2 *There exists an analytic change of coordinate* $(x,y) \mapsto (\tilde{x}, \tilde{y}) = (x, \beta y(1 + O(u))$ *and of parameter* $\eta \mapsto \epsilon$ *bringing the family* (6.1) *to an orbitally prepared form with parameter*

$$\epsilon = 2\pi i p \eta, \tag{6.5}$$

i.e. a form in which the holonomy of the x*-separatrix on the section* $\{x = 1\}$ *is in prepared form and in which the invariant manifold has an equation* $\tilde{u} = \epsilon$, *where* $\tilde{u} = \tilde{x}^p \tilde{y}^q$.

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Proof. The fact that the vector field is non integrable of order 1 leads to the "materialization" of the resonance [12], i.e. to the birth of an analytic invariant manifold of a special form which in the limit $\eta \to 0$ goes to $u = 0$. For $\eta = 0$ there exists a formal change of coordinate $(X, Y) = (x, y + o(u))$ bringing the system to the normal form

$$
\begin{aligned}
\dot{X} &= X \\
\dot{Y} &= Y[-\tfrac{p}{q}(1+\eta) + AU + h(U)],
\end{aligned}
\tag{6.6}
$$

where $h(U) = o(U)$ is a power series in U. In these coordinates the invariant manifold has the form

$$
\eta = k(U) = \sum_{j=1}^{\infty} a_j(\eta) U^j,
\tag{6.7}
$$

with $a_1(0) \neq 0$ and the $a_i(\eta)$ depending analytically on η [12]. Even if the change of coordinates is generically divergent, the invariant manifold is analytic. Coming back to the original variables x and y it has an equation of the form $\eta = k_i(x, y) = a_1(\eta)u + o(u) = u(a_1(\eta) + m(x, y))$ with $m(x, y) = O(u)$. We take a change of coordinates: $(\tilde{x}, \tilde{y}) = (x, y(\frac{a_1(\eta)}{2\pi i p} + \frac{m(x,y)}{2\pi i p})^{1/q})$ in which the invariant manifold has the equation $\epsilon = \tilde{u}$ with $\tilde{u} = \tilde{x}^p \tilde{y}^q$.

This means in particular that the analytic invariant manifold intersects the section $\{x = 1\}$ at $\tilde{y}^q = \epsilon$. Let $\tilde{f}_\eta$ be the holonomy map in the variable $\tilde{y}$. We know that $\tilde{y}^q - \epsilon = 0$ is the equation of the q-periodic points of the holonomy map coming from the intersection of the invariant manifold with $x = 1$. Then the q-th iterate of $\tilde{f}_\eta$ has the form

$$
\tilde{f}_\eta^q(y) = \tilde{y} + \tilde{y}(\tilde{y}^q - \epsilon))(1 + O(\eta) + O(\tilde{y})).
\tag{6.8}
$$

The map is already prepared as $(\tilde{f}_\eta^q)'(0) = \exp(-2\pi i p \eta) = \exp(-\epsilon)$. $\qquad\square$

>From now on we will always limit ourselves to orbitally prepared families of vector fields unfolding a resonant saddle. We can suppose that the family has the form (6.1). We will consider the family as depending of the parameter ϵ where ϵ is given in (6.5) and we will suppose that the holonomy $f_\epsilon(y)$ of the variable y has the form (2.2) in which we replace z by y and p by $-p$. Then its q-th iterate $g_\epsilon(y)$ has the form (3.1).

The variable u plays a special role as the equation of the analytic invariant manifold is given by $u = \epsilon$. Note that changes of coordinates of the form

$$
(\tilde{x}, \tilde{y}) \mapsto (\tilde{x} \exp(h(\tilde{x}, \tilde{y})), \tilde{y} \exp(-\frac{p}{q} h(\tilde{x}, \tilde{y})))
\tag{6.9}
$$

preserve u as $\tilde{u} = u$.

Remark 6.3 In the particular case $p = q = 1$ the two separatrices play identical roles although their holonomy maps are in general not conjugate (in particular they usually do not have the same formal invariant), but related in a more complicated way. A negative rescaling of time, $t \mapsto -t$, allows to exchange them. To study the analytic orbital equivalence of two families we will have to distinguish one separatrix for each family and build the analytic orbital equivalence by extending the conjugacy of the holonomies of these distinguished separatrices.

6.2 Complete invariant of orbital analytic classification for a generic family of vector fields unfolding a generic resonant saddle

Definition 6.4 Two germs of analytic families of vector fields, $\Theta_{\epsilon_1}(x_1, y_1)$ (resp. $\overline{\Theta}_{\epsilon_2}(x_2, y_2)$) unfolding a resonant saddle at the origin for $\epsilon_1 = 0$ (resp. $\epsilon_2 = 0$) are *orbitally equivalent* if there exists a germ of map

$$
K = (h, \Phi, \xi): \quad (\epsilon_1, x_1, y_1) \mapsto (h(\epsilon_1), \Phi(\epsilon_1, x_1, y_1), \xi(\epsilon_1, x_1, y_1))
\tag{6.10}
$$

fibered over the parameter space where

i) $h : \epsilon_1 \mapsto \epsilon_2 = h(\epsilon_1)$ is a germ of an analytic diffeomorphism preserving the origin;

ii) there exists a representative $\Phi_{\epsilon_1}(x_1, y_1) = \Phi(\epsilon_1, x_1, y_1)$ which is an analytic diffeomorphism in (ϵ_1, x_1, y_1) on a small neighborhood of the origin in (ϵ_1, x_1, y_1)-space;

iii) there exists a representative $\xi_{\epsilon_1}(x_1, y_1) = \xi(\epsilon_1, x_1, y_1)$ depending analytically on (ϵ_1, x_1, y_1) in a small neighborhood of the origin in (ϵ_1, x_1, y_1)-space with values in $\mathbb{C}^*$;

iv) the change of coordinates Φ_{ϵ_1} and the scaling of time ξ_{ϵ_1} is an equivalence between $\Theta_{\epsilon_1}(x_1, y_1)$ and $\overline{\Theta}_{h(\epsilon_1)}(x_2, y_2)$ over a ball of small radius $r > 0$:

$$\overline{\Theta}_{h(\epsilon_1)}(\Phi_{\epsilon_1}(x_1, y_1)) = \xi(\epsilon_1, x_1, y_1)(\Phi_{\epsilon_1})_*(\Theta_{\epsilon_1}(x_1, y_1)). \tag{6.11}$$

Theorem 6.5 *We consider a generic family unfolding a generic resonant saddle with hyperbolicity ratio p/q in prepared form. The modulus of analytic classification of the holonomy map of the unfolded vector field as described in Theorem 4.8, namely the two families of equivalence classes*

$$\left((\psi_{1,\epsilon,\pm}^{\infty}, \psi_{1,\epsilon,\pm}^0) / \sim \right)_{\epsilon \in V_{\delta,\pm}}, \tag{6.12}$$

together with $a(0)$ is a complete modulus of orbital analytic classification under orbital equivalence. In the case $p = q = 1$ it is the modulus of the family of resonant saddles with a marked separatrix.

PROOF. The proof uses essentially the fact that the holonomy characterizes the differential equation [16] and [17]. This proof can be carried nearly verbatim over a fixed neighborhood of the origin for values of ϵ in a small neighborhood of the origin.

One direction is obvious: If two orbitally prepared families are equivalent then there exists an equivalence transforming one to the other. In these coordinates they have conjugate holonomies for each value of ϵ over a fixed neighborhood in y-space, hence the same modulus.

Let us now consider the converse. We consider two orbitally prepared families of vector fields $\Theta_{1,\eta}$ and $\Theta_{2,\eta}$, which are generic unfoldings of resonant saddles with hyperbolicity ratio p/q defined on the same neighborhood of the origin containing $|x| < 2$. We suppose that the families of holonomies are conjugate. In particular the parameters η for the vector fields and $\epsilon = 2\pi i \eta$ for the holonomies are the same. For each ϵ in a neighborhood of the origin we need to construct an orbital equivalence between the two vector fields over a neighborhood of the origin $W = B(0, 2) \times U$, which is independent of ϵ: the orbital equivalence will depend analytically on ϵ. As the hononomies $f_{j,\epsilon}$, $j = 1, 2$, are conjugate there exist an analytic change of coordinate $y_2 = \zeta_\epsilon(y_1) = \zeta(y_1, \epsilon)$ conjugating $f_{1,\epsilon}$ with $f_{2,\epsilon}$, i.e. $f_{1,\epsilon} = \zeta_\epsilon^{-1} \circ f_{2,\epsilon} \circ \zeta_\epsilon$. ζ_ϵ is defined from $\{x_1 = 1\}$ to $\{x_2 = 1\}$. We want to perform a change of coordinates on $\Theta_{1,\eta}$ on a whole neighborhood of the origin so that the two holonomies become identical. As $\zeta_\epsilon(0) = 0$ since 0 is a fixed point and since the q-th roots of ϵ are periodic points of period q, then $\zeta_\epsilon(y_1) = e^{\frac{2\pi i k}{q}} y_1 + y_1(y_1^q - \epsilon)\zeta_{1,\epsilon}(y_1)$. We let the change of coordinate on W be given by

$$(\overline{x_1}, \overline{y_1}) = (x_1, y_1(e^{\frac{2\pi i k}{q}} + (u_1 - \epsilon)\zeta_{1,\epsilon}(y_1))) = \Lambda_\epsilon(x_1, y_1). \tag{6.13}$$

The change of coordinates Λ_ϵ transforms the invariant manifold $u_1 = \epsilon$ of $\Theta_{1,\epsilon}$ into $\overline{u}_1 = \epsilon$. Moreover $(\Lambda_\epsilon)_*(\Theta_{1,\eta})$ and $\Theta_{2,\eta}$ have the same holonomy on $\{\overline{x}_1 = 1\}$ and $\{x_2 = 1\}$ respectively.

