LECTURE NOTES ON ERGODICITY OF PARTIALLY HYPERBOLIC SYSTEMS

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1. Lecture I: Introduction

Let $f : M \to M$ be a (pointwise) partially hyperbolic diffeomorphism. Recall what partially hyperbolic means. At every point in M, there exist tangent vectors that are uniformly contracted by the derivative Tf and tangent vectors that are uniformly expanded by Tf. The contracted vectors lie in an invariant subbundle E^s of TM, called the *stable subbundle*, and the expanded vectors lie in an invariant subbundle E^u, called the *unstable subbundle*. The sum of these subbundles $E^u \oplus E^s$ is the hyperbolic part of TM. The rest of TM is also under control: there is a Tf-invariant complement E^c to $E^u \oplus E^s$ in TM, called the *center bundle*. Tangent vectors in E^c may be expanded or contracted under the action of Tf, but they are neither expanded as sharply as vectors in E^u, nor contracted as sharply as vectors in E^s.

As you learned last week, the stable subbundle is uniquely integrable and tangent to a foliation $\mathcal{W}^s$, and the unstable subbundle is uniquely integrable and tangent to a foliation $\mathcal{W}^u$. The center bundle E^c is sometimes, but not always, tangent to a foliation. The stable and unstable foliations are absolutely continuous, while the center foliation (when it exists) can fail to be absolutely continuous.

We will assume throughout these lectures that f is C^2 and preserves a fixed volume m (or a measure equivalent to volume) on M, normalized so that $m(M) = 1$. The most basic property of a measure-preserving dynamical system is ergodicity. We will investigate which partially hyperbolic diffeomorphisms are ergodic.

As Flavio explained to you, if f is an Anosov diffeomorphism, meaning that $E^c = 0$, then f is ergodic:

All Anosov diffeomorphisms are ergodic.

The proof of this fact (due to Anosov and Sinai [A, AS]) uses the *Hopf argument.* We will revisit the Hopf argument in Lecture II, but we recall here the basic idea. Ergodicity of f is equivalent to showing that the forward and backward Birkoff average a of any continuous function are almost everywhere constant. The forward and backward averages are almost everywhere equal. The forward Birkhoff averages of a continuous function are constant along leaves of the stable foliation $\mathcal{W}^s$, and the backward Birkhoff averages are constant along leaves of the unstable foliation $\mathcal{W}^u$. If f is Anosov, then these two foliations are transverse and absolutely continuous. Fubini's Theorem implies that the Birkhoff averages of a continuous are almost everywhere constant. Hence f is ergodic.

Let's now begin our investigation of the ergodicity of partially hyperbolic diffeomorphisms. We will be primarily concerned with what Flavio calls *strong* partial hyperbolicity. That is, the case where both, and not just one, of E^u and E^s are nontrivial. (For important developments in the weak case, see [BonVi, AlBonVi]).

We will continue to keep the Hopf argument in mind. We still have stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$, they are still absolutely continuous. They are no longer transverse, but they are *quasi transverse:* the tangent spaces of their leaves intersect trivially. Forward Birkhoff averages of continuous functions are constant along leaves of $\mathcal{W}^s$, and the backward Birkhoff averages are constant along leaves of $\mathcal{W}^u$. It is so close to the Anosov case that you might expect that the Hopf argument can be tweaked a little to show that every partially hyperbolic diffeomorphism is ergodic. Of course if this were the case, I wouldn't be lecturing on this topic:

Not all partially hyperbolic diffeomorphisms are ergodic.

We can see this via a very simple example.

Example 1. Let $g : N \to N$ be a volume-preserving Anosov diffeomorphism. For concreteness, assume that $N = \mathbf{R}^2/\mathbf{Z}^2$ is the 2-torus, and g is a linear Anosov automorphism, such as the map given by the matrix

$$L = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $V = \mathbf{R}/\mathbf{Z}$ be the circle, and consider the product map

$$f_0 = g \times id : N \times V \to N \times V; \qquad g \times id(x, \theta) = (g(x), \theta).$$

This map preserves volume on the 3-torus $N \times V$, and is partially hyperbolic, with $E^s_{f_0} = E^s_g \oplus \{0\}$, $E^u_{f_0} = E^u_g \oplus \{0\}$, and $E^c_{f_0} = \{0\} \oplus TV$. Clearly, f_0 is not ergodic; any set of the form $N \times A$ is f-invariant, and if A has measure between 0 and 1, then so does $N \times A$.

It is instructive to examine where the Hopf argument goes wrong for this example. Consider the forward and backward Birkhoff averages of a fixed continuous function. As remarked above, these averages are m-a.e. equal, are m-a.e. constant along stable manifolds and m-a.e. constant along unstable manifolds.

We'll be very ambitious for now and drop the "m.-a.e." from these statements. Where do we get? Starting at a point $(x, \theta) \in N \times V$, and considering all points we can get to from (x, θ) along leaves of either the stable or unstable foliations, we arrive at the following set:

$$N \times \{\theta\}.$$

So our Birkhoff averages appear to be constant along sets of the form $N \times \{\theta\}$. These are 0-sets in $N \times V$, and we need constant almost everywhere, so we have not gotten very far with the Hopf argument. This argument can be made precise — see Exercise 1.8 below.

Now let's alter our example slightly.

Example 2. Let g, N, V be as above, and let $h_\alpha : V \to V$ be a rotation by $\alpha \in \mathbf{T}/\mathbf{Z}$. Let $f_\alpha = g \times h_\alpha : N \times V \to N \times V$. As before, f_α is volume-preserving and partially hyperbolic, with the same partially hyperbolic splitting as f_0 (see Exercise 1.7 below). If α is rational, then, as with f_0, it is easy to see that f_α is not ergodic. On the other hand,

if α is irrational, then f_α is ergodic.

We can see this using the Hopf argument as follows. Let φ_+ be the (forward) Birkhoff average of a continuous function φ. Then (Exercise 1.8) φ_+ is m-a.e. constant along sets of the form $N \times \{y\}$, $y \in V$. Let $\hat{\varphi}_+(y)$ be the (m-a.e.) value of the function φ_+ on $N \times \{y\}$. The function $\hat{\varphi}_+ : V \to \mathbf{R}$ is (m-a.e.) invariant under the irrational rotation h_α. Ergodicity of h_α implies that $\hat{\varphi}_+$, and so φ_+ is m-a.e. constant. Hence f_α is ergodic.

In our family of examples $\{f_\alpha \,|\, \alpha \in \mathbf{R}/\mathbf{Z}\}$ we have seen both ergodicity and nonergodicity. Each ergodic example can be approximated by a nonergodic example, and vice versa. Suppose we pick an f_α from this family and attempt to simulate its behavior with an approximate computation. If our computation is not exact we might confuse an ergodic example for a nonergodic one.

For this reason, it is valuable to consider dynamical properties that are *stable under perturbation.*

Definition: Let $\mathcal{P}$ be a property of a m-preserving diffeomorphism $f \in \mathrm{Diff}^r_m(M)$, $r \geq 2$, such as ergodicity. We say that f is *stably* $\mathcal{P}$ if there is a C^r-open neighborhood U of f in $f \in \mathrm{Diff}^r(M)$ such that every $g \in U$ has property $\mathcal{P}$.

Introducing the regularity r into this defintion produces all sorts of expositional headaches, and indeed, this is not quite the standard definition of stably $\mathcal{P}$. Let's live with it for now, and I'll explain how it should be modified later. For the rest of this lecture, assume that r is at least 3.

You have already seen examples of (C^r, $r \geq 2$) of stably ergodic diffeomorphisms. Since all Anosov diffeomorphisms are ergodic, and Anosov diffeomorphisms are stably Anosov, we have:

Anosov diffeomorphisms are stably ergodic.

There are also examples of stably nonergodic diffeomorphisms. KAM (Kolmogorov-Arnol'd-Moser) theory shows that if a (C^3) area-preserving diffeomorphism f of a surface has an elliptic periodic point with the right amount of twist, then f will have elliptic islands in M, preventing ergodicity. Since this twist condition is C^3 open, we obtain:

KAM diffeomorphisms are stably nonergodic.

The partially hyperbolic diffeomorphisms f_α in Examples 1 and 2 are neither stably ergodic nor stably nonergodic. Let's see what happens when we combine a stably ergodic diffeomorphism with a stably nonergodic one in a single partially hyperbolic example. We will use the same product-type construction, this time on the 4-torus.

Example 3. Let $g : N \to N$ be as above, let $V = \mathbf{R}^2/\mathbf{Z}^2$ be the 2-torus, and let $h : V \to V$ be a C^3 diffeomorphism with an elliptic fixed point (a KAM diffeomorphism). See Exercise 1.9 for a specific example. Let $k > 0$ be an integer. If k is sufficiently large, then the diffeomorphism $f = g^k \times h$ is partially hyperbolic. Because h is KAM, f is not ergodic, and any perturbation of f of the form $g' \times h'$ (where g' is a perturbation of g^k and h' is a perturbation of h) is not ergodic. But is f stably nonergodic? That is, does every perturbation of f, including those *not* of the form $g' \times h'$ fail to be ergodic?

In all of our examples so far, this one included, the stable and unstable foliations have been jointly integrable, with $E^u_f \oplus E^s_f$ tangent to the foliation of $N \times V$ by compact sets of the form $N \times \{y\}$, $y \in V$. Thus, when we try to use the Hopf argument starting at a point (x, y) and following hte leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$, we never get further than the 0-set $N \times \{y\}$. This situation seems very special — one foliation ($\mathcal{W}^u$) is determined by the infinite past, and the other ($\mathcal{W}^s$) by the infinite future. Why should these foliations want to cooperate with each other? In a later lecture we'll see why this vague intuition is correct. For now, I will give a concrete example to illustrate why this intuition might be justified.

Example 4. (where things start to get interesting) For our next example, we'll consider a skew product. Let g, N, h and V be as in Example 3. To every $\psi : N \to V$ we can associate a *skew perturbation* f_ψ of $g \times h$ as follows:

$$f_\psi(x, y) = (g(x), h(y) + \psi(x)).$$

Because translation in the torus V is an isometry, it is not hard to see that if $g \times h$ is partially hyperbolic (with respect to the standard Euclidean Riemann structure on the 4-torus $N \times V$), then so is f_ψ. The C^1 distance between f_ψ and $g \times h$ is comparable to the C^1 distance from ψ to the 0-function in V.

