Geometry and dynamics on infinite-type flat surfaces.

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École en Systèmes Dynamiques Contemporains, 2017.
Université de Montréal

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Examples of infinite-type translation surfaces.

Example: polygonal billiards

Consider the dynamical system defined by the frictionless motion of a point inside an Euclidean polygon P where collisions with the boundary are elastic, that is, **each time a point hits a side of the polygon its angle of incidence is equal to its angle of reflection**.

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Convention: the motion of a point ends when reaching a corner.

Conjecture: every **triangular** billiard has a closed trajectory.

"It is fair to say that this 200-year-old problem is widely regarded as impenetrable." **R.E. Schwartz**

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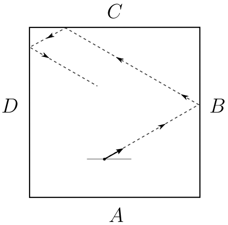
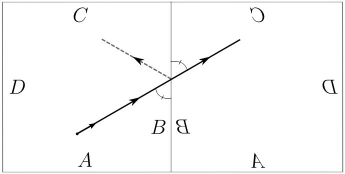
The unfolding trick





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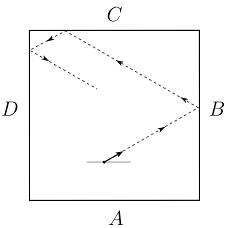

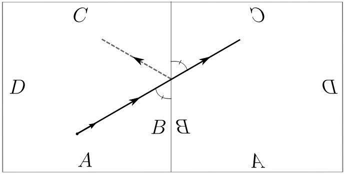

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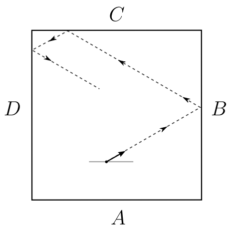

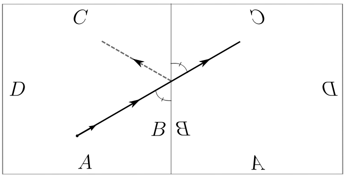

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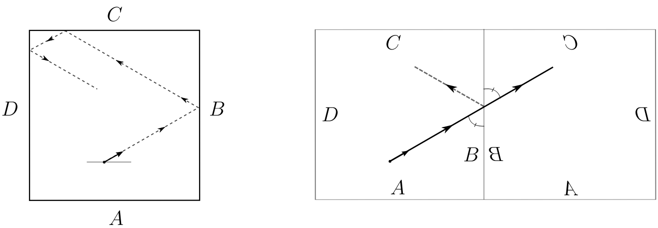

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> **Remark**. By construction, change of coordinates in S_P are **translations**.

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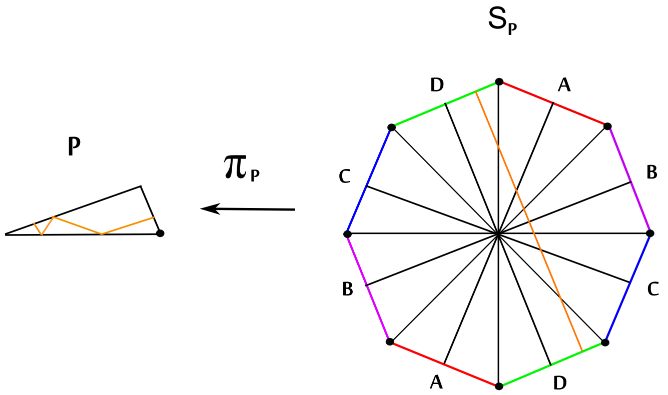

Example:polygonal billiards

Thm. Let F_θ^t be the translation flow on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

1. F_θ^t is periodic iff $\tan(\theta) = \frac{p}{q}$
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A similar statement is valid for the 2-torus arising from the triangle $(\frac{\pi}{2}, \frac{\pi}{8})$. (**Key word**: *Veech's dychotomy*.)

Example:polygonal billiards

Exercise. Let P be a polygon whose interior angles are of the form $\frac{p_i}{q_i}\pi$, $i = 1, \ldots, n$ and $N = lcm(q_1, \ldots, q_n)$. Then S_P is a closed oriented surface whose genus $g(S_P)$ is given by:

$$g(S_P) = 1 + \frac{N}{2}\left(n - 2 - \sum_{1}^{n} \frac{1}{q_i}\right)$$

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These kind of polygons are called *rational polygons*. All other polygons are said to be irrational.

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Natural question: what can be said about the topology of S_P when P is irrational?

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Example:polygonal billiards, irrational case

Theorem. Let P be an irrational polygon. Then S_P has infinite genus and only one end.

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Up to homeo. **there is only one** orientable surface of infinite genus and one end. It is called the **Loch Ness monster** (Phillips-Sullivan/Ghys).

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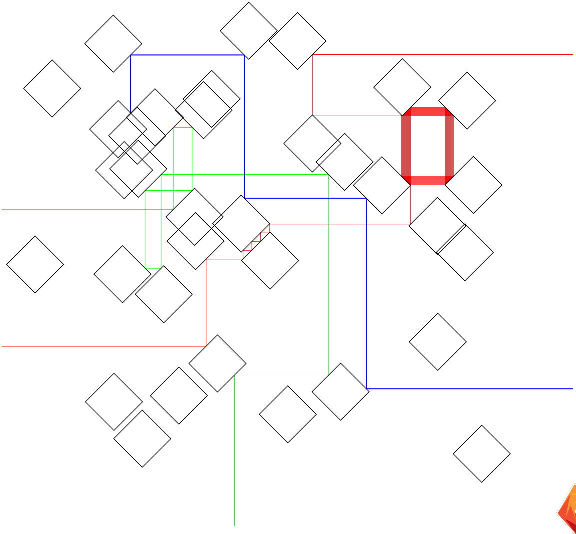
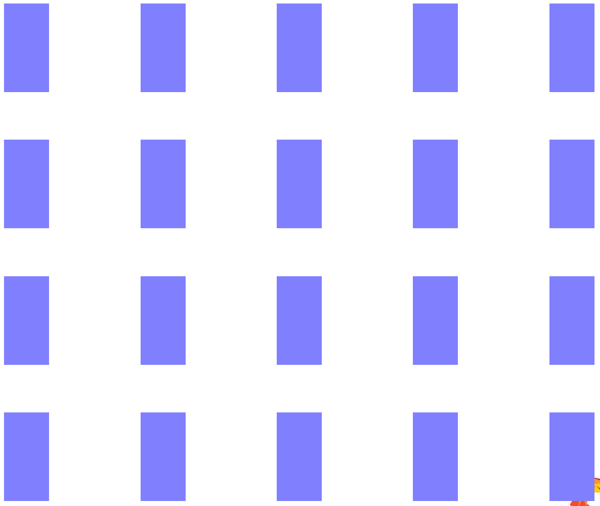
Back to periodic orbits