So we can suppose that $\Theta_{1,\eta}$ and $\Theta_{2,\eta}$ have the same holonomy on $x_j = 1$. The next step consists in constructing a change of coordinate $K_\epsilon(x_1, y_1) = (x_2, y_2)$ transforming the first system to the second. This change of coordinate will of course be the identity (i.e. $K_\epsilon(1, y_1) = (1, y_2)$) on $x_j = 1$. The first step is to extend K_ϵ to $|x_j| = 1$. For that we consider the path $(e^{i\theta}, 0)$, $\theta \in [0, 2\pi]$ which is lifted in the leaf of the foliation through $(1, y_j)$ as $(e^{i\theta}, \alpha_j(\theta, y_j))$. We let

$$K_\epsilon(e^{i\theta}, \alpha_1(\theta, y_1)) = (e^{i\theta}, \alpha_2(\theta, y_1)). \tag{6.14}$$

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The next step consists in extending K_ϵ along lifting of radial paths:

$$r_{x_j} : [0, -\ln|x_j|] \to W, \qquad r_{x_j}(s) = (x_j e^s, 0), \tag{6.15}$$

for $0 < |x_j| < 1$. The lifting in the leaf through (x_j, y_j) is a path $(x_j e^s, \beta_{j,x_j}(s))$ with endpoint $\delta_j(x_j, y_j) = (\frac{x_j}{|x_j|}, \beta_{j,x_j}(-\ln|x_j|))$. The map δ_j is the flow of $\Theta_{j,\eta}$ for the time $-\ln|x_j|$: $\delta_j = \Theta_{j,\eta}^{-\ln|x_j|}$. So the inverse map is well defined. We let

$$\begin{array}{rcl} \delta_1(x_1, y_1) & = & (\frac{x_1}{|x_1|}, \gamma_1(x_1, y_1)) \\ \delta_2^{-1}(\frac{x_2}{|x_2|}, y_2) & = & (x_2, \gamma_2(x_2, y_2)) \end{array} \tag{6.16}$$

Then we let

$$K_\epsilon(x_1, y_1) = (x_1, \gamma_2(\frac{x_1}{|x_1|}, \gamma_1(x_1, y_1))). \tag{6.17}$$

The map K_ϵ is a holomorphic equivalence between the vector fields outside $x_i = 0$. As it is bounded by Lemma 6.6 below it can be extended to $x_i = 0$. It clearly depends analytically on ϵ. $\qquad\square$

Lemma 6.6 *We consider a vector field*

$$\begin{array}{rcl} \dot{x} & = & x \\ \dot{y} & = & y(-\lambda + a(x, y)) \end{array} \tag{6.18}$$

defined on $|x| < 2$ and $y < r$, in which $\lambda \in \mathbb{C}$ is a parameter in a small neighborhood of $\lambda_0 > 0$ where $Re\ \lambda > 0$ and $a(x, y) = O(|xy|)$. We consider a radial path as in (6.15) (we drop the indices). Its lifting through (x, y) has endpoint $\delta_\lambda(x, y) = (\frac{x}{|x|}, \gamma_\lambda(x, y))$. Then there exists a neighborhood $\Lambda \subset \mathbb{C}$ of λ_0 and a neighborhood U of 0 such that for all $\lambda \in \Lambda$, for all x_0 with $0 < |x_0| < 1$ and for all $y_0 \in U$

$$|y_0||x_0|^{3\frac{\lambda_0}{2}} \le |\gamma_\lambda(x_0, y_0)| \le |y_0||x_0|^{\frac{\lambda_0}{2}}. \tag{6.19}$$

Proof. The proof is very close to that of [16] and [17]. From (6.15) we get $\dot{s} = 1$. This yields to the differential equation

$$\frac{dy(s)}{ds} = y(s)(-\lambda + a(x_0 e^s, y(s))) \tag{6.20}$$

with initial condition $y(0) = y_0$. So

$$\frac{d|y(s)|}{ds} = |y(s)|(-Re\ \lambda + \alpha(x_0 e^s, y(s))). \tag{6.21}$$

For $|x| < 2$ and $|y| < r_0$ with r_0 sufficiently small we have $|\alpha(x, y)| < \frac{\lambda_0}{4}$. Then

$$-(\frac{\lambda_0}{4} + Re\ \lambda)s \le \ln\left|\frac{y(s)}{y_0}\right| \le (-Re\ \lambda + \frac{\lambda_0}{4})s. \tag{6.22}$$

Hence

$$|y_0|e^{-(Re\ \lambda + \frac{\lambda_0}{4})s} \le |y(s)| \le |y_0|e^{(-Re\ \lambda + \frac{\lambda_0}{4})s} \tag{6.23}$$

We evaluate this at $s = -\ln|x_0|$ and take $|\lambda - \lambda_0| < \frac{\lambda_0}{4}$ from which (6.19) follows for $|x_0| < 1$. $\qquad\square$

7 The transcritical bifurcation

We consider here the case of a resonant hyperbolic saddle of a real system (6.1) in prepared form. Hence the parameter η is real and also the formal invariant a. Moreover in that case we usually only observe the singular point and not the invariant manifold. We only need to describe the behaviour for $\eta < 0$ and

$\eta > 0$. As $\epsilon = 2\pi i p\eta$ this corresponds to $\epsilon \in i\mathbb{R}^-$ and $\epsilon \in i\mathbb{R}^+$. In the first case the renormalized return map for the q-th power of the holonomy is given by (5.4), namely:

$$k_{0,\epsilon,-} = l_{q,\epsilon,-}^- \circ \psi_{q,\epsilon,-}^\infty \circ \cdots \circ l_{1,\epsilon,-}^- \circ \psi_{1,\epsilon,-}^\infty. \tag{7.1}$$

We see that the renormalized return map, and hence the normalizability of the origin depends on $\psi_{1,\epsilon,-}^\infty$ only. In the second case the renormalized return map is given by (5.2):

$$k_{0,\epsilon,+} = l_{1,\epsilon,+} \circ \psi_{1,\epsilon,+}^0 \circ \cdots \circ l_{q,\epsilon,+}^+ \circ \psi_{q,\epsilon,+}^0. \tag{7.2}$$

The normalizability of the origin depends on $\psi_{1,\epsilon,+}^0$ only. In particular if we start with a semi-normalizable saddle point: ψ_1^0 linear and ψ_1^∞ nonlinear (resp. ψ_1^0 nonlinear and ψ_1^∞ linear) we may observe families in which the origin is non integrable for sequences of negative rational values of η and integrable for $\eta > 0$ (resp. the origin is integrable for $\eta < 0$ and non integrable for sequences of positive rational values of η).

Such examples of families appear in [3] and [5] and raised the first author's interest in the subject. They are families of polynomial vector fields.

8   Modulus of analytic conjugacy for a generic family unfolding a resonant saddle

8.1   Preparation of the family

We consider a generic analytic 1-parameter family of vector fields unfolding a resonant saddle of order 1. As the separatrices are analytic and depend analytically of the parameter the family can always be brought by an analytic change of coordinates to the form

$$X_\epsilon = \begin{cases} \dot{x} = \lambda_1(\epsilon)xh_{1,\epsilon}(x,y) \\ \dot{y} = \lambda_2(\epsilon)yh_{2,\epsilon}(x,y) \end{cases} \tag{8.1}$$

where $\frac{\lambda_2(0)}{\lambda_1(0)} = -\frac{p}{q}$, $\frac{d}{d\epsilon}\left(\frac{\lambda_2}{\lambda_1}\right) \neq 0$ and $h_{i,\epsilon}(x,y) = 1 + O(x,y)$. Modulo some preparation on the orbital form as in the beginning of Section 6.1 we can suppose that

$$\frac{\lambda_2(\epsilon)}{\lambda_1(\epsilon)} = -\frac{p}{q}(1+\eta) \qquad \text{with} \qquad \eta = \frac{\epsilon}{2\pi i p}. \tag{8.2}$$

We can of course suppose that the family

$$\Theta_\epsilon = \frac{X_\epsilon}{\lambda_1(\epsilon)h_{1,\epsilon}(x,y)} \tag{8.3}$$

is in prepared form (6.1) (because the transformations to get (6.1) are in y only and hence preserve the equation $\dot{x} = x$).