The map f_ψ preserves the vertical foliation $\{\{x\} \times V \mid x \in N\}$, permuting its leaves by the action of g. Unless ψ is a constant function, the horizontal foliation $\{N \times \{y\} \mid y \in V\}$ is not preserved. It is then perhaps not surprising that a skew perturbation can break the integrability of $E^u \oplus E^s$. In fact, a skew pertubation can do more than break joint integrability; it can create *accessibility*.

Definition: A partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if any point in M can be reached from any other along an *su-path*, which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$.

While I will not prove it in these lectures, we have the following result:

Theorem 1.1 (Shub-W.)**.** [SW1] *Let $g \times h$ be partially hyperbolic. There exists a function $\psi : N \to V$ such that, for all $\varepsilon \neq 0$ sufficiently small, the map $f_{\varepsilon\psi}$ is accessible, and in fact, stably accessible.*

Consider now a map $f_{\varepsilon\psi}$ given by Theorem 1.1. Let's try the Hopf argument again on this example. As before, consider the Birkhoff averages φ_+ of a continuous function. We'll again be ambitious and drop the m-a.e. considerations from the argument for now; thus we assume that φ_+ is actually constant along leaves of $\mathcal{W}^s$ and leaves of $\mathcal{W}^u$. In this case, if two points are connected by an su-path, then φ_+ must take the same values at those points. Since $f_{\varepsilon\psi}$ is accessible, φ_+ must take the same value at all points; that is, φ_+ must be constant. Hence $f_{\varepsilon\psi}$ must be ergodic.

Notice that the only property of $f_{\varepsilon\psi}$ we used in this "argument" was acccessibility. The "argument" actually shows that every accessible partially hyperbolic diffeomorphism is ergodic. This heuristic argument is simple, but at this time, no one has been able to turn the heuristic into an actual proof that accessibility implies ergodicity. The problem is that a lot is "swept under the rug" when we ignore sets of measure 0 in the Hopf argument.

Despite these difficulties, in 1995, Grayson, Pugh and Shub [GPS] showed that accessibility implies ergdocity for partially hyperbolic maps that satisfy some additional technical hypotheses. Based on this breakthrough, and on later results of Pugh and Shub that further weaken the technical hypotheses [PS2, PS3], Pugh and Shub conjectured:

Conjecture 1.2 (Pugh-Shub)**.** *Let f be a C^2, volume-preserving, partially hyperbolic diffeomorphism. If f is accessible, then f is ergodic.*

In these lectures, I will present the "state of the art" in progress toward proving this conjecture. In particular, I will prove:

Theorem 1.3 (Burns-W.)**.** [BW2] *Let f be C^2, volume-preserving, partially hyperbolic, and center bunched. If f is essentially accessible, then f is ergodic.*

"Center bunched" is a relatively mild technical condition that can be verified in a given example. I will define it in Lecture IV. If $\dim(E^c) = 1$, then this condition is always satisfied. For our examples 3. and 4., this condition is satisfied if k is large enough. "Essentially accessible" is a weakening of acccessibility that I will define in Lecture II.

As an immediate corollary, we have:

Corollary 1.4. *Let f be C^2, volume-preserving, partially hyperbolic, and center bunched. If f is stably essentially accessible, then f is stably ergodic.*

Returning to the issue of accessibility, notice that accessibility is a relatively coarse, topological property of the pair of foliations $\mathcal{W}^s$ and $\mathcal{W}^u$. Motivated in part by considerations in cotrol theory, Pugh and Shub also conjectured that accessibility is a "typical" property among partially hyperbolic diffeomorphisms.

Conjecture 1.5 (Pugh-Shub)**.** *Among the partially hyperbolic diffeomorphisms, stable accessibility is an open-dense property.*

Combining Conjecture 1.2 and Conjecture 1.5, we get:

Conjecture 1.6 (Pugh-Shub)**.** *Among the partially hyperbolic diffeomorphisms, stable ergodicity is an open-dense property.*

This third conjecture I will refer to as the Stable Ergodicity Conjecture. Conjecture 1.5 has recently been proved under the hypothesis that the center bundle is 1-dimensional [HHU]. Combining this fact with our main result, we obtain a proof of Conjecture 1.6 under the hypothesis that the center bundle is 1-dimensional [HHU]. A C^1 version of Conjecture 1.5, with no restriction on the dimension of the center bundle, was proved in [DW].

Before finishing, let's return to our f in Example 3., the product of an Anosov with a KAM. We asked the question: "Is f stably nonergodic?" Putting together the results I've just discussed, we come to the following answer:

If k is sufficiently large, then f can be approximated arbitrarily well by a stably ergodic diffeomorphism.

Far from being stably nonergodic, the f in Example 3 is on the boundary of the stably ergodic diffeomorphisms!

Exercise 1.7. *Let $f = g \times h : M \times V \to M \times V$, where g is Anosov. Find conditions on h that guarantee that f is partially hyperbolic. Show that, in this case, $E^s = E^s_g \oplus \{0\}$, $E^u = E^u_g \oplus \{0\}$, and $E^c = \{0\} \oplus TV$. Thus the subbundle $E^u \oplus E^c$ is integrable, tangent to the foliation:*

$$\{N \times \{y\} \,|\, y \in V\}.$$

Exercise 1.8. *Show that if $f = g \times h$ is partially hyperbolic, where g is Anosov, and $\varphi : N \times V \to \mathbf{R}$ is any continuous function, then the forward and backward Birkhoff averages of φ are almost everywhere constant along sets of the form $N \times \{y\}$, $y \in V$. (hint: Fubini, Fubini)*

Exercise 1.9. *Consider a diffeomorphism $f_\lambda : \mathbf{T}^2 \times \mathbf{T}^2 \to \mathbf{T}^2 \times \mathbf{T}^2$ of the form:*

$$f_\lambda(x, y) = (A(x), g_\lambda(y)),$$

where $A : \mathbf{T}^2 \to \mathbf{T}^2$ is the linear Anosov diffeomorphism given by

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^2,$$

and $g_\lambda : \mathbf{T}^2 \to \mathbf{T}^2$ is a standard map of the form:

$$g_\lambda(z, w) = (z + w, w + \frac{\lambda}{2\pi} \sin(2\pi(z + w))).$$

Show that there is an interval $\Lambda \subset \mathbf{R}$ containing $(-4, 4)$ and contained in $(-6, 6)$ such that, if $\lambda \in \Lambda$, then f_λ is partially hyperbolic with respect to the standard (flat) metric on $\mathbf{T}^2 \times \mathbf{T}^2$.

Exercise 1.10. *Define $\mathcal{E}^r_\mu(M)$ to be the set of $f \in Diff^r_m(M)$ that are ergodic. Show that for any r, $\mathcal{E}^r_\mu(M)$ is a G_δ (in the C^r topology on $Diff^r_\mu(M)$).*

Problem: Classify the partially hyperbolic diffeomorphisms of a given manifold M, or inside a fixed homotopy class. Which manifolds support partially hyperbolic diffeomorphisms?

2. Lecture II: The Hopf Argument

In the 1930's Eberhard Hopf [H] proved that the geodesic flow for a compact, negatively-curved surface is ergodic. His method was to study the Birkhoff averages of continuous functions along leaves of the stable and unstable foliations of the flow. This type of argument has been used since then in increasingly general contexts, and has come to be known as the Hopf Argument. In this lecture, I will describe this argument in some detail.

We say a set A is *saturated* by a foliation $\mathcal{F}$, or $\mathcal{F}$*-saturated* if A consists of whole leaves of $\mathcal{F}$. The Hopf Argument for ergodicity of a partially hyperbolic diffeomorphism can be rephrased in terms of $\mathcal{W}^s$-saturated and $\mathcal{W}^u$-saturated sets. As I mentioned in Lecture I, the Hopf argument necessarily involves sets of measure 0. Thus in order to formulate the Hopf argument properly, it is necessary to define saturation modulo a zero set, a property we call *essential saturation.*

Let $\mathcal{P}$ be a property of a measurable set, such as $\mathcal{W}^s$-saturation or $\mathcal{W}^u$-saturation, or f-invariance. We say that a measurable set A is *essentially* $\mathcal{P}$ if there exists a set B, with $m(A \,\Delta\, B) = 0$ so that B has property $\mathcal{P}$. An essential property is really a property of a measure class. Similarly, a measurable function f is "essentially $\mathcal{P}$" if there is a function g that coincides with f off of a zero set and which has property $\mathcal{P}$. Thus an essentially constant function is one that can be made constant by modifying its values on a zero set, and an essentially invariant function is a function that can be made invariant by modifying its values on a zero set.

Theorem 2.1 (The Hopf Argument)**.** *Let f be partially hyperbolic and volume-preserving. Then f is ergodic if and only if every f-invariant set A that is both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated is trivial, meaning:*

$$m(A) = 0 \text{ or } 1.$$

Proof. If f is ergodic, then every f-invariant set is essentially trivial, so one implication of the theorem is immediate.

The other direction of the theorem is the heart of Hopf's argument. Hopf used Birkhoff's Pointwise Ergodic Theorem in his original argument; we present a more elementary proof that requires a little less machinery, though the basic idea of the proof is the same. [**Remark:** the proof as presented here is **incorrect.** A correct version along these lines appears in the article: Coudene, Y, *The Hopf argument*, J. Mod. Dyn. v. 1, no 1. 147-153.]

We will use von Neumann's Mean Ergodic Theorem, whose proof is a fairly straightforward exercise in Hilbert space operator theory. Recall that any volume-preserving transformation $f : M \to M$ defines an operator on the Hilbert space $L^2(M)$:

$$U_f : L^2(M) \to L^2(M)$$

by:

$$U_f(\varphi) = \varphi \circ f.$$

The operator U_f is *unitary*, meaning that U is continuous and invertible, and for all $\varphi_1, \varphi_2 \in L^2(M)$, we have

$$< U_f(\varphi_1), U_f(\varphi_2) >_2 = < \varphi_1, \varphi_2 >_2 .$$

Theorem 2.2 (von Neumann's Ergodic Theorem)**.** *Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator on a Hilbert space $\mathcal{H}$. Then, for every $\varphi \in \mathcal{H}$, we have:*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} U^n(\varphi) - P(\varphi) \right\| = 0,$$

where $P : \mathcal{H} \to \mathcal{I}$ is the orthogonal projection onto the space of U-invariant elements of $\mathcal{H}$:

$$\mathcal{I} = \{\varphi \in \mathcal{H} \,|\, U(\varphi) = \varphi\}.$$

The proof of Theorem 2.2 is outlined in Exercise 2.5 below.