Conjecture. Let P be an irrational polygon and S_P the corresponding Loch Ness monster obtained by the unfolding-trick. There exists a direction θ for which F_θ^t has a periodic orbit.

Example: The *Ehrenfest*^{2} Wind-tree model (1912)



Example: Hardy-Weber's periodic wind-tree model (1980)



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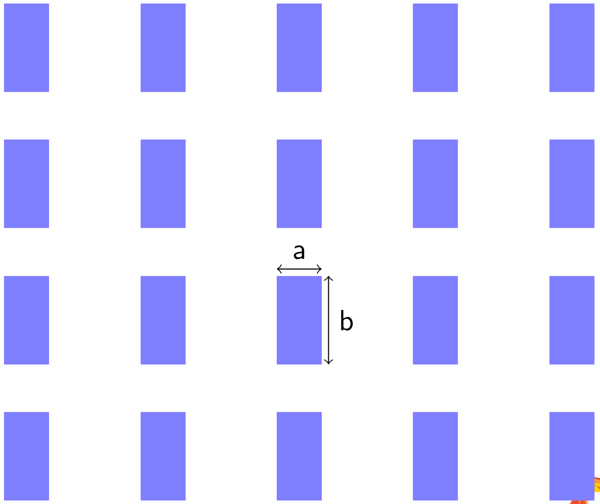

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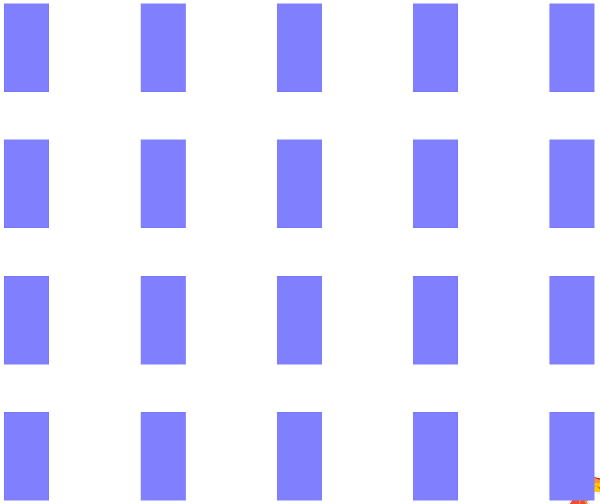

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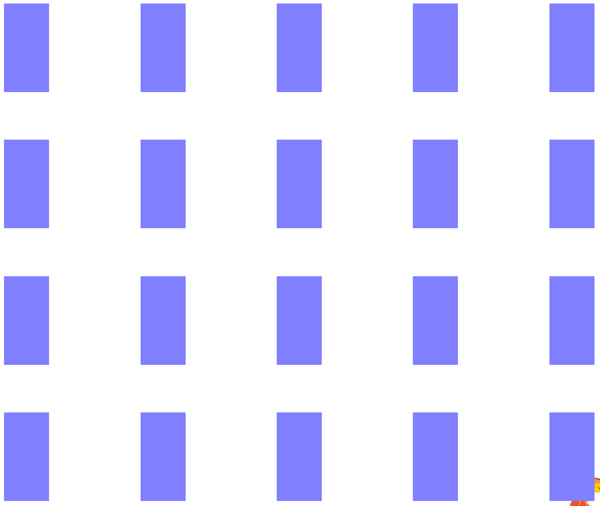

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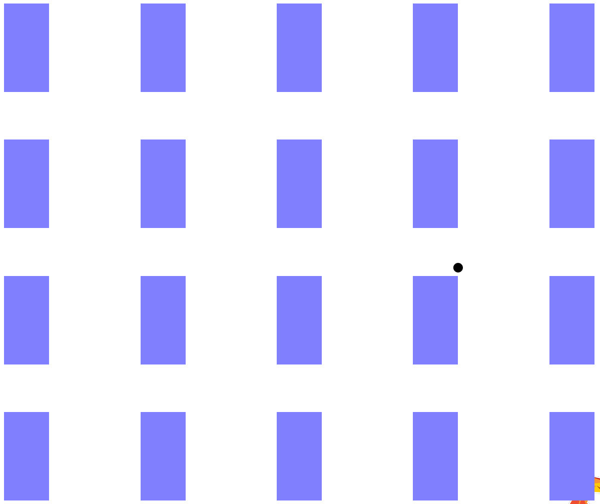

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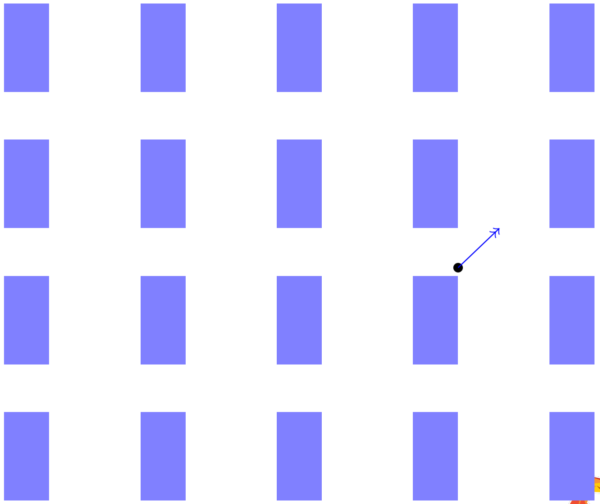