Proposition 8.1 *There exists a change of coordinate depending analytically on ϵ and bringing the family $X_\epsilon = \lambda_1(\epsilon)h_{1,\epsilon}(x,y)\Theta_\epsilon$ with Θ_ϵ given in (6.1) to the prepared form*

$$\lambda_1(\epsilon)k_{1,\epsilon}(x,y)\Theta_\epsilon, \tag{8.4}$$

with $k_{1,\epsilon}(x,y) = 1 + O(u)$.

Proof. A conjugacy given by the flow $X_\epsilon^{\xi_\epsilon(x,y)}$ of X_ϵ for a time $\xi_\epsilon(x,y)$ transforms the vector field X_ϵ to vector field $\lambda_1(\epsilon)k_{1,\epsilon}(x,y)\Theta_\epsilon$ provided

$$\frac{1}{1 + X_\epsilon(\xi_\epsilon)} = \frac{k_{1,\epsilon}}{h_{1,\epsilon}}, \tag{8.5}$$

(see for instance [4]). Hence

$$X_\epsilon(\xi_\epsilon) = \frac{h_{1,\epsilon}}{k_{1,\epsilon}} - 1. \tag{8.6}$$

We need to choose ξ_ϵ in such a way that $k_{1,\epsilon}(x,y) = 1 + O(u)$, with $u = x^p y^q$. For that purpose we write

$$h_{1,\epsilon}(x,y) = 1 + \sum_{i=0}^{q-1} y^i a_{i,\epsilon}(x) + \sum_{j=0}^{p-1} x^j b_{j,\epsilon}(y) + m_\epsilon(x,y) \tag{8.7}$$

with $a_{i,\epsilon}(0) = 0$ and $b_{j,\epsilon}(0) = 0$ and $m_\epsilon(x,y) = O(u)$ and we take

$$\xi_\epsilon(x,y) = \sum_{i=0}^{q-1} y^i \alpha_{i,\epsilon}(x) + \sum_{j=0}^{p-1} x^j \beta_{j,\epsilon}(y). \tag{8.8}$$

Let $n_\epsilon(x,y) = \sum_{i=0}^{q-1} y^i a_{i,\epsilon}(x) + \sum_{j=0}^{p-1} x^j b_{j,\epsilon}(y)$. As

$$\begin{aligned} X_\epsilon(\xi_\epsilon) &= \lambda_1(\epsilon)(1 + n_\epsilon(x,y) + O(u)) \left[\sum_{i=0}^{q-1} \left(x\alpha'_{i,\epsilon}(x) + i\alpha_{i,\epsilon}(x)(-(\frac{p}{q} + \eta) + O(u)) \right) y^i \right. \\ &\quad \left. + \sum_{j=0}^{p-1} \left(j\beta_{j,\epsilon}(y) + y\beta'_{j,\epsilon}(y)(-(\frac{p}{q} + \eta) + O(u)) \right) x^i \right] \end{aligned} \tag{8.9}$$

and writing

$$\frac{n_\epsilon(x,y)}{\lambda_1(\epsilon)(1 + n_\epsilon(x,y) + O(u))} = \sum_{i=0}^{q-1} y^i \gamma_{i,\epsilon}(x) + \sum_{j=0}^{p-1} x^j \delta_{j,\epsilon}(y) + O(u) \tag{8.10}$$

with $\gamma_{i,\epsilon}(0) = \delta_{j,\epsilon}(0) = 0$ (this decomposition is not unique), this yields linear differential equations

$$\begin{aligned} x\alpha'_{i,\epsilon}(x) - i(\frac{p}{q} + \eta)\alpha_{i,\epsilon}(x) &= \gamma_{i,\epsilon}(x) \\ j\beta_{j,\epsilon}(y) - (\frac{p}{q} + \eta)y\beta'_{j,\epsilon}(y) &= \delta_{j,\epsilon}(y) \end{aligned} \tag{8.11}$$

which all have analytic solutions vanishing at the origin. $\qquad\square$

As in the previous section we can assume that we work in a neighborhood of the origin of the form $\{|x| < 2\} \times W$, where W is a neighborhood of the origin in y-space (a scaling in (x,y) preserving u may be necessary to achieve this.) We will always limit ourselves to families $X_\epsilon = \lambda_1(\epsilon)h_{1,\epsilon}(x,y)\Theta_\epsilon$ in prepared forms, i.e. such that $h_{1,\epsilon}(x,y) = 1 + O(u)$.

Proposition 8.2 *The family X_ϵ has four formal invariants:*

i) $\epsilon = 2\pi i p\eta$: the multiplier of the q-th iterate of the holonomy map of the x-separatrix is $\exp(-\epsilon)$;

ii) $a(\epsilon)$, where the multiplier of the q-th iterate of the holonomy of the x-separatrix at the invariant manifold is $\exp(\frac{q\epsilon}{1+a(\epsilon)\epsilon})$.

iii) $t_1(\epsilon) = \frac{2\pi i}{\lambda_1(\epsilon)}$ is the time spent along the loop $x = x_0 e^{i\theta}, y = 0$ when $\theta \in [0, 2\pi]$. This time is independent of x_0. In particular we can take $x_0 = 1$.

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iv) $t_2(\epsilon)$ *is the time spent along the loop* $x = x_0 e^{i\theta}, u = \epsilon$, *when* $\theta \in [0, 2\pi q]$. *We can of course assume that*

$$t_2(\epsilon) = \frac{2\pi i q}{\lambda_1(\epsilon)(1 + \epsilon C(\epsilon))}. \tag{8.12}$$

Again it is independent of x_0. *In particular we can take* $x_0 = 1$. $C(\epsilon)$ *is completely determined by* $t_2(\epsilon)$ *and admits a continuous extension at* $\epsilon = 0$.

The first two invariants depend only of the orbital analytic classification and the last two involve time, so they are only invariant under conjugacy.

The problem we are interested in is the classification of analytic families under conjugacies (the definition is similar to Definition 6.4 above except that we do not allow to divide by a nonzero function: see below). For that purpose we compare our family to an adequate model family. As above the comparison with the model family will generically be ramified and we will find the modulus from this ramification. The modulus will have two parts, the first part being the modulus of analytic orbital classification and a second part dealing with the time.

8.2   Comparing the family to a model family

This step is an intermediate step in deciding if two families are conjugate. Indeed a conjugacy, it is exists will be found by composing conjugacies of each family to the model family. The conjugacy is first defined on a section of the separatrix and then extended. Here we limit ourselves to the definition of the conjugacy on a section.

We compare our family to a model family

$$Y_\epsilon = \begin{cases} \dot{x} = \lambda_1(\epsilon)x(1 + C(\epsilon)u) \\ \dot{y} = -\lambda_1(\epsilon)\frac{p}{q}y(1 + \eta)\frac{(1 + B(\epsilon)u)}{(1 + A(\epsilon)u)}(1 + C(\epsilon)u) \end{cases} \tag{8.13}$$

where $A(\epsilon)$ and $B(\epsilon)$ are chosen so that $u = \epsilon$ be the invariant manifold. Hence $2i\pi p(A - B - \eta B) = 1$, which determines $B(\epsilon)$ as a function of $A(\epsilon)$:

$$B(\epsilon) = \frac{A(\epsilon)}{1 + \eta} - \frac{1}{2\pi i p(1 + \eta)}. \tag{8.14}$$

Then the family can be rewritten

$$Y_\epsilon = \begin{cases} \dot{x} = \lambda_1(\epsilon)x(1 + C(\epsilon)u) \\ \dot{y} = -\lambda_1(\epsilon)\frac{p}{q}y\left[1 - \frac{1}{2\pi i p}\frac{u - \epsilon}{1 + A(\epsilon)u}\right](1 + C(\epsilon)u). \end{cases} \tag{8.15}$$

Taking $v = u^{\frac{1}{q}}$ yields the equation

$$\frac{dv}{dx} = \frac{1}{2\pi i q}\frac{v(v^q - \epsilon)}{x(1 + A(\epsilon)v^q)}. \tag{8.16}$$

To calculate the q-th iterate of the holonomy we take $x = \exp(2\pi i q \theta)$ with $\theta \in [0, 1]$, which yields

$$\frac{dv}{d\theta} = \frac{v(v^q - \epsilon)}{1 + A(\epsilon)v^q}. \tag{8.17}$$

The coefficients $A(\epsilon), \lambda_1(\epsilon), C(\epsilon)$ are chosen so that the family has the same four analytic invariants described in Proposition 8.2 as (8.1). This goes in the following way:

ii) We simply have $A(\epsilon) = a(\epsilon)$ as the q-th iterate of the holonomy is the time-one map of (8.16);

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iii) We have $t_1(\epsilon) = \frac{2\pi i}{\lambda_1(\epsilon)}$;

iv) $t_2(\epsilon) = \frac{2\pi i q}{\lambda_1(\epsilon)(1+\epsilon C(\epsilon))}$.