Applying Theorem 2.2 to the operator $U_f : L^2(M) \to L^2(M)$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} U_f^n = P,$$

(in the strong topology) where P is the projection onto the U_f-invariant (i.e., f-invariant) functions:

$$\mathcal{I}_f = \{\varphi \in L^2(M) \mid \varphi \circ f = \varphi \quad (a.e.)\}.$$

Since f is invertible, and f^{-1} is volume-preserving, we can also apply Theorem 2.2 to the operator $U_{f^{-1}} : L^2(M) \to L^2(M)$. Notice that the spaces $\mathcal{I}_f$ of U_f-invariant functions and $\mathcal{I}_{f^{-1}}$ of $U_{f^{-1}}$-invariant functions coincide, since:

$$\varphi \circ f = \varphi \quad (a.e.) \quad \Longleftrightarrow \quad \varphi \circ f^{-1} = \varphi \quad (a.e.).$$

Thus, the projections onto $\mathcal{I}_f$ and $\mathcal{I}_{f^{-1}}$ coincide, and we have:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} U_f^n = P = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} U_{f^{-1}}^n.$$

Now suppose that f is not ergodic. Then the space $\mathcal{I}_f = P(L^2(M))$ is nontrivial — it contains more than just the essentially constant functions. The space of continuous functions $C^0(M)$ is dense in $L^2(M)$, and P is continuous, so there exists $\varphi \in C^0(M)$ such that $P(\varphi)$ is not essentially constant. Fix such a φ, and for $n \geq 0$, let

$$\varphi_n = \frac{1}{n}\sum_{i=1}^{n} U_f^n(\varphi) = \frac{1}{n}\sum_{i=1}^{n} \varphi \circ f^n.$$

Theorem 2.2 implies that $\varphi_n \to P(\varphi)$ in L^2. We now use the basic fact that a convergent sequence in L^2 converges almost everywhere along a subsequence. Thus there exists a subsequence $n_j \to \infty$ such that

$$\varphi_{n_j} \to P(\varphi)$$

almost everywhere in M.[1] Let

$$\varphi_+ = \limsup_{n_j \to \infty} \varphi_{n_j}$$

(the limsup guarantees that φ_+ is defined everywhere).

Lemma 2.3. *φ_+ has the following properties:*

(a) $\varphi_+ = P(\varphi)$ *(a.e.); in particular, φ_+ is essentially f-invariant,*
(b) *for all $x, y \in M$,*

$$y \in \mathcal{W}^s(x) \quad \Longrightarrow \quad \varphi_+(x) = \varphi_+(y).$$

Proof of Lemma 2.3. Part (a) follows immediately from the fact that $\varphi_{n_j} \to P(\varphi)$ almost everywhere in M. Part (b) is the key step in Hopf's argument. Suppose that $y \in \mathcal{W}^s(x)$. Because φ is continuous, and $d(f^i(x), f^i(y)) \to 0$ as $i \to \infty$, we have

$$\lim_{i\to\infty} |\varphi(f^i(x)) - \varphi(f^i(y))| = 0.$$

From this it is easy to see that $\varphi_+(x) = \varphi_+(y)$. The details are left to Exercise 2.6 below. $\diamond$

By exactly the same argument that produced φ_+, this time using f^{-1}, we obtain a function φ_- such that $\varphi_- = P(\varphi)$ (*a.e.*), and

$$y \in \mathcal{W}^u(x) \quad \Longrightarrow \quad \varphi_-(x) = \varphi_-(y).$$

To summarize our argument to this point, we have:

Proposition 2.4. *If f is not ergodic, then there exist essentially invariant functions φ_+, φ_- such that:*

[1] Although we don't need it, Birkhoff's Ergodic Theorem implies that in fact it is not necessary to take a subsequence; $\lim \varphi_n$ exists almost everywhere.

(1) *φ_+, φ_- are not essentially constant functions,*
(2) *$\varphi_+ = \varphi_-$, almost everywhere,*
(3) *for all $x, y \in M$,*

$$y \in \mathcal{W}^s(x) \implies \varphi_+(x) = \varphi_+(y), \text{ and}$$
$$y \in \mathcal{W}^s(x) \implies \varphi_-(x) = \varphi_-(y).$$

To complete the proof of Theorem 2.1, assuming f is not ergodic, we exhibit an essentially invariant set A that is both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated, with

$$0 < m(A) < 1.$$

We find A as a sublevel set of $P(\varphi)$. For $a \in \mathbf{R}$, let

$$A_+(a) = \varphi_+^{-1}(-\infty, a), \quad \text{and} \quad A_-(a) = \varphi_-^{-1}(-\infty, a).$$

Proposition 2.4 implies that $A_+(a)$ and $A_-(a)$ are essentially invariant, $A_+(a)$ is $\mathcal{W}^s$-saturated, $A_-(a)$ is $\mathcal{W}^u$-saturated, and

$$m(A_+(a) \,\Delta\, A_-(a)) = 0.$$

Furthermore, there exists an $a \in \mathbf{R}$ such that

$$0 < m(A_+(a)) = m(A_-(a)) < 1.$$

Fix such an a, and let

$$A = A_+(a) \cap A_-(a).$$

Then A is essentially invariant, essentially $\mathcal{W}^s$-saturated, essentially $\mathcal{W}^u$-saturated, and

$$0 < m(A) < 1.$$

This completes the proof.$\diamond$

Exercise 2.5. *Prove von Neumann's Ergodic Theorem, as follows.*

(1) *Let $\mathcal{I} = \{\varphi \in \mathcal{H} \,|\, U(\varphi) = \varphi\}$ be the eigenspace of U-invariant elements. Prove that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} U^n(\varphi) = \varphi = P(\varphi),$$

for all $\varphi \in \mathcal{I}$.

(2) *Let* $\mathcal{B} = \{\psi - U(\psi) \mid \psi \in \mathcal{H}\}$ *be the set of* coboundaries *for* U. *Prove that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} U^n(\varphi) = 0 = P(\varphi),$$

for all $\varphi \in \mathcal{B}$.

(3) *Show that the orthogonal complement of* $\mathcal{I}$ *in* $\mathcal{H}$ *is* $\overline{\mathcal{B}}$, *the closure of* $\mathcal{B}$.

(4) *Conclude that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} U^n = P,$$

in the strong topology.

Exercise 2.6. *Show that if* $x \in \mathcal{W}^s(y)$, *and* $\varphi \in C^0(M)$, *then, for any subsequence* $n_j \to \infty$,

$$\limsup_{n_j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1} \varphi(f^i(x)) = \limsup_{n_j\to\infty} \frac{1}{n_j}\sum_{i=0}^{n_j-1} \varphi(f^i(y)).$$

2.0.1. *Application: ergodicity of Anosov diffeomorphisms.* Recall how the Hopf argument can be used to prove that every C^2, volume-preserving Anosov diffeomorphism is ergodic. An Anosov diffeomorphism is a partially hyperbolic diffeomorphism for which E^c is trivial. The stable and unstable foliations $\mathcal{W}^s$ and $\mathcal{W}^u$ are therefore:

(a) absolutely continuous, and
(b) transverse.

This means that in local volume-preserving (not necessarily smooth) coordinates on M, the leaves of $\mathcal{W}^s$ are horizontal, s-dimensional planes in $\mathbf{R}^n$, and the leaves of $\mathcal{W}^u$ are vertical, u-dimensional planes in $\mathbf{R}^n$, where $n = u + s$. In these coordinates, volume is Lebesgue measure in $\mathbf{R}^n$.

Suppose that A is any measurable set in M that is both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated. In these local coordinates, this translates into essential saturation of A with respect to both the foliation by horizontal planes and the foliation by vertical planes.

Fubini's theorem implies that A has full measure or measure 0 in this coordinate system. (The details are left as an exercise). Since M is connected, A has full measure or measure 0 in M. Theorem 2.1 implies that f is ergodic.

2.1. **Ergodicity of partially hyperbolic diffeomorphisms.** Suppose that $f : M \to M$ is partially hyperbolic and volume-preserving. The Hopf argument (Theorem 2.1) tells us that f is ergodic if and only if every set A that is essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated is trivial. Let us call a set that is both essentially $\mathcal{W}^s$-saturated and essentially $\mathcal{W}^u$-saturated *bi-essentially saturated.* The "bi" refers to the two foliations $\mathcal{W}^s$ and $\mathcal{W}^u$.

The property of accessibility can also be rephrased in terms of $\mathcal{W}^s$ and $\mathcal{W}^u$ saturated sets. We call a set that is both $\mathcal{W}^s$-saturated and $\mathcal{W}^u$-saturated *bi-saturated.* It is easy to see that f is accessible if and only if the only bi-saturated subsets of M are the empty set and M itself. As mentioned in the previous lecture, there is a property slightly weaker than accessibility, called essential accessibility.

Definition: f is *essentially accessible* if every measurable set that is essentially bi-saturated is trivial.

If it seems difficult to visualize the difference between accessibility and essential accessibility, it might be helpful to keep in mind the following two examples.

- The time-1 map of a geodesic flow $\varphi_t : T^1S \to T^1S$ for a negatively curved surface S is **accessible**. In fact, starting at any point (vector) $v \in T^1S$, we can reach any point in a neighborhood of v by a *su*-path with 4 short *legs* (subpaths of $\mathcal{W}^u$ and $\mathcal{W}^s$).
- An ergodic automorphism of a torus is **essentially accessible,** but not accessible (unless it is Anosov). The stable and unstable foliations are jointly integrable, so there is a foliation $\mathcal{W}^{us}$ whose leaves are tangent to $E^u \oplus E^s$. The leaves of $\mathcal{W}^{us}$ are cosets of a dense Lie subgroup of the torus. The only measurable $\mathcal{W}^{us}$-saturated sets are trivial.