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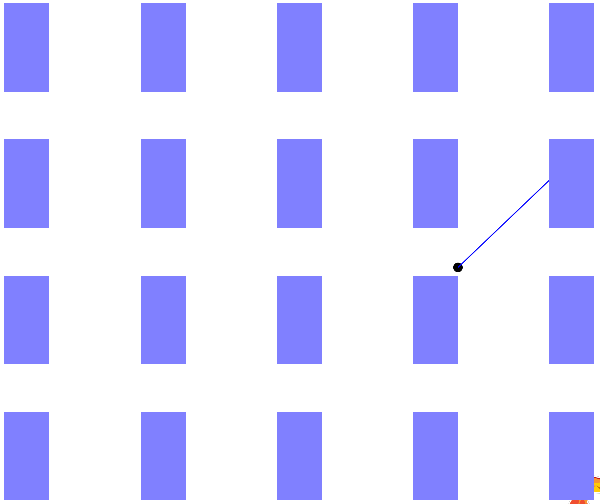

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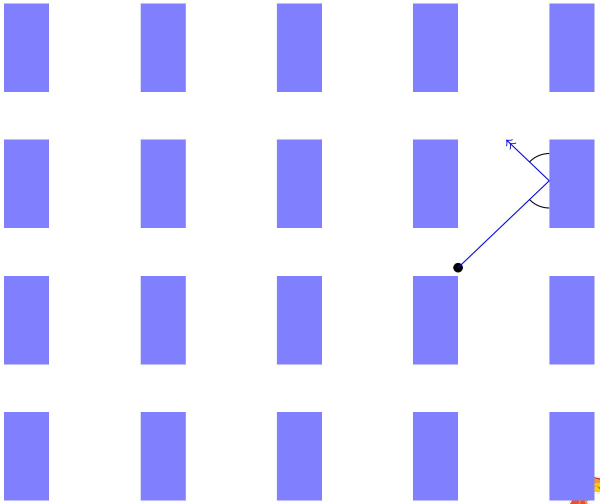

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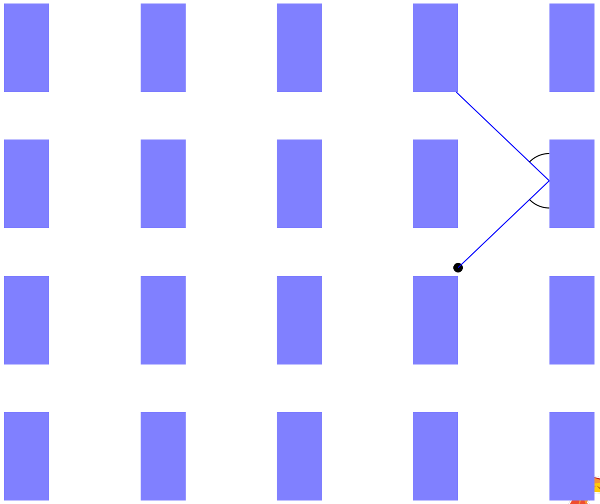

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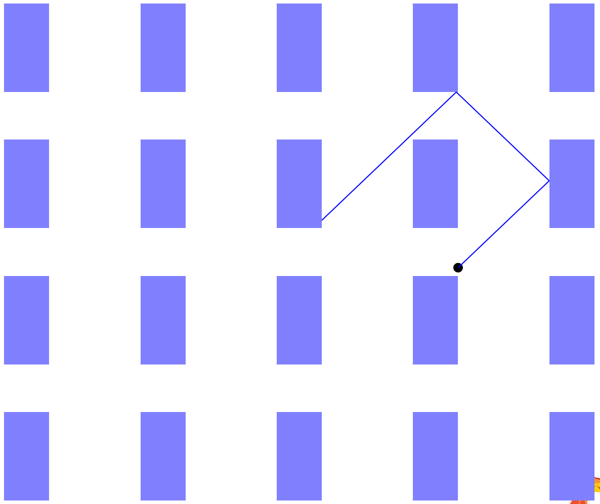

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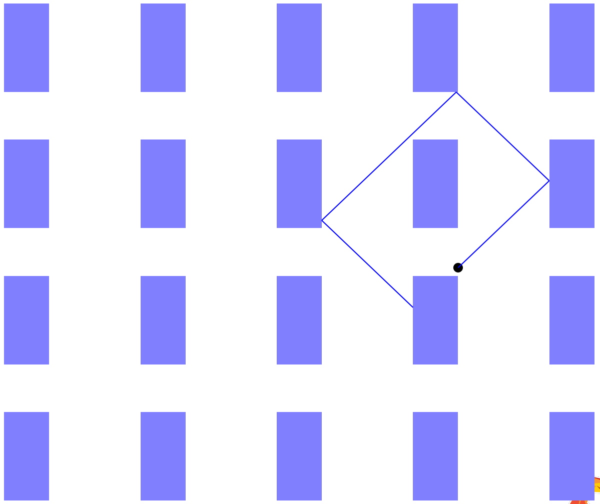

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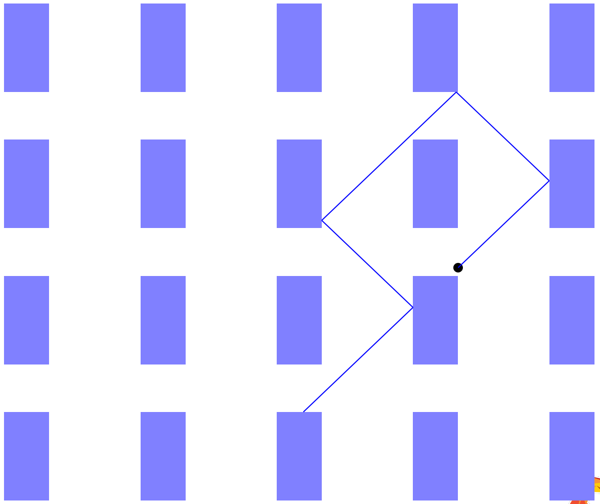