We look for a conjugacy of a certain type, namely the flow of X_ϵ for a certain time $\xi_\epsilon(x,y)$. This flow map preserves the leaves over a fixed neighborhood of the origin for sufficiently small ϵ. It is shown in [4], [24] and [25] that X_ϵ is mapped to Y_ϵ by an analytic diffeomorphism, H_ϵ being the flow map $X_\epsilon^{\xi_\epsilon(x,y)}$ of X_ϵ for some time $\xi_\epsilon(x,y)$, if

$$\frac{1}{1 + X_\epsilon(\xi_\epsilon)} = \frac{1 + C(\epsilon)u}{h_{1,\epsilon}}. \tag{8.18}$$

Hence we must construct a solution of (8.18). Let

$$\tau(\epsilon) = \frac{h_{1,\epsilon}}{1 + C(\epsilon)u} - 1. \tag{8.19}$$

Thus we want to construct $\xi_\epsilon(x,y)$ satisfying

$$X_\epsilon(\xi_\epsilon) = \tau_\epsilon. \tag{8.20}$$

The construction goes in two steps. We first construct a solution $\xi_\epsilon(1,y)$ over the section $\Sigma = \{x = 1\}$: this step will be sufficient to define the modulus of conjugacy. We should then extend it to a full neighborhood of the origin but we will see that this second step is not necessary to solve the problem of deciding when two families are conjugate. Let $y \in \Sigma$ such that $f_\epsilon^q(y) \in \Sigma$, where f_ϵ is the holonomy map of the x-separatrix and f_ϵ^q its q-th iterate. Then the function $\xi_\epsilon(1,y) = \xi_{1,\epsilon}(y)$ must satisfy

$$\xi_{1,\epsilon}(f_\epsilon^q(y)) - \xi_{1,\epsilon}(y) = \int_{\gamma_q(y)} \tau_\epsilon \, dt \tag{8.21}$$

where f_ϵ is the holonomy map, $\gamma_q(y)$ is the lifting of the curve $\gamma_{q,0} = \{(x = e^{i\theta}, y), \theta \in [0, 2\pi q]\}$, to the leaf through $(1,y)$ joining $(1,y)$ and $(1, f_\epsilon^q(y))$ and dt is the time-form of X_ϵ.

Before discussing the first step, namely the construction of a solution to (8.21) let us give the definition of a time-form of a vector field.

Definition 8.3 A *time form dt* of a vector field X is a 1-form such $i_X dt = 1$.

Remark 8.4 (i) Let ω be a form such that $i_X(\omega) = 0$. Such a form is called a dual form to X. Then the time form is uniquely determined up to the addition of a dual form to X.

(ii) It is easily checked that if dt_X is a time form for X and $Y = \phi^*(X)$ for some change of coordinates then $\phi^*(dt_X)$ is a time form for Y.

Lemma 8.5 *Let*

$$k_\epsilon(y) = \int_{\gamma(y)} \tau_\epsilon \, dt. \tag{8.22}$$

where $\gamma(y)$ is the lifting of the curve $\gamma_0 = \{(x = e^{i\theta}, y), \theta \in [0, 2\pi]\}$, to the leaf through $(1,y)$ joining $(1,y)$ and $(1, f_\epsilon(y))$. The function $k_\epsilon(y)$ of (8.22) is analytic in y. Moreover there exists positive constants $c_1(\epsilon), c_2(\epsilon)$ such that

$$\begin{cases} |k_\epsilon(y)| \le c_1(\epsilon)|y| & near\ y = 0 \\ |k_\epsilon(y)| \le c_2(\epsilon)|y - \epsilon^{1/q}| & near\ y \le \epsilon^{1/q}. \end{cases} \tag{8.23}$$

PROOF. This comes from the fact that Y_ϵ has the same formal time invariants as X_ϵ. Indeed, if dt_{X_ϵ} (resp dt_{Y_ϵ}), is the time-form of X_ϵ (resp. Y_ϵ) and if γ_ϵ is a trajectory of X_ϵ (hence of Y_ϵ), then

$$\int_{\gamma_\epsilon} dt_{Y_\epsilon} - dt_{X_\epsilon} = \int_{\gamma_\epsilon} \tau_\epsilon \, dx \tag{8.24}$$

and this integral vanishes for $y = 0$ and $y^q = \epsilon$, yielding the estimate (8.23). □

We now examine the holonomy on $x = 1$ with $z = y$ as a coordinate and we lift to the Z-plane by composition with (3.7): $z = p_\epsilon(Z)$.

Theorem 8.6 *Let*

$$\begin{aligned} K_\epsilon &= k_\epsilon \circ p_\epsilon \\ F_\epsilon &= p_\epsilon^{-1} \circ f_\epsilon \circ p_\epsilon. \end{aligned} \tag{8.25}$$

Let $Q_{j,\epsilon,\pm}^+$ be a translation domain and $\Phi_{j,\epsilon,\pm}^\pm$ be a Fatou coordinate on it. Let σ be the permutation on $\{1,\dots,q\}$ given by $j \mapsto j + p \,(mod\, q)$ and σ^i its i-th iterate.

(1) There exists a unique holomorphic function $\tilde{\Xi}_{j,\epsilon,\pm}^\pm$ defined on $\Phi_{j,\epsilon,\pm}^\pm(Q_{j,\epsilon,\pm}^\pm)$ satisfying:

$$\tilde{\Xi}_{j,\epsilon,\pm}^\pm(W+1) - \tilde{\Xi}_{j,\epsilon,\pm}^\pm(W) = \sum_{i=0}^{q-1} \tilde{K}_{\sigma^i(j),\epsilon,\pm}^\pm \left(W + \frac{i}{q} \right), \tag{8.26}$$

where

$$\tilde{K}_{j,\epsilon,\pm}^\pm(W) = K_\epsilon \circ (\Phi_{j,\epsilon,\pm}^\pm)^{-1}(W), \tag{8.27}$$

and such that its limit is zero at the lower end of the strip and its limit at the upper end of the strip exists. The function $\tilde{\Xi}_{j,\epsilon,\pm}^\pm$ depends analytically on $\epsilon \in V_{\delta,\pm}$ for $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$.

(2) The different $\tilde{\Xi}_{j,\epsilon,\pm}^\pm$ are related by the following recurrence relations:

$$\tilde{\Xi}_{\sigma(j),\epsilon,\pm}^\pm(W) = \tilde{\Xi}_{j,\epsilon,\pm}^\pm \left(W - \frac{1}{q} \right) + \tilde{K}_{j,\epsilon,\pm}^\pm \left(W - \frac{1}{q} \right). \tag{8.28}$$

(3) For $\epsilon \neq 0$ the $\tilde{\Xi}_{j,\epsilon,\pm}^+$ are related to the $\tilde{\Xi}_{j,\epsilon,\pm}^-$ by means of:

$$\begin{cases} \tilde{\Xi}_{j,\epsilon,+}^- = \tilde{\Xi}_{j,\epsilon,+}^+ \circ L_{j,\epsilon,+} = \tilde{\Xi}_{j,\epsilon,+}^+ \circ \Phi_{j,\epsilon,+}^+ \circ T_{-q\alpha} \circ (\Phi_{j,\epsilon,+}^-)^{-1} \\ \tilde{\Xi}_{j,\epsilon,-}^- = \tilde{\Xi}_{j+1,\epsilon,-}^+ \circ L_{j,\epsilon,-} = \tilde{\Xi}_{j+1,\epsilon,+}^+ \circ \Phi_{j+1,\epsilon,-}^+ \circ T_{-q\alpha} \circ (\Phi_{j,\epsilon,-}^-)^{-1}, \end{cases} \tag{8.29}$$

where $L_{j,\epsilon,\pm}$ are the Lavaurs translations defined in (4.15) and (4.16).

(4) The different functions $\tilde{\Xi}_{j,\epsilon,-}^\pm$ (resp. $\tilde{\Xi}_{j,\epsilon,+}^\pm$) all have the same limit at the upper end of the strip.