Recall the main result I am discussing:

Theorem 2.7 (Burns-Wilkinson)**.** *Let f be C^2, volume-preserving, partially hyperbolic, and center bunched. If f is essentially accessible, then f is ergodic.*

The Hopf Argument says that if all bi-essentially saturated sets are trivial, then f is ergodic. Essential accessibility means that all essentially bi-saturated sets are trivial. To prove this theorem, it thus suffices to show:

If f is center bunched, then every bi-essentially saturated set is essentially bi-saturated.

How might we proceed in trying to prove this? Suppose we are given a set A that is bi-essentially saturated. We would like to modify it on a zero set to produce a set that is bi-saturated. We know we can modify A on a zero set to produce a new set, A^s, that is $\mathcal{W}^s$-saturated (We call such a set A^s a $\mathcal{W}^s$*-saturate* of A). We now would like to modify A^s on a zero-set to make it $\mathcal{W}^u$-saturated, still preserving $\mathcal{W}^s$-saturation. How? We know that $\mathcal{W}^u$ (and $\mathcal{W}^s$) are absolutely continuous. Since A^s is essentially $\mathcal{W}^u$-saturated, this means that for *almost every* $p \in A^s$, the leaf $\mathcal{W}^u(p)$ intersects A^s in a set of full leaf measure (i.e. Riemannian volume in $\mathcal{W}^u(p)$). We might try to discard those points in A^s whose $\mathcal{W}^u$- leaves don't meet A^s in a set of full leaf measure and throw into A^s all those $\mathcal{W}^u$-leaves that meet A^s in a set of full leaf measure. In doing so, we have modified A^s on a zero set and produced a $\mathcal{W}^u$-saturated set A^u (a $\mathcal{W}^u$*-saturate* of A) . But there is no guarantee that this new set A^u is $\mathcal{W}^s$-saturated.

The problem is that there are *many* ways to modify an essentially $\mathcal{W}^s$-saturated set on a zero-set to produce an essential $\mathcal{W}^s$-saturate, and there are many ways to modify an essentially $\mathcal{W}^u$-saturated set on a zero-set to produce an essential $\mathcal{W}^u$-saturate. All of these methods consist of throwing in some leaves and removing others. We need to find a method that throws away the right leaves to produce a bi-saturated set from an bi-essentially saturated set.

2.2. **Density points.** The Lebesgue Density Theorem gives a way to assign a *canonical* representative to every equivalence class of Lebesgue measurable set. If μ is a measure and A and B are μ-measurable sets with $\mu(B) > 0$, we define the *density of A in B* by:

$$\mu(A : B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

Now suppose that M is a Riemannian manifold, and m is normalized volume. A point $x \in M$ is a *Lebesgue density point* of a measurable set

$A \subseteq M$ if

$$\lim_{r\to 0} m(A : B(x,r)) = 1,$$

where $B(x,r)$ is the geodesic ball of radius r centered about x. Recall the Lebesgue Density Theorem:

Theorem 2.8 (Lebesgue Density Theorem)**.** *If A is a measurable set, and $DP(A)$ is the set of Lebesgue density points of A, then*

$$m(A \,\Delta\, DP(A)) = 0.$$

Denote by $DP(A)$ the set of Lebesgue density points of a measurable set A. If A and B differ by a 0-set, then $DP(A) = DP(B)$.

Theorem 1.3 follows from the following:

Theorem 2.9. *Let f be C^2, partially hyperbolic, and center bunched. If A is bi-essentially saturated, then $DP(A)$ is bi-saturated. Thus, every bi-essentially saturated set is essentially bi-saturated.*

Consequently, if f is essentially accessible and preserves volume, then f is ergodic.

3. Lecture III

We are now at the heart of the proof of Theorem 1.3. We are given a partially hyperbolic diffeomorphism $f : M \to M$. To prove that f is ergodic, we will show that if A is a bi-essentially saturated set, then the set of density points $DP(A)$ is bi-saturated. To do so, we will eventually have to assume that the center bunching condition holds (which we have still not defined). We will define this condition when it arises naturally in the argument (in Lecture V.)

There are two steps to the argument:

Step 1. Show that every Lebesgue density point of a bi-essentially saturated set is a julienne density point. To define julienne density points, we construct, for each $x \in M$, a special kind of sequence of sets $J_n(x)$, called juliennes. A julienne density point is like a Lebesgue density point, except that we measure density inside of juliennes rather than inside of balls. Juliennes are defined using dynamical objects such as invariant foliations.

Step 2. Show that the set of julienne density points of a bi-essentially saturated set is bi-saturated. Because juliennes are constructed dynamically, they are not distorted excessively when we slide them along leaves of the stable and unstable foliations. This fact, combined with absolute continuity of the $\mathcal{W}^s$ and $\mathcal{W}^u$ foliations, allows us to show that julienne density points are not destroyed by sliding along leaves of these foliations.

3.1. **What is a julienne?** A julienne is a dynamically-defined object especially designed to measure the density of a bi-essentially saturated set. There are two-types of juliennes: *cu*-juliennes and *cs*-juliennes. A *cu*-julienne is a manifold approximately tangent to the subbundle $E^{cu} = E^c \oplus E^u$, and transverse to the $\mathcal{W}^s$-foliation. Analogously, a *cs*-julienne is a manifold approximately tangent to the subbundle $E^{cs} = E^c \oplus E^s$, and transverse to the $\mathcal{W}^u$-foliation. If there are invariant foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ tangent to the subbundles E^{cu} and E^{cs}, respectively, then we can use these foliations to define *cu*- and *cs*-juliennes, respectively. If these foliations do not exist, then we must do more work to construct juliennes. This is carried out in [BW2].

To simplify the discussion, let's assume that there are foliations $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ tangent to E^{cu} and E^{cs}, respectively. If such foliations exist, we say that f is *dynamically coherent.* There are many examples of dynamically coherent partially hyperbolic diffeomorphisms. All of the examples in Lecture I are dynamically coherent, for example (see Exercise 3.1). If f is dynamically coherent, then:

- There is an invariant foliation $\mathcal{W}^c$, called a *center foliation*, tangent to E^c, whose leaves are found by intersecting the leaves of $\mathcal{W}^{cu}$ with the leaves of $\mathcal{W}^{cs}$.
- The leaves of $\mathcal{W}^{cs}$ are subfoliated by the leaves of $\mathcal{W}^c$ (this is by construction of $\mathcal{W}^c$). That is, each leaf of $\mathcal{W}^{cs}$ is the union of leaves of $\mathcal{W}^c$. Similarly, the leaves of $\mathcal{W}^{cu}$ are subfoliated by the leaves of $\mathcal{W}^c$.
- The leaves of $\mathcal{W}^{cs}$ are also subfoliated by the leaves of $\mathcal{W}^s$ (this is a consequence of the unique integrability of $\mathcal{W}^s$). Similarly, the leaves of $\mathcal{W}^{cu}$ are subfoliated by the leaves of $\mathcal{W}^u$

As Flavio mentioned, not every partially hyperbolic diffeomorphism is dynamically coherent: the center subbundle can fail to be integrable.

Assuming now that f is dynamically coherent, let's construct some juliennes. We introduce some convenient notation. If $x \in M$, and n is an integer, then x_n denotes the point $f^n(x)$. If $\mathcal{F}$ is a foliation, and d is a metric on M. then $\mathcal{F}_d(x, r)$ will denote the connected component of

x in the intersection of the ball $B(x, r)$ with the leaf $\mathcal{F}(x)$ containing x. When the metric d is the standard (background) Riemannian metric we will drop the d subscript and write $\mathcal{F}(x, r)$.

Recall from Flavio's lectures the definition of (pointwise) partial hyperbolicity. We have a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, γ and $\hat{\gamma}$ with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{1}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{2}$$

$$\gamma(p) < \|Tfv\| < \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{3}$$

$$\hat{\nu}(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{4}$$

We will assume for simplicity that ν, $\hat{\nu}$, γ and $\hat{\gamma}$ are constant functions. These constants give bounds on the growth rates of vectors in the subbundles in the partially hyperbolic splitting. After integration, growth rates of vectors translate into growth rates of curves. Thus, by rescaling our Riemannian metric if necessary (the details are in [BW]), we may assume that, for all $n \geq 0$, and all $p \in M$:

- if $x, y \in \mathcal{W}^s(p, 2)$, then
$$d(x_n, y_n) \leq \nu^n d(x, y),$$
- if $x_i, y_i \in \mathcal{W}^{cs}(p_i, 2)$ for $i = 0, \ldots, n-1$, then
$$d(x_n, y_n) \leq \gamma^n d(x, y),$$
- if $x, y \in \mathcal{W}^u(p, 2)$, then
$$d(x_{-n}, y_{-n}) \leq \hat{\nu}^n d(x, y),$$
- if $x_{-i}, y_{-i} \in \mathcal{W}^{cu}(p_i, 2)$ for $i = 0, \ldots, n-1$, then
$$d(x_{-n}, y_{-n}) \leq \hat{\gamma}^n d(x, y).$$

We will sometimes refer to $\mathcal{F}(x, 2)$ as a *local leaf* of $\mathcal{F}$. We may also assume that inside any d-ball of radius 2, the foliations have no topology (any two local leaves from any pair of transverse foliations intersect in at most one point).

Fix $x \in M$. We are now ready to descibe the juliennes through x. To construct the biggest julienne $J_0^{cu}(x)$ through x, we start with a ball $\mathcal{W}^c(x, 1)$ in the center manifold of x. Through each $y \in \mathcal{W}^c(x, 1)$, we attach the ball $\mathcal{W}^u(y, 1)$. We obtain a subset of $\mathcal{W}^{cu}(x)$ that looks crudely like a cube. The next julienne $J_1^{cu}(x)$ will be contained in the first. To construct $J_1^{cu}(x)$, we start with a ball $\mathcal{W}^c(x,\sigma)$ of radius $\sigma < 1$. Through each point $y \in \mathcal{W}^c(x,\sigma)$ we attach the set $f^{-1}\mathcal{W}^u(x_1,\tau)$, where $\tau < 1$. Since $\tau < 1$, and f^{-1} contracts distances in $\mathcal{W}^u$, the diameter of the set $f^{-1}\mathcal{W}^u(x_1,\tau)$ is strictly less than 1 — it is not hard to see that $J_1^{cu}(x) \subsetneq J_0^{cu}(x)$. More generally, for $n \geq 0$, we define:

$$J_n^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x,\sigma^n)} f^{-n}\left(\mathcal{W}^u(x_n,\tau^n)\right).$$

The juliennes we have constructed depend on the choice of σ, τ. We will specify later how to choose these constants. The point x is called the *center* of the julienne $J_n^{cu}(x)$, and n is called its *rank*.