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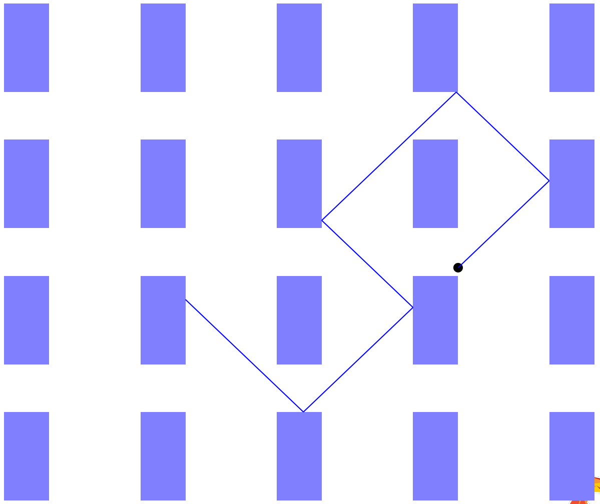

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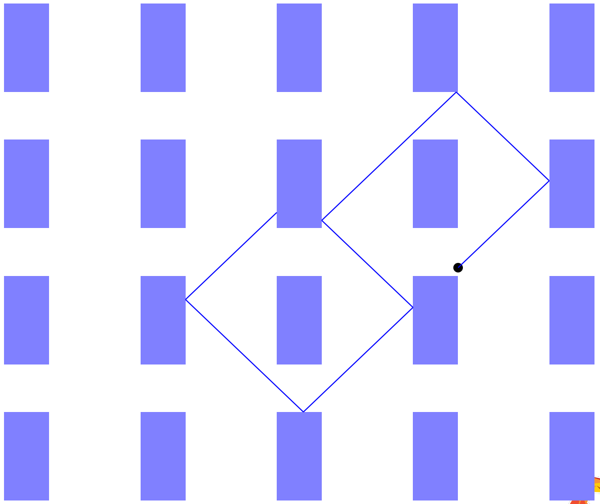

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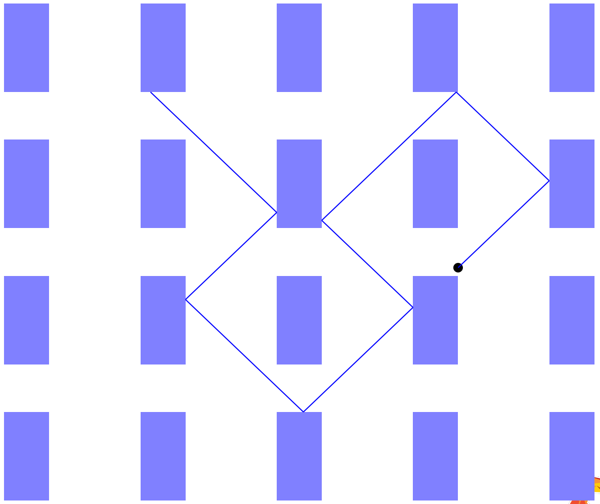

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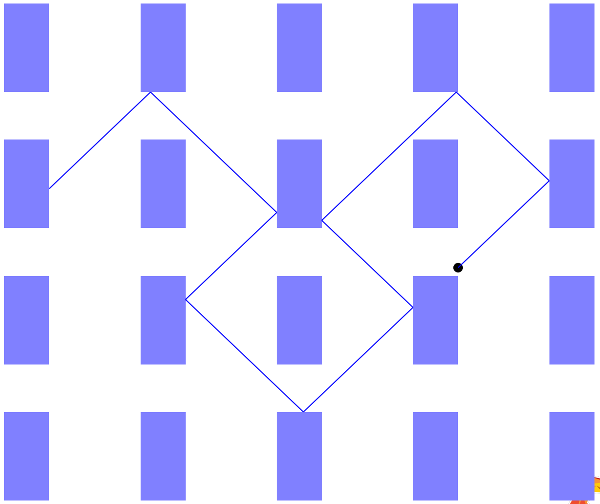

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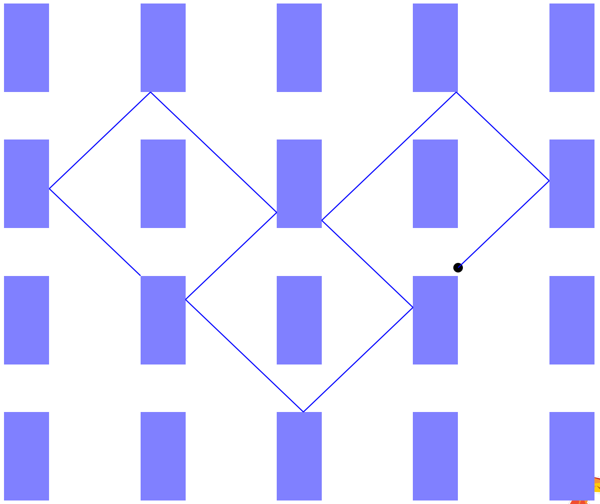

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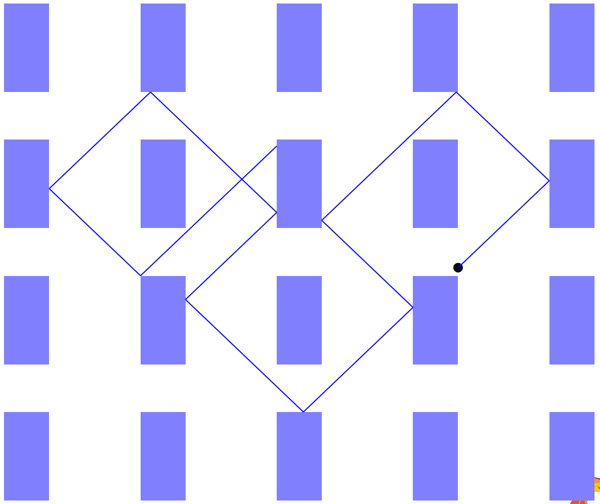