PROOF. For all the proof we remove the lower indices. For the first part of the proof we also drop the upper indices $\pm$. Let

$$\tilde{R}_{j,\epsilon}(W) = \sum_{i=0}^{q-1} \tilde{K}_{\sigma^i(j),\epsilon} \left(W + \frac{i}{q} \right). \tag{8.30}$$

Let $\hat{C} \subset \Phi_{j,\epsilon}(Q_{j,\epsilon})$ be a closed strip of width 1 with boundary $\ell \cup T_1(\ell)$, where ℓ is a line with same direction as $e^{i\beta}$, $\beta \in (\delta, \pi - \delta)$, and $T_1(W) = W + 1$. By Lemma 8.5, the definition of p_ϵ and the limit behaviour of the Fatou coordinates in (3.30) we have that $\tilde{R}_{j,\epsilon}(W) \to 0$ as $Re(We^{-i\beta}) \to \pm\infty$.

Let U_0 be an open neighborhood of $\hat{C}$. Let $\tilde{s}$ of the form

$$\tilde{s}(W) = s(Re(W) - \cot\beta \, Im(W)) = s(-\frac{Im(We^{-i\beta})}{\sin\beta}) \tag{8.31}$$

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where $s: \mathbb{R} \to \mathbb{R}$ is C^∞, monotone increasing and satisfies

$$s(x) = \begin{cases} 0 & x \leq x_1 \\ 1 & x \geq x_2 \end{cases} \tag{8.32}$$

with $x_1 < x_2$. The values x_1 and x_2 are chosen so that $\tilde{s} \equiv 0$ (resp. $\tilde{s} \equiv 1$) on a neighborhood V_1 (resp. V_2) of ℓ (resp. $T_1(\ell)$). Let $\hat{\Xi}_{j,\epsilon}(W) = \tilde{s}(W)\tilde{R}_{j,\epsilon}(W-1)$.) Then $\hat{\Xi}_{j,\epsilon}$ is a C^∞-solution of (8.26).

The first derivatives of $\tilde{s}$ are bounded as $Re(We^{-i\beta}) \to \pm\infty$. Hence the function $\hat{\Xi}_{j,\epsilon}$ satisfies

$$\lim_{Re(We^{-i\beta}) \to \pm\infty} \hat{\Xi}_{j,\epsilon}(W) = 0. \tag{8.33}$$

We must correct $\hat{\Xi}_{j,\epsilon}$ to an analytic function. As $\tilde{R}_{j,\epsilon}$ is analytic in W we have that

$$\overline{\partial}\hat{\Xi}_{j,\epsilon}(W+1) - \overline{\partial}\hat{\Xi}_{j,\epsilon}(W) = \overline{\partial}\tilde{R}_{j,\epsilon} = 0, \tag{8.34}$$

where $\overline{\partial} = \frac{\partial}{\partial \overline{W}}$. Hence $\overline{\partial}\hat{\Xi}_{j,\epsilon}$ is periodic of period 1. Moreover

$$\lim_{Re(We^{-i\beta}) \to \pm\infty} \overline{\partial}\hat{\Xi}_{j,\epsilon}(W) = 0, \tag{8.35}$$

as $\overline{\partial}\hat{\Xi}_{j,\epsilon}(W) = \overline{\partial}\tilde{s}(W)\tilde{R}_{j,\epsilon}(W-1)$. We look for a function H_ϵ, periodic of period 1, such that

$$\overline{\partial}\hat{\Xi}_{j,\epsilon} = \overline{\partial}H_\epsilon. \tag{8.36}$$

Then an analytic solution to (8.26) will be given by

$$\tilde{\Xi}_{j,\epsilon} = \hat{\Xi}_{j,\epsilon} - H_\epsilon. \tag{8.37}$$

To find H_ϵ we let

$$m_\epsilon(z) = \hat{\Xi}_{j,\epsilon}\left(-\frac{1}{2\pi i}\ln z\right). \tag{8.38}$$

The function $\frac{\partial m_\epsilon}{\partial \overline{z}} = \frac{\overline{\partial}\hat{\Xi}_{j,\epsilon}}{\partial \overline{W}}\frac{\partial \overline{W}}{\partial \overline{z}} = \frac{1}{2\pi i \overline{z}}\frac{\overline{\partial}\hat{\Xi}_{j,\epsilon}}{\partial \overline{W}}$ is C^∞ on $\mathbb{C}^*$ and bounded in the neighborhood of 0 (resp. $O(\frac{1}{\overline{z}^2})$ in the neighborhood of ∞) on $\mathbb{CP}^1$ because of (8.23). We will show the existence of a function n_ϵ defined on $\mathbb{CP}^1$ which will be C^∞ on $\mathbb{C}^* = \mathbb{CP}^1 \setminus \{0, \infty\}$ and C^0 on $\mathbb{CP}^1$ and such that on $\mathbb{C}^*$

$$\frac{\partial m_\epsilon}{\partial \overline{z}} = \frac{\partial n_\epsilon}{\partial \overline{z}}. \tag{8.39}$$

The function H_ϵ we are looking for is

$$H_\epsilon(W) = n_\epsilon(e^{-2\pi i W}). \tag{8.40}$$

Indeed H_ϵ is periodic of period 1 and satisfies (8.36) since

$$\frac{\partial H_\epsilon}{\partial \overline{W}} = \frac{\partial n_\epsilon}{\partial z}\frac{\partial z}{\partial \overline{W}} + \frac{\partial n_\epsilon}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial \overline{W}} = 2\pi i \overline{z}\frac{\partial n_\epsilon}{\partial \overline{z}} = \frac{\partial \hat{\Xi}_{j,\epsilon}}{\partial \overline{W}}. \tag{8.41}$$

To construct n_ϵ we now consider a covering of $\mathbb{CP}^1$ by two connected open sets $U_1 = \{z; |z| < \rho_1\}$ and $U_2 = \{z; |z| > \rho_2\}$ with $0 < \rho_2 < \rho_1$. Let $U_1^* = U_1 \setminus \{0\}$ and $U_2^* = U_2 \setminus \{\infty\}$. Moreover the closure $\overline{U}_i$ of each U_i can be identified to a compact set of $\mathbb{C}$. On each U_i we will construct in Lemma 8.7 functions $n_{i,\epsilon}$ with are C^0 on U_i and C^∞ on U_i^* and such that $\overline{\partial}n_{i,\epsilon} = \overline{\partial}m_\epsilon$ on U_i^*. The difference $n_{12,\epsilon} = n_{1,\epsilon} - n_{2,\epsilon}$ is analytic on $U_1 \cap U_2 = \{z; \rho_2 < |z| < \rho_1\}$, hence it can be expanded as a Laurent series:

$$n_{12,\epsilon} = \sum_{j \in \mathbb{Z}} a_j(\epsilon)z^j. \tag{8.42}$$

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The holomorphic functions

$$\begin{cases} r_{1,\epsilon}(z) = \sum_{j=0}^{\infty} a_j(\epsilon) z^j & z \in U_1 \\ r_{2,\epsilon}(z) = -\sum_{j=-\infty}^{-1} a_j(\epsilon) z^j & z \in U_2 \end{cases} \tag{8.43}$$

are analytic on their domain and satisfy $n_{12,\epsilon} = r_{1,\epsilon} - r_{2,\epsilon}$. The function n_ϵ we are looking for is given by

$$n_\epsilon(z) = n_{i,\epsilon}(z) - r_{i,\epsilon}(z) \qquad \text{for } z \in U_i. \tag{8.44}$$

Any other solution differs from this one by a global holomorphic function on $\mathbb{CP}^1$, i.e. a constant. In particular $n_\epsilon(z)$ is uniquely defined if we ask that $n_\epsilon(0) = 0$.

We then extend $\tilde{\Xi}_{j,\epsilon}$ by iterating (8.26) on $\Phi_{j,\epsilon}(Q_{j,\epsilon})$. The function defined here depends analytically on $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$: this comes from the explicit construction of $n_{j,\epsilon}$ and $r_{j,\epsilon}$.

For the proof of (2) we construct $\tilde{\Xi}_{1,\epsilon}$ as in (1). The other $\tilde{\Xi}_{j,\epsilon}$ are constructed from $\tilde{\Xi}_{1,\epsilon}$ by means of (8.28) and yield solutions of (8.26). By uniqueness of solutions of (8.26) they must be the solutions constructed in (1).

The proof of (3) follows since F_ϵ and G_ϵ commute with $T_{q\alpha}$, K_ϵ is invariant under $T_{q\alpha}$ and $\phi_{j,\epsilon}^+ \circ T_{-q\alpha} \circ (\phi_{j,\epsilon}^-)^{-1}$ commutes with T_1.