What does the set $J_n^{cu}(x)$ look like? The juliennes become exponentially small as n tends to ∞; in particular, $\bigcap_{n\geq 0} J_n^{cu}(x) = \{x\}$, and this intersection is nested. While the first julienne $J_0^{cu}(x)$ looks roughly like a cube, as n increases, $J_n^{cu}(x)$ becomes increasingly elongated in some directions relative to others, depending on our choice of σ, τ. If, for example, $\tau < \sigma$, then $J_n^{cu}(x)$ will be exponentially thinner in the unstable direction than in the center direction. This is the origin of the term “julienne,” a term in French cooking (see [GPS]).

The julienne $J_n^{cu}(x)$ looks like a ball or radius σ^n in the center direction. In the unstable direction, $J_n^{cu}(x)$ is also ball-like, but in a diferent metric than the standard Riemannian metric, called the d_n*-metric.*

Definition: Let n be an integer. For $x, y \in M$, let

$$d_n(x,y) = \sup_{i=0,\dots,n} d(x_i,y_i).$$

Note that $d_0 = d$.

It is easy to see that d_n defines a metric on M, for each n. These metrics are used in the definition of topological entropy of f. See, e.g., [Ma]. It is an easy exercise (Exercise 3.2) to show that

$$J_n^{cu}(x) = \bigcup_{y \in \mathcal{W}^c(x,\sigma^n)} \mathcal{W}^u_{d_n}(y,\tau^n).$$

To conclude:

A cu-julienne is the union of d_n-balls in $\mathcal{W}^u$ of radius τ^n passing through a d_0-ball in $\mathcal{W}^c$ of radius σ^n.

Similarly, though I'll leave the details of the definition to the reader, we have:

A cs-julienne is the union of d_{-n}-balls in $\mathcal{W}^s$ of radius τ^n passing through a d_0-ball in $\mathcal{W}^c$ of radius σ^n.

Finally, we explain what it means for x to be a julienne density point. We define *cu*-julienne density points; *cs*-density points are defined analogously (replacing s for u).

Definition: Let A be a bi-essentially saturated set. Then x is a julienne density point of A if, for every stable saturate A^s of A, we have:

$$\lim_{n\to\infty} m^{cu}(A^s : J_n(x)) = 1,$$

where m^{cu} is the induced Riemannian measure on $\mathcal{W}^{cu}$.

Exercise 3.1. *Verify that if $f : N\times V \to M\times V$ is a partially hyperbolic skew product of the form:*

$$f(x,y) = (g(x), h(x,y)),$$

where g is Anosov and $h : N\times V \to V$, then f is dynamically coherent.

Exercise 3.2. *Prove that d_n is a metric. Show that*

$$J_n^{cu}(x) = \bigcup_{y\in\mathcal{W}^c(x,\sigma^n)} \mathcal{W}^u_{d_n}(y,\tau^n).$$

Exercise 3.3. *Let A be a bi-essentially saturated set. Show that x is a julienne density point of A if and only if, for* some *stable saturate A^s of A, we have:*

$$\lim_{n\to\infty} m^{cu}(A^s : J_n(x)) = 1.$$

*(**Hint:** $\mathcal{W}^s$ is absolutely continuous.)*

4. Lecture IV

In this lecture, we begin to describe to the proof of:

Step 1. Show that every Lebesgue density point of a bi-essentially saturated set is a julienne density point.

We will show this for *cu*-juliennes. The proof for *cs*-juliennes is completely analogous. Recall that we are using x_i to denote $f_i(x)$. If a_n and b_n are sequences of real numbers, the notation $a_n \asymp b_n$ means that there exists a constant $C \geq 1$ such that $\frac{a_n}{b_n} \in [C^{-1}, C]$, for all $n \geq 0$, and $a_n \precsim b_n$ means that there exists a constant $C \geq 1$ such that $a_n \leq C b_n$, for all $n \geq 0$.

4.1. **Density sequences.** We examine first what it means for a point x to be a Lebesgue density point of a bi-essentially saturated set A. Our goal is to weaken and reformulate this condition until we can use cu-juliennes to define Lebesgue density.

Let $X_0 \supset X_1 \supset X_2 \supset \cdots$ be a nested sequence of sets such that $\bigcap_n X_n = \{x\}$. We use the notation $X_n \searrow x$ to denote this, and sometimes we will write $X_n(x)$ for X_n to make transparent the role of x.

We say that $X_n \searrow x$ is a *Lebesgue density sequence* if, for every measurable set A,

$$x \text{ is a Lebesgue density point of } A \quad \Leftrightarrow \quad \lim_{n\to\infty} m(A : X_n) = 1.$$

Similarly, we say that $X_n \searrow x$ is an *su-density sequence at x* if, for every measurable bi-essentially saturated set A,

$$x \text{ is a Lebesgue density point of } A \quad \Leftrightarrow \quad \lim_{n\to\infty} m(A : X_n(x)) = 1.$$

Every Lebesgue density sequence is (obviously) an su-density sequence, but there is no reason to expect the converse implication to hold.

Suppose x is a Lebesgue density point for a measurable set A. Recall this means that

$$\lim_{r\to 0} m(A : B(x,r)) = 1. \tag{5}$$

Clearly, if (5) holds, then for any subsequence $r_n \to 0$, we have

$$\lim_{n\to\infty} m(A : B(x,r_n)) = 1, \tag{6}$$

but for which subsequences r_n does (6) imply (5)? An elementary argument (Exercise 4.8) shows that r_n cannot approach 0 too quickly. In terms of the volumes of the balls used in calculating the density $m(A : B(x,r_n))$, we need that there is a constant $\delta > 0$ such that:

$$m(B(x,r_{n+1})) \geq \delta m(B(x,r_n)), \tag{7}$$

for all n, which translates into a similar condition on the radii r_n. Condition (7) is satisfied, for example, if we choose $r_n = \sigma^n$, for some $\sigma < 1$. We formulate this property (7) of the sequences $B(x,\sigma^n)$ as a general definition.

Definition: $X_n \searrow x$ is *volumetrically regular*, or *v-regular* for short, if there exists $\delta > 0$ such that, for all $n \geq 0$, we have:

$$\frac{m(X_{n+1})}{m(X_n)} \geq \delta.$$

We have just seen:

the balls $B(x, \sigma^n) \searrow x$ form a v-regular Lebesgue density sequence.

It is well-known that the round balls $B(x, r)$ used in computing Lebesgue density points can be replaced by cubes, or by any shape invariant under conformal dilation. Denoting the cube in M centered at x of side radius r by $Q(x, r)$, we have, for example, that for any measurable set A,

$$\lim_{n\to\infty} m(A : B(x, \sigma^n)) \quad \Leftrightarrow \quad \lim_{n\to\infty} m(A : Q(x, \sigma^n)) = 1.$$

Thus the cubes $Q(x, \sigma^n) \searrow x$ are a Lebesgue density sequence. The reason why is a simple geometric fact: cubes and balls *internest:*

Definition: Two sequences $X_n \searrow x$ and $Y_n \searrow x$ are *internested* if there exists a $k > 0$ such that

$$X_{n+k} \subset Y_n \quad \text{and} \quad Y_{n+k} \subset X_n,$$

for all $n \geq 0$.

Internested sequences are important to us for the following reason:

Proposition 4.1. *If $X_n \searrow x$ is v-regular and $X_n \searrow x$ and $Y_n \searrow x$ are internested, then $Y_n \searrow x$ is v-regular and, for every A, we have:*

$$\lim_{n\to\infty} m(A : X_n) = 1 \iff \lim_{n\to\infty} m(A : Y_n) = 1.$$

The proof is an exercise, more or less in chasing definitions. We'll use this proposition to construct density sequences. An immediate corollary is:

Corollary 4.2. *If $X_n \searrow x$ is a v-regular Lebesgue (resp. su) density sequence, and $X_n \searrow x$ and $Y_n \searrow x$ are internested, then $Y_n \searrow x$ is a v-regular Lebesgue (resp. su) density sequence.*

We now give the background material necessary for constructing *su*-density sequences at x. Recall the constants σ, τ used in the construction of juliennes. We now put an additional constraint on the choices of $\sigma, \tau < 1$:

$$\sigma\nu < \tau < \sigma\hat{\nu}^{-1}, \qquad \sigma < \hat{\gamma}.$$

Notice that it is always possible to choose σ, τ satisfying these definitions, for any partially hyperbolic system, since $\nu < 1 < \hat{\nu}^{-1}$. Fix a number κ close to 1, greater than both σ and $\sigma\hat{\gamma}^{-1}$. All sequences will be constructed inside a fixed **environment** around x, using a fixed set of **building blocks**, and one fixed **construction tool**.

Environment. Each set X_n in any *su*-density seqeunce $X_n \searrow x$ we construct will be contained inside the d_n-ball:

$$B_{d_n}(x, \kappa^n) = \{y \,|\, d_n(x, y) < \kappa^n\}.$$

These balls have the important feature that

Hölder-continuous f^n-cocycles have bounded distortion inside $B_{d_n}(x, \kappa^n)$.

I'll discuss this property in Lemma 4.6 below. It is used to show that our building blocks are nicely behaved.

Building blocks. All of the sequences $X_n \searrow x$ we construct will be built out of two basic types of building blocks. For each $n \geq 0$ and $y \in B_{d_n}(x, \kappa^n)$, we will use:

- d**-balls:** $\mathcal{W}^u(y, \sigma^n)$, $\mathcal{W}^c(y, \sigma^n)$, and $\mathcal{W}^s(y, \sigma^n)$,
- d_n**-balls:** $\mathcal{W}^u_{d_n}(y, \tau^n)$.