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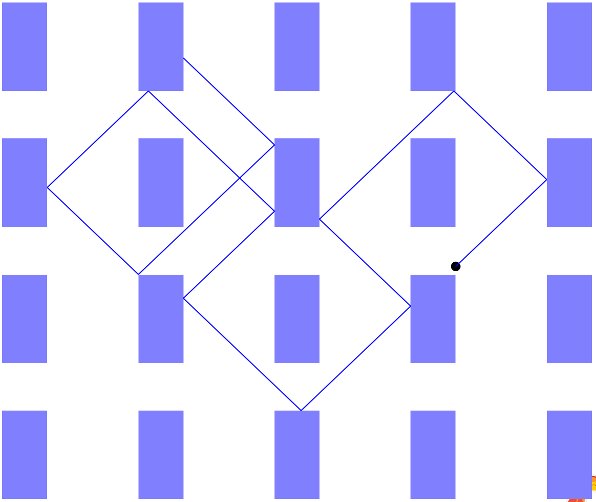

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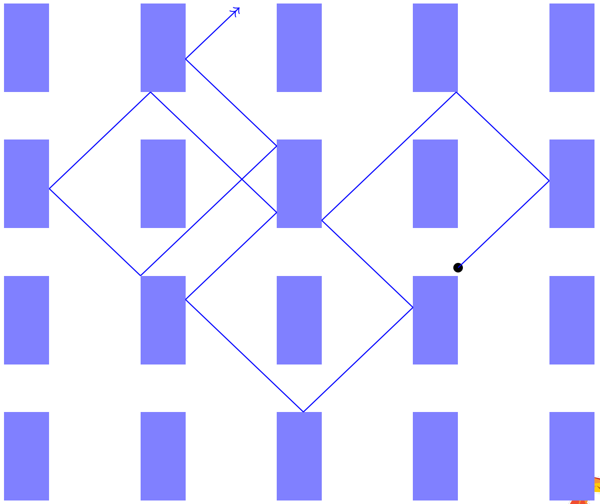

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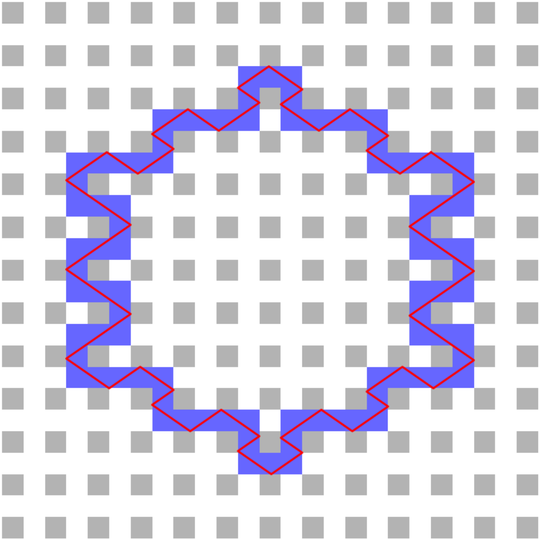

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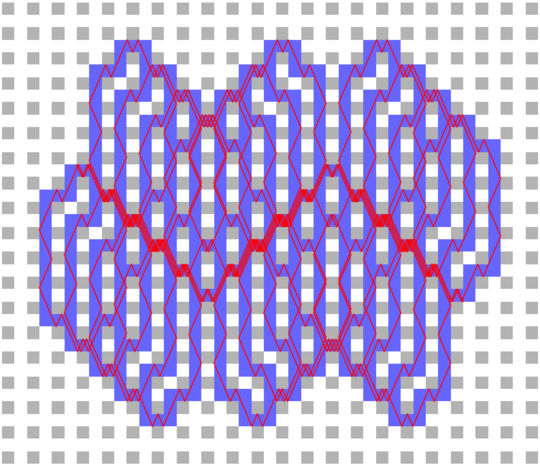

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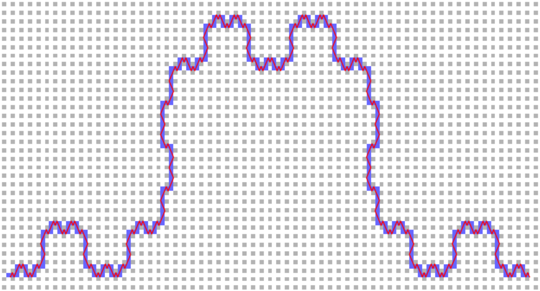
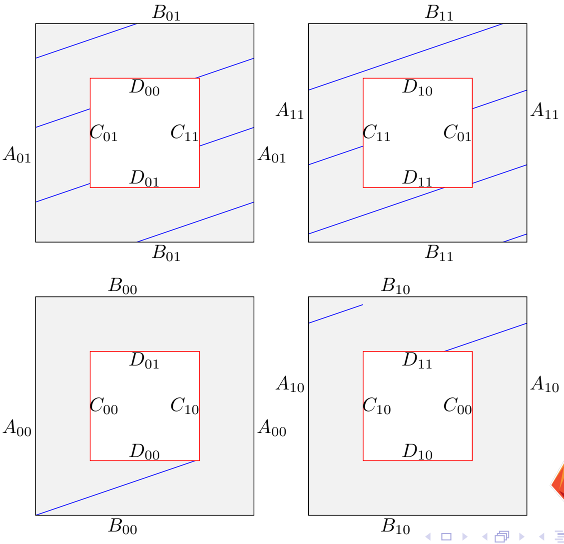


Example: Hardy-Weber's periodic wind-tree model is a covering of this genus 5 surface