To prove (4) we first remark that $\lim_{Re(We^{-i\beta})\to+\infty} \tilde{\Xi}_{j,\epsilon}^\pm$ exists since $\lim_{Re(We^{-i\beta})\to\pm\infty} \hat{\Xi}_{j,\epsilon} = 0$, from (3.30) and from $\lim_{Re(We^{-i\beta})\to+\infty} H_\epsilon(W) = n_\epsilon(\infty) \in \mathbb{C}$. The fact that it is the same for all $\tilde{\Xi}_{j,\epsilon}^\pm$ comes from (8.28) and (8.29). $\qquad\square$

Lemma 8.7 *There exist functions $n_{i,\epsilon}$ with are C^0 on U_i and C^∞ on U_i^*, and such that*

$$\frac{\partial n_{i,\epsilon}}{\partial \overline{z}} = \frac{\partial m_\epsilon}{\partial \overline{z}} \tag{8.45}$$

on U_i^*.

PROOF. We first construct $n_{1,\epsilon}$. Let $\delta_1 > 0$ be small and let $U_{1,\delta_1} = \{z; \delta_1 < |z| < \rho_1\}$. Let b_{ϵ,δ_1} be C^∞ on U_1 such that

$$b_{\epsilon,\delta_1} = \begin{cases} \frac{\partial m_\epsilon}{\partial \overline{z}} & z \in U_{1,\delta_1} \\ 0 & |z| < \frac{\delta_1}{2}. \end{cases} \tag{8.46}$$

Let

$$u_{\epsilon,\delta_1}(z) = \iint_{\mathbb{C}} \frac{b_{\epsilon,\delta_1}(\zeta)}{\zeta - z} \, d\zeta \wedge d\overline{\zeta}. \tag{8.47}$$

By Theorem 1.2.2 of [9] the function u_{ϵ,δ_1} is C^∞ on $\mathbb{C}$ and analytic outside U_{1,δ_1}. The function $n_{1,\epsilon}$ we are looking for will be

$$n_{1,\epsilon} = \lim_{\delta_1 \to 0} u_{\epsilon,\delta_1}. \tag{8.48}$$

It will be C^∞ on U_1^* and C^0 at the origin provided we show that

$$\iint_{|\zeta|<\delta_1} \frac{\frac{\partial m_\epsilon}{\partial \overline{\zeta}}(\zeta)}{\zeta} \, d\zeta \wedge d\overline{\zeta} = -2i \int_0^{2\pi} e^{-i\theta} \left(\int_0^{\delta_1} \frac{\partial m_\epsilon}{\partial \overline{z}}(re^{i\theta}) dr \right) d\theta \tag{8.49}$$

tends to 0 as δ_1 tends to zero. This comes from the fact that $\frac{\partial m_\epsilon(z)}{\partial \overline{z}}$ is bounded in the neighborhood of the origin.

The construction of $n_{2,\epsilon}$ is similar. The result follows from the fact that $\frac{\partial m_\epsilon(z)}{\partial \overline{z}}$ is $O(\frac{1}{\overline{z}^2})$ in the neighborhood of ∞. $\qquad\square$

Proposition 8.8 *The functions*

$$\begin{cases} \tilde{M}^0_{j,\epsilon,\pm} = \tilde{\Xi}^-_{j,\epsilon,\pm} - \tilde{\Xi}^+_{j+1,\epsilon,\pm} \circ (\Psi^0_{j,\epsilon,\pm})^{-1} \\ \tilde{M}^\infty_{j,\epsilon,\pm} = \tilde{\Xi}^-_{j,\epsilon,\pm} - \tilde{\Xi}^+_{j,\epsilon,\pm} \circ (\Psi^\infty_{j,\epsilon,\pm})^{-1} \end{cases} \tag{8.50}$$

are periodic of period 1. They are defined on the same domains as the corresponding $\Psi^{0,\infty}_{j,\epsilon,\pm}$, *namely* $Q^-_{j,\epsilon,\pm} \cap \Psi^0_{j,\epsilon,\pm}(Q^+_{j+1,\epsilon,\pm})$ *(resp.* $Q^-_{j,\epsilon,\pm} \cap \Psi^\infty_{j,\epsilon,\pm}(Q^+_{j,\epsilon,\pm})$*) for* $\tilde{M}^0_{j,\epsilon,\pm}$ *(resp.* $\tilde{M}^\infty_{j,\epsilon,\pm}$*).*

Moreover they are all determined by $\tilde{M}^{0,\infty}_{1,\epsilon,\pm}$ *since*

$$\begin{aligned} \tilde{M}^0_{\sigma(j),\epsilon,\pm}(W) &= \tilde{M}^0_{j,\epsilon,\pm}(W - \tfrac{1}{q}), \\ \tilde{M}^\infty_{\sigma(j),\epsilon,\pm}(W) &= \tilde{M}^\infty_{j,\epsilon,\pm}(W - \tfrac{1}{q}). \end{aligned} \tag{8.51}$$

Let

$$\zeta^{0,\infty}_{j,\epsilon,\pm} = \tilde{M}^{0,\infty}_{j,\epsilon,\pm} \circ E^{-1}. \tag{8.52}$$

The functions $\zeta^0_{j,\epsilon,\pm}$ *(resp* $\zeta^\infty_{j,\epsilon,\pm}$*) are germs of analytic functions defined respectively in the neighborhood of 0 (resp.* ∞*) on* $\mathbb{CP}^1$ *and vanishing at 0 (resp.* ∞*) provided the different* $\tilde{\Xi}^\pm_{j,\epsilon,\pm}$ *have the same limit in the direction* $Re(We^{-i\beta}) \to -\infty$ *(resp.* $Re(We^{-i\beta}) \to \infty$*). This last fact follows from the recurrence relations* (8.28) *and* (8.29)*. The functions* $\zeta^{0,\infty}_{j,\epsilon,\pm}$ *depend analytically on* $\epsilon \in V_{\delta,\pm}$ *for* $\epsilon \neq 0$ *and continuously on* ϵ *near* $\epsilon = 0$.

Definition 8.9 The families of 2-tuples $(\zeta^0_{1,\epsilon,\pm}, \zeta^\infty_{1,\epsilon,\pm})|_{\epsilon \in V_{\delta,\pm}}$ is the *time-part of the modulus.*

8.3   The modulus of analytic conjugacy for families

Definition 8.10 Two germs of analytic families of vector fields, $X_{\epsilon_1}(x_1, y_1)$ (resp. $\overline{X}_{\epsilon_2}(x_2, y_2)$) unfolding a resonant saddle at the origin for $\epsilon_1 = 0$ (resp. $\epsilon_2 = 0$) are analytically conjugate if there exists a germ of analytic diffeomorphism

$$K = (h, H): \quad (\epsilon_1, x_1, y_1) \mapsto (h(\epsilon_1), H(\epsilon_1, x_1, y_1)) \tag{8.53}$$

fibered over the parameter space, where

  i) $h : \epsilon_1 \mapsto \epsilon_2 = h(\epsilon_1)$ is a germ of analytic diffeomorphism preserving the origin;

 ii) if we let $H_{\epsilon_1}(x_1, y_1) = H(x_1, y_1, \epsilon_1)$, the change of coordinates H_{ϵ_1} is a conjugacy between $X_{\epsilon_1}(x_1, y_1)$ and $\overline{X}_{h(\epsilon_1)}(x_2, y_2)$ over a ball of small radius $r > 0$:

$$\overline{X}_{h(\epsilon_1)}(H_{\epsilon_1}(x_1, y_1)) = (H_{\epsilon_1})_*(X_{\epsilon_1}(x_1, y_1)). \tag{8.54}$$

Definition 8.11   1. We consider the set $\mathcal{D} = \mathrm{Diff}^0 \times \mathrm{Diff}^\infty \times \mathcal{H}^0_0 \times \mathcal{H}^\infty_0$, where Diff^0 (resp. Diff^∞) is the set of germs of analytic diffeomorphims in the neighborhood of 0 (resp. ∞) on $\mathbb{CP}^1$ fixing 0 (resp. ∞) and $\mathcal{H}^0_0$ (resp. $\mathcal{H}^\infty_0$) is the set of germs of holomorphic functions in the neighborhood of 0 (resp. ∞) on $\mathbb{CP}^1$ sending 0 (resp. ∞) to 0.

   2. We consider the following equivalence relation on $\mathcal{D}$

$$(\psi^0, \psi^\infty, \zeta^0, \zeta^\infty) \equiv (\overline{\psi}^0, \overline{\psi}^\infty, \overline{\zeta}^0, \overline{\zeta}^\infty) \iff \exists C \in \mathbb{C}^* \begin{cases} \psi^0(Cw) = C\overline{\psi}^0(w) \\ \psi^\infty(Cw) = C\overline{\psi}^\infty(w) \\ \zeta^0(Cw) = \overline{\zeta}^0(w) \\ \zeta^\infty(Cw) = \overline{\zeta}^\infty(w). \end{cases} \tag{8.55}$$

We note by $[(\psi^0, \psi^\infty, \zeta^0, \zeta^\infty)]$ the equivalence class of an element of $\mathcal{D}$.