Recall, for example, that *cu*-juliennes are constructed by taking a union of d_n-balls $\mathcal{W}^u_{d_n}(y, \tau^n)$ through each point y in the central d-ball $\mathcal{W}^c(y, \sigma^n)$.

Our choice of σ, τ gives these sets two important properties:

(1) The diameter of the first n iterates of any set taken Z_n from this list is exponentially small, less than a constant times κ^n:

$$\text{diam } (f^i(Z_n)) \precsim \kappa^n, \qquad i = 0, \ldots, n-1.$$

(**Exercise:** Verify this.)

(2) If Z_n and Z'_n are two sets of the same type, constructed inside leaves of the foliation $\mathcal{F}$ (e.g. two d_n-balls $\mathcal{W}^u_{d_n}(y, \tau^n)$ and $\mathcal{W}^u_{d_n}(y', \tau^n)$ in the foliation $\mathcal{W}^u$), then

$$m^{\mathcal{F}}(Z_n) \asymp m^{\mathcal{F}}(Z'_n),$$

where $m^{\mathcal{F}}$ is the induced Riemannian metric on the leaves of $\mathcal{F}$.

I'll discuss the second property in more detail below (see Corollary 4.7).

Construction tool. Every set $X_n(x)$ in the sequences $X_n \searrow x$ we consider will be a local fibration of the form:

$$X_n(x) = \bigcup_{y \in B_n(x)} F_n(y),$$

where $B_n \searrow x$ is called the *base* of $X_n \searrow x$, and $F_n(y)$ is the *fiber* of X_n over $y \in X_n(x)$. We call $X_n \searrow x$ a *fibered sequence.* The fibers in our constructions will always consist of building blocks taken from a fixed foliation, usually either $\mathcal{W}^u$ or $\mathcal{W}^s$.

For example, the *cu*-juliennes $J^{cu}_n(x) \searrow x$ have base $\mathcal{W}^c(x, \sigma^n)$ and fiber $\mathcal{W}^u_{d_n}(y, \tau^n)$ over each $y \in \mathcal{W}^c(x, \sigma^n)$.

I'll now discuss the key property of these fibered sequences. To do so, we need to revisit the property of absolute continuity.

4.2. **Absolute continuity revisited.** There is some subtlety in defining absolute continuity of a foliation, which I will ignore. See [BW2] for a careful discussion. There are two consequences of absolute continuity I'll use, both of which hold for the foliations $\mathcal{W}^u$ and $\mathcal{W}^s$. The first consequence, discussed by Flavio, involves holonomy maps:

Proposition 4.3. *Let p, p' be two points on the same leaf of the foliation $\mathcal{W}^s$, chosen so that the leaf distance between p and p' is ≤ 1 (or any other fixed constant). Then the $\mathcal{W}^s$-holonomy map*

$$h^s : \mathcal{W}^{cu}(p, 1) \to \mathcal{W}^{cu}(p')$$

(defined by sliding along leaves of $\mathcal{W}^s$ from $\mathcal{W}^{cu}(p)$ to $\mathcal{W}^{cu}(p')$) is uniformly absolutely continuous: for every measurable set $A \subset \mathcal{W}^{cs}(p, 1)$,

$$m^{cu}(h(A)) \asymp m^{cu}(A).$$

Similarly, the $\mathcal{W}^u$-holonomy map between $\mathcal{W}^{cs}$-transversals is uniformly absolutely continuous.

The proof of Proposition 4.3 goes back to Anosov and Sinai, and can be found, in varying levels of generality in [A, AS, PS1, BP] and (in the pointwise partially hyperbolic setting) in the forthcoming preprint [AV].

The second consequence of absolute continuity of $\mathcal{W}^u$ and $\mathcal{W}^s$ allows us to estimate volumes of fibered sequences with absolutely continuous fibers. We need a definition first.

Definition: Let $X_n \searrow x$ be a fibered sequence with base $B_n \searrow x$ and fibers F_n. Suppose that the fibers of F_n lie entirely in leaves of the foliation $\mathcal{W}^u$ (respectively, entirely in leaves of $\mathcal{W}^s$). Then $X_n \searrow x$ has *volumetrically uniform fibers*, or *v-uniform fibers* for short, if, for every $y, y' \in X_n(x)$,

$$m^u(F_n(y)) \asymp m^u(F_n(y')),$$

where m^u is induced Riemannian volume on the leaves of $\mathcal{W}^u$ (respectively,

$$m^s(F_n(y)) \asymp m^s(F_n(y')),$$

where m^u is induced Riemannian volume on the leaves of $\mathcal{W}^s$). Notice that the v-uniform fibers property makes no claim about the geometry of the fibers of $X_n \searrow x$. While the volumes of two fibers must be comparable, one fiber could be shaped like a ball, and the other like a long, thin and winding snake.

The following proposition is a corollary of Proposition 4.3. A proof can be found in [BW2], following ideas in [BS].

Proposition 4.4. *Let $X_n \searrow x$ be a fibered sequence with base $B_n \searrow x$ lying in $\mathcal{W}^{cu}(x)$ and fibers F_n lying in leaves of $\mathcal{W}^s$. Assume that the fibers are v-uniform. Then*

$$m(X_n) \asymp m^{cu}(B_n) m^s(F_n),$$

where $m^s(F_n)$ is the volume of any fiber of X_n, which is well-defined up to a uniform multiplicative constant.

Similarly, if the base of $X_n \searrow x$ lies in $\mathcal{W}^{cs}(x)$ and the fibers lie in leaves of $\mathcal{W}^u$, then

$$m(X_n) \asymp m^{cs}(B_n) m^u(F_n).$$

Proposition 4.4 brings to mind the grade-school refrain: "volume equals base times height" — provided that "height" is measured in an absolutely continuous direction. More accurately, Proposition 4.4 is reminscent of Cavalieri's Principle, which states that the volume of a 3-dimensional solid may computed by taking two-dimensional slices of the solid. If the areas of these slices are equal, then volume of the solid is equal to the product of the slice area and the total height of the the slices.

4.3. **The key property of fibered sequences.** Using Proposition 4.4 we can now state:

Proposition 4.5. *Let $X_n \searrow x$ be a fibered sequence with base $B_n \searrow x$ contained in $\mathcal{W}^{cu}(x)$ and fibers F_n lying in leaves of $\mathcal{W}^s$. Assume that the fibers of F_n are v-uniform. Let A be an essentially $\mathcal{W}^s$ saturated set (e.g. a bi-essentially saturated set). Then, for any $\mathcal{W}^s$-saturate A^s of A, we have:*

$$\lim_{n\to\infty} m(A : X_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} m^{cu}(A^s : B_n) = 1.$$

Similarly, if the base $B_n \searrow x$ lies in $\mathcal{W}^{cs}(x)$ and the fibers F_n lie in leaves of $\mathcal{W}^u$ and are v-uniform, then for any essentially $\mathcal{W}^u$-saturated set A and any esential $\mathcal{W}^u$-saturate A^u of A, we have:

$$\lim_{n\to\infty} m(A : X_n) = 1 \quad \Leftrightarrow \lim_{n\to\infty} \quad m^{cs}(A^u : B_n) = 1.$$

Proof. Let $A' = M \setminus A$ be the complement of A, and let $(A^s)' = M \setminus A^s$ be the complement of a $\mathcal{W}^s$-saturate A^s of A. Note that $(A^s)'$ is a $\mathcal{W}^s$-saturate of A'. We'll show that

$$m(A' : X_n) \asymp m^{cu}((A^s)' : B_n), \tag{8}$$

which implies that

$$\lim_{n\to\infty} m(A' : X_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} m^{cu}((A^s)' : B_n) = 0.$$

This in turn implies the proposition.

Since $(A^s)'$ is $\mathcal{W}^s$-saturated, the sequence $(A^s)' \cap X_n \searrow x$ is a fibered sequence with base $B_n \cap (A^s)' \searrow x$ and v-uniform fibers F_n. By Proposition 4.4,

$$m((A^s)' \cap X_n) \asymp m^{cu}((A^s)' \cap B_n) m^s(F_n)$$

Since $m(X_n) \asymp m^{cu}(B_n)m^s(F_n)$, we obtain:

$$\begin{aligned} m(A' : X_n) &= m((A^s)' : X_n) \\ &= \frac{m((A^s)' \cap X_n)}{m(X_n)} \\ &\asymp \frac{m^{cu}((A^s)' \cap B_n)m^s(F_n)}{m^{cu}(B_n)m^s(F_n)} \\ &= \frac{m^{cu}((A^s)' \cap B_n)}{m^{cu}(B_n)} \end{aligned}$$

$$= m^{cu}((A^s)' : B_n),$$

establishing (8).$\diamond$

4.4. **The d_n metric and distortion.** Recall that we are trying constructing density sequences in a fixed environment $B_{d_n}(x, \sigma^n)$ out of building blocks consisting of balls and d_n-balls, using the fibered sequence construction. We will use Proposition 4.4 to go from one nested sequence to another. To do this, we need our fibers to be v-uniform. The technique we use here is fundamental in measurable smooth dynamics, and you've already seen it in the proof of absolute continuity of $\mathcal{W}^u$ and $\mathcal{W}^s$. See [Ma], for example, for many other applications of this type of argument.