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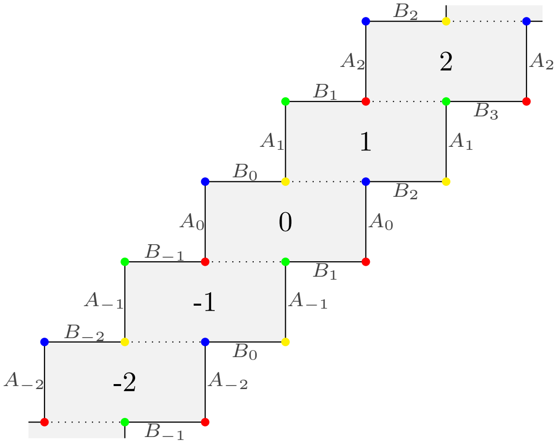
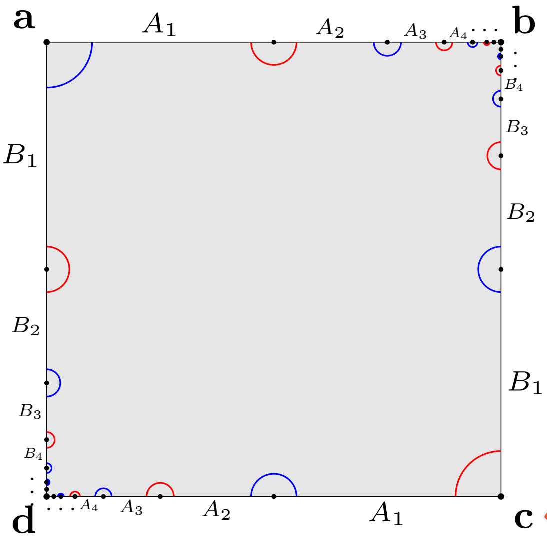
Example: the infinite staircase



Example: baker's surface



Example: Thurston-Veech-McMullen-Hooper's construction of pseudo-Anosovs

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INPUT: (1) G bipartite graph with a ribbon structure. $V(G) = H \sqcup V$. (2) A posivite eigenfunction $f : V(G) \to \mathbb{R}^+$ of the adjacency operator.

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$$Hor : E(G) \to H, \qquad Vert : E(G) \to V$$

These and the ribbon structure define two permutations on the set of edges of G:

$$\mathcal{E}(e) = p_{Hor(e)}(e), \qquad \mathcal{N}(e) = p_{Vert(e)}(e).$$

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GLUEING RULES (using translations). For each edge $e \in E(G)$:

1. Glue the right side of R_e to the left side of $R_{\mathcal{E}(e)}$ and
2. the top side of R_e to the bottom side of $R_{\mathcal{N}(e)}$.

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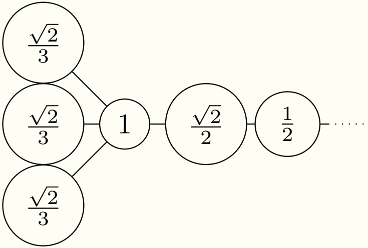
Example: T-V-McM-H's construction of p.Anosovs

Exercise.

1. Suppose that $S = S(G, f)$ is a Thurston-Veech-McMullen-Hooper surface. Show that:

$$\mathrm{Area(S)} = \frac{\lambda}{2} \sum_{\mathrm{v} \in \mathrm{V(G)}} \mathrm{f(v)}^2$$

2. Perform the construction for:



Main definitions and general aspects.

Three definitions: constructive

Definition. Let $\mathcal{P}$ be an at most countable set of Euclidean polygons and $f : E(\mathcal{P}) \to E(\mathcal{P})$ a pairing of edges. Let M be $\bigsqcup_{P\in\mathcal{P}} P/\sim$ deprived of all vertices of infinite degree. If M is connected we call it *the translation surface obtained from the family of polygons* $\mathcal{P}$.

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For each vertex $v \in P \in \mathcal{P}$ we denote by $\alpha_v \in (0, 2\pi)$ the interior angle of P at v. For each finite degre vertex $v \in P$ there exists positive integer k_v so that

$$\sum_{w\in\pi^{-1}(\pi(v))} \alpha_w = 2k_v\pi. \quad (1)$$

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If $k_v > 1$, the point $\pi(v) \in M$ is called a (finite) *conical singularity of angle* $2\pi k_v$ while if $k = 1$ it is called a *regular point*.

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Three definitions: geometric

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Definition. A *translation surface* is a pair $(S, \mathcal{T})$ made of a connected topological surface S and a maximal translation atlas $\mathcal{T}$ on $S \setminus \Sigma$, where:

1. Σ is a discrete subset of S and
2. every $z \in \Sigma$ is a finite conical singularity*.

The maximal translation atlas $\mathcal{T}$ is called a *translation surface structure* on S and its charts are called the *flat charts* or *flat coordinates*.

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Translation atlases provide structure

At any point x of $M \setminus \Sigma$ we can pull-back using flat charts the Euclidean metric of $\mathbb{C}$. This gives rise to a globally well-defined *flat* metric μ on $M \setminus \Sigma$.

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Translation atlases provide structure

At any point x of $M \setminus \Sigma$ we can pull-back using flat charts the Euclidean metric of $\mathbb{C}$. This gives rise to a globally well-defined *flat* metric μ on $M \setminus \Sigma$.This metric provides a notion of distance and area. If $z = x + iy$ is a flat chart on $M \setminus \Sigma$ then μ is given in these coordinates by $dx^2 + dy^2$ and the area by $dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$.We denote by $\widehat{M}$ the metric completion of $M \setminus \Sigma$ w.r.t. (the intrinsic metric induced by) μ. We also have a well defined Lebegue measure λ on M.

Definition. A point $p \in \widehat{M}$ is called a *conical singularity* if there exists a neighborhood U of p in $\widehat{M}$ such that $U \setminus p$ is isometric to a flat cyclic covering $\widetilde{U}_k$ of degree k over the punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\}$, for some $\varepsilon > 0$ and $k \in \{2, 3, \ldots, \infty\}$. If the degree k of the covering is finite we say that p is a *conical singularity of finite angle* $2\pi k$ and if $k = \infty$ we say that p *is an infinite angle singularity*.

Conical and wild singularities

Definition. A point $p \in \widehat{M}$ is called a *conical singularity* if there exists a neighborhood U of p in $\widehat{M}$ such that $U \setminus p$ is isometric to a flat cyclic covering $\widetilde{U}_k$ of degree k over the punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < \varepsilon\}$, for some $\varepsilon > 0$ and $k \in \{2, 3, \ldots, \infty\}$. If the degree k of the covering is finite we say that p is a *conical singularity of finite angle* $2\pi k$ and if $k = \infty$ we say that p *is an infinite angle singularity*.