3. We call $\mathcal{N}$ the set of equivalence classes of $\mathcal{D}$.

Theorem 8.12 *We consider a germ of family of vector fields $X_\epsilon = \lambda_1(\epsilon) h_{1,\epsilon} \Theta_\epsilon$ as in (8.4) in prepared form. Then the families $(\mathcal{M}_{\epsilon,\pm})|_{\epsilon \in V_{\delta,\pm}}$, with*

$$\mathcal{M}_{\epsilon,\pm} = [(\psi^0_{1,\epsilon,\pm}, \psi^\infty_{1,\epsilon,\pm}, \zeta^0_{1,\epsilon,\pm}, \zeta^\infty_{1,\epsilon,\pm})] \in \mathcal{N} \tag{8.56}$$

together with the formal invariants $a(0)$, $t_1(\epsilon), t_2(\epsilon)$, is a complete modulus of classification under analytic conjugacy. Moreover it is possible to choose families of representatives $(\psi^0_{1,\epsilon,\pm}, \psi^\infty_{1,\epsilon,\pm}, \zeta^0_{1,\epsilon,\pm}, \zeta^\infty_{1,\epsilon,\pm})|_{\epsilon \in V_{\delta,\pm}}$ depending analytically of ϵ for $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$.

PROOF. If two germs of families of vector fields are analytically conjugate it is already known that they have the same orbital part of the modulus. Also they can both be brought through an analytic change of coordinates to the respective forms $X_\epsilon = h_{1,\epsilon} \Theta_\epsilon$ and $\overline{X}_\epsilon = \overline{h}_{1,\epsilon} \Theta_\epsilon$, with same Θ_ϵ. We know that there is a change of coordinates H_ϵ transforming X_ϵ into $\overline{X}_\epsilon$. This change of coordinates induces a conjugacy between the holonomies (indeed H_ϵ sends $\{x = 1\}$ to a section Σ transversal to the flow of $\overline{X}_\epsilon$ and we compose with a transition map for $\overline{X}_\epsilon$ from Σ to $\{x = 1\}$.) We can apply a change of coordinates $H_{1,\epsilon} = (x, \beta y + o(|x, y|))$ to $\overline{X}_\epsilon$ as in (6.13) so that the two vector fields $\hat{X}_\epsilon = (H_{1,\epsilon})_*(\overline{X}_\epsilon)$ and X_ϵ have the same holonomies on $x = 1$. Then $H_{1,\epsilon} \circ H_\epsilon$ preserves the holonomy. Hence it induces a symmetry of the holonomy. The symmetries of the holonomy have been described in Proposition 4.18. A symmetry of the holonomy can be extended to a symmetry $H_{2,\epsilon}$ of the foliation described Θ_ϵ, by the same method as in the proof of Theorem 6.5. Then $H_{2,\epsilon}$ is a change of coordinates preserving the foliation described by Θ_ϵ and transforming X_ϵ into $\tilde{X}_\epsilon$. Then $\tilde{X}_\epsilon$ and $\hat{X}_\epsilon$ have the same holonomies and moreover the conjugacy induced by $H_{1,\epsilon} \circ H_\epsilon \circ (H_{2,\epsilon})^{-1}$ between the holonomies is the identity. Then the map $H_{1,\epsilon} \circ H_\epsilon \circ (H_{2,\epsilon})^{-1}$ preserves the leaves and is hence given by the flow $\tilde{X}^{\xi_\epsilon}_\epsilon$ of the $\tilde{X}_\epsilon$ for some time function ξ_ϵ. It follows that the two families of vector fields have the same time part of the modulus.

The converse requires more work. We take two germs of analytic families of vector fields which have the same modulus. We then know that they are orbitally equivalent, so we can always suppose that they have the same prepared orbital form with same canonical parameter ϵ: $X_\epsilon = \lambda_1(\epsilon) h_{1,\epsilon} \Theta_\epsilon$ and $\overline{X}_\epsilon = \lambda_1(\epsilon) \overline{h}_{1,\epsilon} \Theta_\epsilon$ over each of the two sectors $V_{\delta,\pm}$. For the rest of the proof we drop the lower indices $\pm$ referring to the sectors.

We will look for a conjugacy of a certain type namely the flow of X_ϵ for a certain time $\chi_\epsilon(x, y)$. This flow map preserves the leaves over a fixed neighborhood of the origin for sufficiently small ϵ. It is shown in [4], [24] and [25] that X_ϵ is mapped to $\overline{X}_\epsilon$ under the flow of X_ϵ for some time $\chi_\epsilon(x, y)$ if

$$X_\epsilon(\chi_\epsilon) = \frac{\overline{h}_{1,\epsilon}}{h_{1,\epsilon}} - 1 = \tau_\epsilon \tag{8.57}$$

(see [4]). Hence we must construct a solution of (8.57).

We first choose a function $\chi_{1,\epsilon}(y)$ defined on $\{x = 1\}$ and satisfying

$$\chi_{1,\epsilon}(f(y)) - \chi_{1,\epsilon}(y) = \int_{\gamma(y)} \tau_\epsilon \, dt_{X_\epsilon}, \tag{8.58}$$

where we integrate over $\gamma(y)$ which is the lifting of the curve $x = \exp(2\pi i \theta)$, $\theta \in [0, 2\pi]$ and dt_{X_ϵ} is the time form of X_ϵ. We then extend it to a neighborhood of $(x, y) = (0, 0)$ except on $x = 0$ by considering trajectories $(x(t), y(t))$ of X_ϵ through $(1, y_0)$. By passing to the Z-variable, where $Z = p_\epsilon^{-1}(y)$, it suffices to find a function $\hat{\chi}_{1,\epsilon}(Z)$ such that

$$\hat{\chi}_{1,\epsilon}(F(Z)) - \hat{\chi}_{1,\epsilon}(Z) = K_\epsilon(Z) - \overline{K}_\epsilon(Z). \tag{8.59}$$

To construct explicitly $\chi_{1,\epsilon}$ we first define functions

$$\begin{aligned} \Xi^\pm_{j,\epsilon} &= \tilde{\Xi}^\pm_{j,\epsilon} \circ \Phi^\pm_{j,\epsilon} \\ \overline{\Xi}^\pm_{j,\epsilon} &= \tilde{\overline{\Xi}}^\pm_{j,\epsilon} \circ \Phi^\pm_{j,\epsilon} \end{aligned} \tag{8.60}$$

on $(\Phi^{\pm}_{j,\epsilon})^{-1}(Q_{j,\epsilon})$ and in U. (Note that X_ϵ and $\overline{X}_\epsilon$ have the same Fatou coordinates $\Phi^{\pm}_{j,\epsilon}$.)

If $q = 1$ it is natural to take $\chi_{1,\epsilon} = \xi_{1,\epsilon} - \overline{\xi}_{1,\epsilon}$ which is well defined because the two vector fields have the same modulus. Then we would have

$$
\begin{aligned}
\chi_{1,\epsilon}(f(y)) - \chi_{1,\epsilon}(y) &= \int_{\gamma(y)} \left(\frac{h_{1,\epsilon}}{1 + C(\epsilon)u} - 1 \right) dt_{X_\epsilon} - \left(\frac{\overline{h}_{1,\epsilon}}{1 + C(\epsilon)u} - 1 \right) dt_{\overline{X}_\epsilon} \\
&= \int_{\gamma(y)} \left(\frac{h_{1,\epsilon}}{1 + C(\epsilon)u} - 1 \right) \frac{dx}{\lambda_1(\epsilon)x h_{1,\epsilon}} - \left(\frac{\overline{h}_{1,\epsilon}}{1 + C(\epsilon)u} - 1 \right) \frac{dx}{\lambda_1(\epsilon)x \overline{h}_{1,\epsilon}} \quad (8.61) \\
&= \int_{\gamma(y)} \frac{h_{1,\epsilon} - \overline{h}_{1,\epsilon}}{\lambda_1(\epsilon)x} \frac{dx}{h_{1,\epsilon}\overline{h}_{1,\epsilon}} = \int_{\gamma(y)} \left(\frac{h_{1,\epsilon}}{\overline{h}_{1,\epsilon}} - 1 \right) dt_{X_\epsilon}.
\end{aligned}
$$

In the case $q > 1$ we need to be more subtle and adjust to the fact that the functions $\xi^{\pm}_{j,\epsilon}$ are defined by integrating over a time corresponding to the q-th iterate of the holonomy map. The details are as follows. We first show that the function

$$
N_\epsilon(Z) = \Xi^{\pm}_{j,\epsilon}(Z) - \overline{\Xi}^{\pm}_{j,\epsilon}(Z), \quad Z \in (\Phi^{\pm}_{j,\epsilon})^{-1}(Q^{\pm}_{j,\epsilon}) \quad (8.62)
$$

is well defined. Indeed, let

$$
\begin{aligned}
M^{0,\infty}_{j,\epsilon} &= \tilde{M}^{0,\infty}_{j,\epsilon} \circ \Phi^{-}_{j,\epsilon} \\
\overline{M}^{0,\infty}_{j,\epsilon} &= \overline{\tilde{M}}^{0,\infty}_{j,\epsilon} \circ \Phi^{-}_{j,\epsilon}.
\end{aligned} \quad (8.63)
$$

Then $M^{0,\infty}_{j,\epsilon} = \overline{M}^{0,\infty}_{j,\epsilon}$. Moreover

$$
\begin{aligned}
M^{0}_{j,\epsilon} &= \Xi^{-}_{j,\epsilon} - \Xi^{+}_{j+1,\epsilon} \\
M^{\infty}_{j,\epsilon} &= \Xi^{-}_{j,\epsilon} - \Xi^{+}_{j,\epsilon}.
\end{aligned} \quad (8.64)
$$

All this together implies that N_ϵ is well defined.