There are two types of fibers we might use: balls in $\mathcal{W}^s$ or $\mathcal{W}^u$, and d_n-balls in $\mathcal{W}^u$. It is easy to see that two balls of the same radius have comparable volumes: this follows from compactness of M and the continuity of the foliations $\mathcal{W}^s$ and $\mathcal{W}^u$. Thus for *any* $y, y' \in M$, we have

$$m^u(\mathcal{W}^u(y, \sigma^n)) \asymp m^u(\mathcal{W}^u(y', \sigma^n)) \text{ and } m^s(\mathcal{W}^s(y, \sigma^n)) \asymp m^s(\mathcal{W}^s(y', \sigma^n)).$$

The situation with d_n-balls is considerably more delicate. The d_n-ball $\mathcal{W}^u_{d_n}(y, \tau^n)$ is really the iterate under f^{-n} of the ball $\mathcal{W}^u(y_n, \tau^n)$. The volume of $\mathcal{W}^u_{d_n}(y, \tau^n)$ is thus given by the formula:

$$m^u(\mathcal{W}^u_{d_n}(y, \tau^n)) = \int_{\mathcal{W}^u(y_n, \tau^n)} \mathrm{Jac}^u(f^{-n})\, dm^u, \tag{9}$$

where, for $k \in \mathbf{Z}$, the function $\mathrm{Jac}^u(f^k)$ is the "unstable Jacobian" of f^k — the Jacobian of the restriction of f^k to the leaves of $\mathcal{W}^u$:

$$\mathrm{Jac}^u(f^k)(p) = \det(T_p f^k|_{E^u}).$$

Suppose for a minute we knew that:

$$\text{Jac}^u(f^{-n})(p) \asymp \text{Jac}^u(f^{-n})(p'), \tag{10}$$

for all $p, p' \in M$. Combining (9) with (10), we would then get:

$$\frac{m^u(\mathcal{W}^u_{d_n}(y, \tau^n))}{m^u(\mathcal{W}^u_{d_n}(y', \tau^n))} \asymp \frac{m^u(\mathcal{W}^u(y_n, \tau^n))}{m^u(\mathcal{W}^u(y'_n, \tau^n))}.$$

for all $y, y' \in M$, as $n \to \infty$. Since (as we saw above)

$$\frac{m^u(\mathcal{W}^u(y_n, \tau^n))}{m^u(\mathcal{W}^u(y'_n, \tau^n))} \asymp 1,$$

for any y, y', we could then obtain:

$$m^u(\mathcal{W}^u_{d_n}(y, \tau^n)) \asymp m^u(\mathcal{W}^u_{d_n}(y, \tau^n)), \tag{11}$$

for any y, y', which is the type of estimate we need for v-uniformity.

There is no hope in general, however, that (10) can hold for *all* $y, y' \in M$. The Chain Rule implies that the unstable Jacobian is a multiplicative cocycle, i.e., for all $k \geq 0$,

$$\text{Jac}^u(f^k)(x) = \text{Jac}^u(f)(x) \cdot \text{Jac}^u(f)(x_1) \cdots \text{Jac}^u(f)(x_{k-1}),$$

and for all $k < 0$,

$$\text{Jac}^u(f^k)(x) = \frac{1}{\text{Jac}^u(f^{-k})(x_{-k})}.$$

Thus, unless $\text{Jac}^u(f)$ is identically equal to a constant, the values of $\text{Jac}(f^n)$ will fluctuate on M, and this fluctuation (also called *distortion*) will be unbounded as a function of n.

Fortunately, proving (11) for *all* $y, y' \in M$ is not only more than we can do, it is more than we need to do. Since the environment in which we construct our density sequences is the sequence of d_n-balls:

$$B_{d_n}(x, \kappa^n) = \{y \,|\, d_n(x, y) < \kappa^n),$$

we need only establish (11) for all $y, y' \in B_{d_n}(x, \sigma^n)$. Here we indicate how this is done.

If $\alpha : M \to \mathbf{R}$ is a positive real-valued function, and $n \geq 0$ is an integer, then we denote by α_n the function:

$$\alpha_n(p) = \alpha(p)\alpha(p_1) \cdots \alpha(p_{n-1}).$$

As observed above, if $\alpha = \text{Jac}^u(f)$, then $\alpha_n = \text{Jac}^u(f^n)$.

Lemma 4.6. *Let α be a Hölder continuous function, let $p \in M$, and let $\kappa < 1$. Then, for all $p, p' \in B_n(p, \kappa^n)$,*

$$\alpha_n(p) \asymp \alpha_n(p').$$

Proof. The function $\log \alpha$ is also Hölder continuous; ket $\theta \in (0,1)$ be its Hölder exponent. Let $H > 0$ be the Hölder constant of $\log \alpha$, so that for all $x, y \in M$:

$$|\log \alpha(x) - \log \alpha(y)| \leq H d(x,y)^\theta.$$

Then $\alpha_n(p) \asymp \alpha_n(p')$ if:

$$|\log \alpha_n(p) - \log \alpha_n(q)| \leq c,$$

for some $c > 0$. Expanding $\log \alpha_n$ as a series, we obtain:

$$\begin{aligned} |\log \alpha_n(p) - \log \alpha_n(q)| &\leq \sum_{i=0}^{n-1} |\log \alpha(p_i) - \log \alpha(q_i)| \\ &\leq H \sum_{j=0}^{n-1} d(p_j, q_j)^\theta. \\ &\leq H n \kappa^{n\theta}. \end{aligned}$$

Since $Hn\kappa^{n\theta} \to 0$ as $n \to \infty$, this establishes the result. $\diamond$

Corollary 4.7. *d_n-balls in $\mathcal{W}^u$ of radius τ^n lying inside of $d_n(x, \kappa^n)$ form a v-uniform family.*

Proof. Since E^u is a Hölder continuous bundle (as Flavio explained) and f is C^2, the function $\alpha = \mathrm{Jac}^u(f)$ is Hölder continuous. The corollary follows from (9) and Lemma 4.6. The details are left as an exercise.

Exercise 4.8. *Show that (6) implies (5) for every measurable set A if and only if there exists a constant $\delta > 0$ such that $r_{n+1} \geq \delta r_n$.*

Exercise 4.9. *Prove Proposition 4.1.*

Exercise 4.10. *Show that cu-juliennes $J_n^{cu}(x) \searrow x$ form a v-regular sequence (with respect to the induced Riemannian volume m^{cu} on $\mathcal{W}^{cu}(x)$). Hint: the restriction of $\mathcal{W}^u$ to $\mathcal{W}^{cu}(x)$ is an absolutely continuous foliation.*

5. Lecture V

We now finish the proof of our Main Theorem (reformulated). Having set everything up in the previous lecture, the proof of Step 1 is now fairly direct. Recall we have chosen $\sigma, \tau < 1$ such that:

$$\sigma\nu < \tau < \sigma\hat{\nu}^{-1}, \qquad \sigma < \hat{\gamma}.$$

Suppose that x is a Lebesgue density point of a bi-essentially saturated set A. We want to show that x is a julienne density point of A.

Let $X_n \searrow x$ be the v-regular Lebesgue density sequence

$$X_n = B_d(x, \sigma^n).$$

We use the foliations $\mathcal{W}^u, \mathcal{W}^c, \mathcal{W}^s$ to build up a cube-like sequence $Y_n \searrow x$ that is comparable to X_n. We first construct its base $B_n \searrow x$, setting

$$B_n = \bigcup_{y \in \mathcal{W}^s(x,\sigma^n)} \mathcal{W}^c(y, \sigma^n).$$

It is not hard to see (from the continuity and transversality of the foliations $\mathcal{W}^s, \mathcal{W}^c$) that B_n is comparable to the ball $\mathcal{W}^{cs}(x, \sigma^n)$. Now define $Y_n \searrow x$ by:

$$Y_n = \bigcup_{y \in B_n} \mathcal{W}^u(y, \sigma^n).$$

Now it is a straightforward exercise to show that $X_n \searrow x$ and $Y_n \searrow x$ are comparable. Hence $Y_n \searrow x$ is a v-regular Lebesgue density sequence. Since x is a Lebesgue density point of A, we have:

$$\lim_{n\to\infty} m(A : Y_n) = 1. \tag{12}$$

Next, we define $Z_n \searrow x$ using the same base as $Y_n \searrow x$, but different fibers:

$$Z_n = \bigcup_{y \in B_n} \mathcal{W}^c_{d_n}(y, \tau^n).$$

Because we have chosen $\sigma < \hat{\gamma}$, the sequence $Z_n \searrow x$ lies in our environment $B_{d_n}(x, \asymp \kappa^n)$: the d_n-distance from a point $z \in Z_n$ to x is less than or equal to the d_n-distance from z to B_n, which is less than τ^n plus the radius of B_n in the d_n-metric, which, as we saw in the previous lecture, is less than κ^n. The d_n-distance from z to x is $\precsim \kappa^n$, since $\tau^n + \kappa^n \precsim \kappa^n$.

Since $Z_n \searrow x$ lies in our environment, Corollary 4.7 implies that its fibers $\mathcal{W}^c_{d_n}(y, \tau^n)$ are v-uniform. Since A is essentially W^u-saturated, Proposition 4.5 implies that

$$(13) \qquad \lim_{n\to\infty} m(A : Z_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} m^{cs}(A^u : B_n) = 1,$$

where A^u is any $\mathcal{W}^u$-saturate of A. Going back to the sequence $Y_n \searrow x$, we note that it has v-uniform fibers, too: the balls $\mathcal{W}^u(y, \sigma^n)$. Applying Proposition 4.5 to $Y_n \searrow x$, we have for any $\mathcal{W}^u$ saturate A^u of A,

$$(14) \qquad \lim_{n\to\infty} m(A : Y_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} m^{cs}(A^u : B_n) = 1.$$

Now (13) and (14) imply that

$$(15) \qquad \lim_{n\to\infty} m(A : Y_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} m(A : Z_n) = 1,$$

and so (12) implies that

$$(16) \qquad \lim_{n\to\infty} m(A : Z_n) = 1.$$

We construct one more nested sequence, this time using cu-juliennes for the base sequence. Define $W_n \searrow x$ by:

$$W_n = \bigcup_{y \in J^{cu}_n(x)} \mathcal{W}^s(x, \sigma^n).$$

The fibers of W_n, being balls in $\mathcal{W}^s$, are v-uniform, and so Proposition 4.5 implies that if A is any bi-essentially saturated set, then, for any $\mathcal{W}^s$-saturate A^s of A, we have the equivalence:

$$(17) \qquad m(A : W_n) \to 1 \quad \Leftrightarrow \quad m^{cu}(A^s : J^{cu}_n(x)) = 1.$$

If we can show that

$$(18) \qquad \lim_{n\to\infty} m(A : Z_n) = 1 \quad \Leftrightarrow \quad \lim_{n\to\infty} m(A : W_n) = 1,$$

then we are done, since (16) and (17) then imply that x is a julienne density point, which is the desired conclusion.

The sequence $W_n \searrow x$ is v-regular (Exercise 5.5). By Proposition 4.1, (18) holds if $Z_n \searrow x$ and $W_n \searrow x$ are internested.