Definition. A point $p \in \widehat{M} \setminus M$ which is not a conical singularity is called a *wild singularity*. The subset of $\widehat{M}$ formed by all conic and wild singularities is called *the set of singularities of* M and will be denoted by $Sing(M)$.

Translation atlases provide symmetries

Definition. A homeomorphism $f : M \to M$ which in flat charts is an $\mathbb{R}$-affine map is called *an affine automorphism*. Let $\mathrm{Aff}_+(M)$ be the group of affine automorphisms of M which preserve orientation.

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$$D : \mathrm{Aff}_+(M) \to GL_2^+(\mathbb{R})$$

is called the *Veech group* of M. We denoted it by $\Gamma(M)$.

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Translation atlases provide dynamics

Definition. For each direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ the vector field $\frac{dz}{zt} = e^{i\theta}$ in $\mathbb{C}$ is translation invariant. The pull-back of this vector field through flat charts is well-defined on $M \setminus Sing(M)$ and the associated "flow" F_θ^t is called the *translation/straightline/geodesic in direction* θ.

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Veech's dychotomy

Thm.(Veech '89). Let M be a compact translation surface such that $\Gamma(M)$ is a lattice. Then F_θ^t is either *periodic** or uniquely ergodic.

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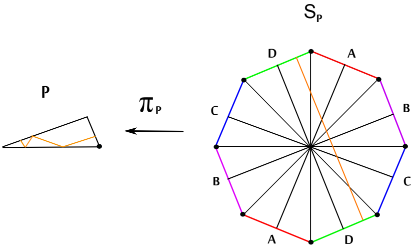

Three definitions:analytic

Definition. A translation surface is a pair (X, ω) formed by a Riemann surface X and a holomorphic 1–form ω on X which is not identically zero.

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Infinite type

A translation surface $M = (X, \omega)$ is said to be of *finite type* if X is of finite type* (as a Riemann surface) and M has finite area. If M is not of finite type we say it is of infinite type.

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A translation surface whose fundamental group is not finitely generated is of infinite type. We will focus on this kind of infinite type surfaces.

How do we classify them topologically?

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Ends of a topological space

Definition. Let $U_1 \supseteq U_2 \supseteq \dots$ be an infinite sequence of non-empty connected open subsets of X such that for each $i \in \mathbb{N}$ the boundary ∂U_i of U_i is compact and $\bigcap_{i \in \mathbb{N}} \overline{U_i} = \emptyset$.

Two such sequences $U_1 \supseteq U_2 \supseteq \dots$ and $U'_1 \supseteq U'_2 \supseteq \dots$ are said to be equivalent if for every $i \in \mathbb{N}$ there exist j such that $U_i \supseteq U'_j$ and viceversa. The corresponding equivalence classes are also called *topological ends* of X and we will denote it by $\mathrm{Ends}(X)$.

Ends of a surface

An end $[U_1 \supseteq U_2 \supseteq \dots]$ is called planar if there exists U_i of genus zero.

Definition. We define $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$ as the set of all ends which are not planar.

Thm. (Kerékjártó-Richards, '63). Two non-compact orientable surfaces S and S' of the same genus are homeomorphic if and only if they have the same genus $g \in \mathbb{N} \cup \{0, \infty\}$, and both $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$ and $\mathrm{Ends}_\infty(S') \subset \mathrm{Ends}(S')$ are homeomorphic as nested topological spaces, that is, there exists a homeomorphism $h : \mathrm{Ends}(S) \to \mathrm{Ends}(S)'$ such that $h(\mathrm{Ends}_\infty(S)) = \mathrm{Ends}_\infty(S')$

Ends of a surface

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Thm.(Kerékjártó-Richards'63). Let $C' \subset C$ be a nested pair of closed subset of the Cantor set. Then there exist a surface S such that $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$ is homeomorphic to $C' \subset C$ as nested pair of topological spaces.

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Affine symmetries and Veech groups.

Veech groups: compact case

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2. $\Gamma(M)$ is never co-compact.

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Open questions:

1. Does there exist $\Gamma(M)$ of the second kind?
2. Does there exist $\Gamma(M)$ cyclic and hyperbolic?

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Thm. Let P be an irrational polygon whose interior angles are $\lambda_j \pi$, $j = 1, \ldots, n$. Then the group of rotations $z \to e^{2\pi\lambda_j i} z$ has finite index $\Gamma(S_P)$.

Veech groups and billiards

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Question. Can any *countable* subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ be the Veech group of an infinite type translation surface?

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Thm.(Przytycki-Schmithüsen-V.) Every countable subgroup $G < \mathrm{GL}_2^+(\mathbb{R})$ is the Veech group of a translation surface M_G *homeomorphic* to the Loch Ness monster.

Veech groups of LNM

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Remarks

- The translation surface M_G has infinite area.
- The dynamic of the translation flow F_θ^t in M_G is uninteresting.

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Remarks

- The translation surface M_G has infinite area.
- The dynamic of the translation flow F_θ^t in M_G is uninteresting.

The Veech group of the infinite staircase is a lattice and, as we will see later, in this particular example we have an analog of Veech's dychotomy.

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Open questions

Question. Let $C_\infty \subset C$ be a nested couple of closed subspaces of the Cantor set and G any countable subgroup of $\mathrm{GL}_2^+(\mathbb{R})$. Is it possible to find a translation surface M such that $Ends_\infty(S) = C_\infty$, $Ends(S) = C$ and whose Veech group is G?

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Question. Does there exist a translation surface M homeomorphic to Jacob's ladder and whose Veech group is $\mathrm{SL}_2(\mathbb{Z})$? a lattice?

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Veech groups in finite area

Thm.(Hooper,2015) Let M be a Thurston-Veech translation surface constructed from a bipartite graph G and a positive eigenfunction f with eigenvalue λ. Then the group H_λ generated by the matrices

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

is a subgroup of $\Gamma(M)$.

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Proposition. The Veech group of Baker's surface is a non-elementary Fuchsian group of the second kind.

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Thm.(Treviño,2013). Let M be a translation surface of infinite genus and finite area whose Veech group is a lattice. Then for a.e. direction F_θ^t is ergodic.