The function $\chi_{1,\epsilon}$ is given by:

$$
\chi_{1,\epsilon}(y) = \hat{\chi}_{1,\epsilon} \circ p^{-1}_\epsilon \quad (8.65)
$$

with

$$
\begin{aligned}
\hat{\chi}_{1,\epsilon}(Z) &= \Xi^{\pm}_{\sigma(j),\epsilon}(F_\epsilon(Z)) - \Xi^{\pm}_{j,\epsilon}(Z) - [\overline{\Xi}^{\pm}_{\sigma(j),\epsilon}(F_\epsilon(Z)) - \overline{\Xi}^{\pm}_{j,\epsilon}(Z)] \\
&= N_\epsilon(F_\epsilon(Z)) - N_\epsilon(Z),
\end{aligned} \quad (8.66)
$$

for $Z \in (\Phi^{\pm}_{j,\epsilon})^{-1}(Q_{j,\epsilon})$. Since N_ϵ is well defined, so is $\hat{\chi}_{1,\epsilon}$, and hence $\chi_{1,\epsilon}$. Moreover (8.59) is satisfied since

$$
\begin{aligned}
\Xi^{\pm}_{\sigma(j),\epsilon}(F_\epsilon(Z)) - \Xi^{\pm}_{j,\epsilon}(Z) &= \tilde{\Xi}^{\pm}_{\sigma(j),\epsilon} \circ \Phi^{\pm}_{\sigma(j),\epsilon} \circ F(Z) - \tilde{\Xi}^{\pm}_{j,\epsilon} \circ \Phi^{\pm}_{j,\epsilon}(Z) \\
&= \tilde{\Xi}^{\pm}_{j,\epsilon} \circ T_{-\frac{1}{q}} \circ \Phi^{\pm}_{\sigma(j),\epsilon} \circ F(Z) - \tilde{\Xi}^{\pm}_{j,\epsilon} \circ \Phi^{\pm}_{j,\epsilon}(Z) \\
&\quad + \tilde{K}_{j,\epsilon} \circ T_{-\frac{1}{q}} \circ \Phi^{\pm}_{\sigma(j)} \circ F(Z) \\
&= K_\epsilon(Z),
\end{aligned} \quad (8.67)
$$

because of (4.7), and similarly

$$
\overline{\Xi}^{\pm}_{\sigma(j),\epsilon}(F_\epsilon(Z)) - \overline{\Xi}^{\pm}_{j,\epsilon}(Z) = \overline{K}_\epsilon(Z), \quad (8.68)
$$

yielding (8.59).

We must now extend $\chi_{1,\epsilon}(y)$ to a function $\chi_\epsilon(x, y)$. Indeed we consider a solution curve $(x(t), y(t))$ of T_ϵ with $(x(0), y(0)) = (1, y_0)$ and we let

$$
\chi_\epsilon(x(t), y(t)) = \int_0^t \tau_\epsilon(x(s), y(s)) ds. \quad (8.69)
$$

We must show that χ_ϵ is well defined outside of $x = 0$ and bounded in the neighborhood of $x = 0$, thus having an analytic extension at $x = 0$. This is done in Lemma 8.13 below.

The fact that χ_ϵ is well defined comes from (8.58).

On each of the sectors $V_{\epsilon,\pm}$ we have constructed a function $\chi_{\epsilon,\pm}$ depending analytically of $\epsilon \neq 0$ and continuously on ϵ near $\epsilon = 0$. The last step of the proof is to show that the two functions $\chi_{\epsilon,\pm}$ glue together in a uniform function χ_{ϵ}, which is therefore holomorphic. It suffices to show that the two functions coincide on the intersection of the two sectors $V_{\epsilon,\pm}$. Indeed on the intersection of the two sectors we have two functions $\chi_{\epsilon,+}$ and $\chi_{\epsilon,-}$ such that the flow $X_{\epsilon}^{\chi_{\epsilon,\pm}}$ of the vector field X_{ϵ} for a time $\chi_{\epsilon,\pm}$ is a conjugacy between X_{ϵ} and $\overline{X}_{\epsilon}$. Then the flow $X_{\epsilon}^{\chi_{\epsilon,+} - \chi_{\epsilon,-}}$ is a conjugacy between X_{ϵ} and itself. Hence the function $\chi_{\epsilon,+} - \chi_{\epsilon,-}$ is a solution of (8.58) in the particular case $\tau_{\epsilon} \equiv 0$. Constant functions are obviously solutions of (8.58) when $\tau_{\epsilon} \equiv 0$. To conclude that $\chi_{\epsilon,+} - \chi_{\epsilon,-}$ is constant we need some form of unicity. This is obtained by applying Theorem 8.6 with the particular function $K_{\epsilon} \equiv 0$. We know that the solutions $\tilde{\Xi}_{j,\epsilon}$ of the equation (8.26) are unique up to a constant, hence they are precisely the constant functions. As moreover we have that $\chi_{\epsilon,\pm}(1,0) = 0$ then $\chi_{\epsilon,+}(1,y) - \chi_{\epsilon,-}(1,y) \equiv 0$ and finally $\chi_{\epsilon,+} = \chi_{\epsilon,-}$ using (8.69). $\square$

Lemma 8.13 *The function $\chi_{\epsilon}(x,y)$ defined in a neighborhood of the origin minus the axis $x = 0$ is uniformly bounded.*

PROOF. As a natural trajectory from a point (x_0, y_0) with $x_0 \neq 0$ to $(1, y(T(x_0, y_0)))$ we first take a trajectory γ_1 with time $t = t'\lambda_1(\epsilon)$ with t' real, $t' \in [0, T_1(x_0, y_0)]$, until we reach a point (x_1, y_1) with $|x_1| = 1$, and we follow by a lifting γ_2 of $(e^{i\theta}, 0)$ with initial condition (x_1, y_1) until we reach $(1, y(T(x_0, y_0)))$. The time from (x_1, y_1) to $(1, y(T(x_0, y_0)))$ is uniformly bounded. As the function τ_{ϵ} is uniformly bounded this part of the integral is bounded. We concentrate on the integral on the first path. There we have that for small $\eta_1 > 0$ there exists $\eta_2 > 0$ such that for $|u| < \eta_2$

$$|x|(1 - \eta_1) \leq \frac{d|x|}{dt'} \leq |x|(1 + \eta_1). \tag{8.70}$$

Hence

$$\left| \int_{\gamma_1} \tau_{\epsilon} \, dt \right| \leq |\lambda_1(\epsilon)| \int_{\gamma_1} \frac{\tau_{\epsilon}}{\frac{d|x|}{dt'}} \, d|x|. \tag{8.71}$$

The result follows as $\tau_{\epsilon} = O(u)$. $\square$

9 Directions for further research

We mention some natural directions for further research:

1. The first is to identify precisely the modulus space for germs of analytic families of diffeomorphisms unfolding a germ of generic resonant diffeomorphism. This is equivalent to give the set of families ψ_{ϵ} of 2-tuples of $(Diff^{\infty} \times Diff^{0})$ which can be realized as the modulus of a generic family unfolding a resonant difformorphism. The difficulty in this problem comes from the unknown behaviour in ϵ of the $\psi_{\epsilon,j,\pm}^{0,\infty}$ at $\epsilon = 0$. We conjecture that there is probably some form of 1-summability in ϵ, a stronger conclusion than only continuity in ϵ near $\epsilon = 0$.

2. The second is to generalize the previous results for higher codimensions. The Fatou coordinates for individual vector fields have already been constructed by Oudkerk ([18] and more recent work).

3. Any resonant germ of diffeomorphism can be realized as the holonomy map of a resonant saddle of a 2-dimensional vector field [20]. Can any germ of family of diffeomorphisms unfolding a germ of resonant diffeomorphism be realized as the family of holonomy maps of a germ of family of 2-dimensional vector fields unfolding a resonant saddle?

4. Finally the last question is to identify the modulus space for germs of generic families of vector fields unfolding a germ of vector field with a generic resonant saddle.

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