Veech groups in finite area and dynamics

Thm.(Treviño,2013). Let M be a translation surface of infinite genus and finite area whose Veech group is a lattice. Then for a.e. direction F_θ^t is ergodic.

Question Does there exist an infinite type and finite area translation surface whose Veech group is $\mathrm{SL}_2(\mathbb{Z})$? a lattice? of the first kind?

CENTRO DE CIENCIAS MATEMÁTICAS

Dynamical properties of the translation flow (recurrence and ergodicity).

$\mathbb{Z}$-covers

A (connected) cover $\pi : \tilde{M} \to M \setminus \Sigma$ with deck transf. group isomorphic to $\mathbb{Z}$ is called a $\mathbb{Z}$-cover. These are determined by non-primitive classes in $H_1(M, \Sigma; \mathbb{Z})$.

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Every translation surface has a *holonomy map*:

$$\textbf{hol} : H_1(M, \Sigma; \mathbb{Z}) \to \mathbb{R}^2$$

It is defined by developing a representative of the class in the plane and then taking the difference of the starting and end points.

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Recurrence in $\mathbb{Z}$-covers

Thm.(Hooper-Weiss.'12) Let $\tilde{M} \to M$ be a $\mathbb{Z}$-cover defined by a class $c \in H_1(M, \Sigma; \mathbb{Z})$ such that $\mathbf{hol}(c) = 0$. If θ is a direction for which F_θ^t is ergodic, then $\tilde{F}_\theta^t$ is recurrent.

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A classical theorem by Kerckhoff-Masur-Smillie implies:

Cor. Let $\tilde{M} \to M$ be a $\mathbb{Z}$-cover defined by a class $c \in H_1(M, \Sigma; \mathbb{Z})$ such that $\mathbf{hol}(c) = 0$.Then $\tilde{F}_\theta^t$ is recurrent for a.e. direction θ.

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Skew-products over IET's

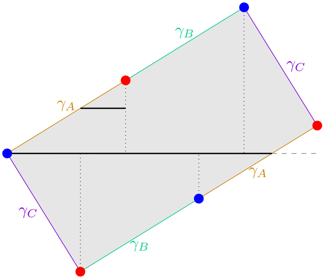
Let $T : I \to I$ be a IET and $f : I \to \mathbb{Z}$ a measurable function. The dyn. system on $I \times \mathbb{Z}$ defined by:

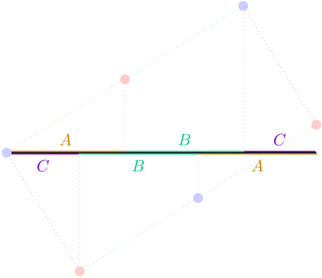
$$T_f(x, n) = (T(x), n + f(x))$$

is called a *skew-product*.

Thm.(Atkinson-Krygin) Suppose that f is integrable and T is ergodic. If $\int_I f \, d\lambda = 0$ then the skew product T_f is recurrent.

Recurrence in the staircase





Recurrence in general Abelian coverings

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Thm.(Avila-Hubert) Let $P_{a,b}$ be a periodic wind-tree model. Then for a.e. direction θ the billiard flow on $P_{a,b}$ is recurrent.

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Thm.(Avila-Hubert) Let $P_{a,b}$ be a periodic wind-tree model. Then for a.e. direction θ the billiard flow on $P_{a,b}$ is recurrent.

Question. Does there existe $\tilde{M} \to M$ a $\mathbb{Z}^d$-cover, $d \geq 2$, defined by a cycle $c \in H_1(M, \Sigma; \mathbb{Z}^d)$ such that M is a square-tiled surface, $Re(\mathbf{hol}(c)) = 0$ and the translation flow is dissipative in almost every direction?

Ergodicity. Case study: the infinite staircase

Thm.(Hopper-Hubert-Weiss) Let $\widetilde{F}_\theta^t$ be the translation flow on the infinite staircase M.

1. If $\theta = \frac{p}{q}$ with p or q EVEN, then $\widetilde{F}_\theta^t$ decomposes M into infinitely many cylinders,
2. If $\theta = \frac{p}{q}$ with p and q ODD, then $\widetilde{F}_\theta^t$ decomposes M into two infinite strips, and
3. If θ is irrational, then $\widetilde{F}_\theta^t$ is ergodic w.r.t. Lebesgue.

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3. If θ is irrational, then $\widetilde{F}_\theta^t$ is ergodic w.r.t. Lebesgue.
4. For every irrational direction θ, the locally finite Borel ergodic measures for the flow in direction of slope θ are precisely the **Maharam measures**.

Ergodicity: $\mathbb{Z}$-covers

Thm.(Hubert-Weiss) Let $\widetilde{M} \to M$ be a $\mathbb{Z}$-cover for which $\Gamma(\widetilde{M})$ is a lattice and has an infinite strip. Then for a.e. direction θ, the flow $\widetilde{F}_\theta^t$ is ergodic.

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Definition. Let T_f be a $\mathbb{Z}$-valued skew-product over an IET. An element $N \in \mathbb{Z}$ is called an **essential value** if for every $A \subset I$ of positive measure, there exists $n \in \mathbb{Z}$ such that $T_f^n(A \times 1) \cap A \times N$ has positive measure.

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Proposition. With the aforementioned hypothesis, if there exist an essential value different from zero then T_f is ergodic.

Ergodicity: $\mathbb{Z}^2$-covers and beyond

Thm.(Fraczek-Ulcigrai) For all parameteres (a, b) and for a.e. direction θ the translation flow F_θ^t on the **periodic** wind-tree model **is not** ergodic.

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Thm.(Málaga-Troubetzkoy) There are families of non-periodic wind-tree models for which for a.e. direction θ the translation flow F_θ^t **is** ergodic.

Ergodicity: $\mathbb{Z}^2$-covers and beyond

Thm.(Hooper) Let M be a Thurston-Veech surface constructed from a bipartite infinite graph which has no vertices of valance one. Then for a "small" set of directions $\theta \in \Theta$ it is possible to classify **all** locally finite ergodic invariant measures.

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