To get to a point $z \in Z_n$ from x, we first go out a distance $\precsim \sigma^n$ in $\mathcal{W}^{cs}(x)$ (to get to a point $y \in B_n$), and then we move a d_n-distance $\precsim \tau^n$ in $\mathcal{W}^u(y)$ to get to z.

On the other hand, to get to a point $w \in W_n$, we first go a distance $\precsim \sigma^n$ in $\mathcal{W}^c(x)$, we then move a d_n-distance $\precsim \tau^n$ in the $\mathcal{W}^u$-direction, and then we move a distance $\precsim \sigma^n$ in the $\mathcal{W}^s$-direction.

We would like to show that z and w are roughly in the same place. There are three components to their locations: central, stable, and unstable. The first two components are measured using the ordinary d metric, and the third, using the d_n metric. The order in which these components are measured can in theory make a difference; we need to show that in fact, it does not make a difference.

Since the d_0-radius of the d_n ball in the unstable direction is less than $\hat{\nu}^n\tau^n$, and $\hat{\nu}\tau < \sigma$, all d-distances under consideration are $\precsim \sigma^n$. We can therefore ignore the central and stable components of the two points z and w — they are both $\precsim \sigma^n$. The real question, then, is whether the $\mathcal{W}^u$ components of z and w are the same, and this must be measured in the d_n metric.

By drawing a picture, you should convince yourself that the following lemma is the essence of what we need to prove:

Lemma 5.1. *Consider three points p, p', q, with $p \in \mathcal{W}^c(x, \sigma^n)$, $p' \in \mathcal{W}^u_{d_n}(p, \tau^n)$, $q' \in \mathcal{W}^s(p', \sigma^n)$. Let q be the unique point of intersection:*

$$\{q\} = \mathcal{W}^u(q, 1) \cap \mathcal{W}^{cu}(x, 1).$$

Then

$$d_n(q, q') \precsim \tau^n.$$

Proof of Lemma 5.1. Apply f^n to the four points p, p', q', q to obtain p_n, p'_n, q'_n, q_n. Then $d(p_n, p'_n) \precsim \tau^n$, and $d(p'_n, q'_n) \precsim \nu^n d(p, q') \precsim \nu^n\sigma^n$. Both the distance $d(p_n, p'_n)$ from p_n to p'_n and the distance $d(q_n, q'_n)$ from q_n to q'_n measure the distance between the manifolds $\mathcal{W}^{cs}(p_n)$ $(= \mathcal{W}^{cs}(x_n))$ and $\mathcal{W}^{cs}(p'_n)$. These manifolds are drawn from a uniform family of manifolds – the foliation $\mathcal{W}^{cs}$. Therefore, if $d(p_n, q'_n) \precsim \tau^n$ and either $d(p_n, q_n) \precsim \tau^n$ or $d(p'_n, q'_n) \precsim \tau^n$ then both $d(p_n, q_n)$ and $d(p'_n, q'_n)$ are $\precsim \tau^n$. (This can be proved rigorously, but probably the best way to convince yourself is to draw a picture). But we know that

$$d(p'_n, q'_n) \precsim \nu^n\sigma^n \ll \tau^n,$$

and

$$d(p_n, p'_n) \precsim \tau^n,$$

and so

$$d(q_n, q'_n) \precsim \tau^n.$$

This implies the conclusion.$\diamond$

5.1. **Proof of Step 2.** We now turn to the proof of:

Step 2. Show that the set of julienne density points of a bi-essentially saturated set is bi-saturated.

Let x be a julienne density point of a bi-essentially saturated set A. Let $x' \in \mathcal{W}^s(x,1)$. Let $h^s : \mathcal{W}^{cu}(x,1) \to \mathcal{W}^{cu}(x')$ be the stable holonomy map. It is absolutely continuous (Proposition 4.3). In general it is not smooth, or even Lipschitz continuous.

Let $V_n = h^s(J_n^{cu}(x))$, so $V_n \searrow x'$. Absolute continuity of h^s and $\mathcal{W}^s$-saturation of A^s imply (exercise): that

$$\lim_{n\to\infty} m^{cu}(A^s : V_n) = 1.$$

We would like to show that

$$\lim_{n\to\infty} m^{cu}(A^s : J_n^{cu}(x')) = 1.$$

By reasoning similar to that in Lecture IV, it suffices to show:

(1) $J_n^{cu}(x') \searrow x'$ is v-regular, and
(2) $V_n \searrow x'$ and $J_n^{cu}(x') \searrow x'$ are internested.

To prove 1., we apply Cavalieri's principle to juliennes (Exercise 4.10).

To prove 2., we need to restrict our choice of σ, τ. In order to be able to choose a σ, τ satisfying these restrictions, we will need to impose a center bunching condition. Let's explore what this center bunching condition has to be.

To prove 2., it is really necessary to draw a picture. On the one hand, we have a point p' in the julienne $J_n^{cu}(x)$. The point p' lies at a d_n-distance $\precsim \tau^n$ from a point $p \in \mathcal{W}^c(x, \sigma^n)$. On the other hand, we have the holonomy image $q = h^s(p') \in V_n$. The (ordinary, d) distance from p' to q is $\precsim 1$. To show that q lies in a julienne $J_k^{cu}(x')$, with $k \asymp n$, we need to prove:

Lemma 5.2. *Consider three points p, p', q, with $p \in \mathcal{W}^c(x, \sigma^n)$, $p' \in \mathcal{W}^u_{d_n}(p, \tau^n)$, $q' \in \mathcal{W}^s(p', 1)$. Let q be the unique point of intersection:*

$$\{q\} = \mathcal{W}^u(q, 1) \cap \mathcal{W}^{cu}(x, 1).$$

Then

$$d_n(q, q') \precsim \tau^n.$$

Notice the similarity between Lemma 5.2 and Lemma 5.1. The number "σ^n" in Lemma 5.1 is replaced by "1" in Lemma 5.2. The proof of Lemma 5.2 is in fact the same as the proof of Lemma 5.1, but instead of requiring that

$$\nu\sigma < \tau,$$

we must require:

$$\nu < \tau. \tag{19}$$

Lemma 5.2 is however not all we need to show 2.: we must also show that the point q in Lemma 5.2 lies at a distance $\precsim \sigma^n$ from x'. (We did not have to worry about this fact in our previous proof because we were considering holonomy at a very small scale, $\precsim \sigma^n$. This time the holonomy is at a relatively huge scale — $\precsim 1$.) Re-examining the proof of Lemma 5.1, it becomes clear that we need to be able to choose σ, τ so that

$$\tau\gamma^{-1} < \sigma. \tag{20}$$

Combining (19) and (20) with the previous constraints on σ, τ, we find that it is necessary to assume:

$$\nu < \gamma\hat{\gamma}.$$

This is (one half) of the center bunching condition. The other half, which arises in proving that cs-density points are preserved under $\mathcal{W}^u$-holonomy, is:

$$\hat{\nu} < \gamma\hat{\gamma}.$$

5.2. **The role of center bunching.** Finally we define the center bunching condition. Recall the functions $\nu < \gamma < \hat{\gamma}^{-1} < \hat{\nu}^{-1}$ used to define partial hyperbolicity. At each point $p \in M$, $\nu(p)$ is an upper bound on the contraction of vectors in $E^s(p)$, $\gamma(p)$ and $\hat{\gamma}^{-1}(p)$ bound the contraction/expansion of vectors in $E^c(p)$, from below and above, respectively, and $\hat{\nu}^{-1}(p)$ is a lower bound on the expansion of vectors in $E^u(p)$.

Definition: f is *center bunched* if the functions $\nu, \hat{\nu}, \gamma$, and $\hat{\gamma}$ can be chosen so that:

$$\nu < \gamma\hat{\gamma} \qquad \text{and} \qquad \hat{\nu} < \gamma\hat{\gamma}. \tag{21}$$

What does this condition mean? Center bunching is automatically satisfied when the action of Tf on E^c is conformal. In that case, we can pick γ, and $\hat{\gamma}^{-1}$ to be as close to each other as we like, so that $\gamma\hat{\gamma}$ is nearly 1. Then center bunching is just a restatement of the partial hyperbolicity condition:

$$\nu < 1 \qquad \text{and} \qquad \hat{\nu} < 1.$$

For example, in the case where $\dim(E^c) = 1$, the action of Tf the center bundle is conformal, and so center bunching is always satisfied.

If the action of Tf on E^c is not conformal, then center bunching condition requires that the action of Tf on E^c be close enough to conformal that the hyperbolicity of f dominates the nonconformality of Tf on E^c.

We have already seen how the center bunching condition is needed in the proof of 2.: $V_n \searrow x$ and $J_n^{cu}(x) \searrow x$ are internested. There is one more role it plays in the proof this fact, which is summarized in the following theorem:

Theorem 5.3. [PSW, PSWc] *If $\nu < \gamma\hat{\gamma}$, then the $\mathcal{W}^s$-holonomy map $h^s : \mathcal{W}^{cu}(x,1) \to \mathcal{W}^{cu}(x')$ is Lipschitz* when restricted to the center manifold $\mathcal{W}^c(x,1)$.

Similarly, If $\hat{\nu} < \gamma\hat{\gamma}$, then the $\mathcal{W}^u$-holonomy map $h^u : \mathcal{W}^{cs}(x,1) \to \mathcal{W}^{cu}(x')$ is Lipschitz when restricted to the center manifold $\mathcal{W}^c(x,1)$.

As a corollary, we obtain the proof of 2., which completes the proof of Theorem 1.3:

Proposition 5.4. *Let f be center bunched, and suppose σ, τ are chosen so that:*

$$\nu < \tau < \sigma\gamma, \qquad \sigma < \min\{1, \hat{\gamma}\}.$$

Then $h^s(\mathcal{W}^c(x,\sigma^n)) \searrow x'$ and $\mathcal{W}^c(x',\sigma^n) \searrow x'$ are internested, as are $V_n = h^s(J_n(x)) \searrow x'$ and $J_n^{cu}(x') \searrow x'$.

Proof. Combine Lemma 5.2 with Theorem 5.3. The details are left to an energetic reader!$\diamond$

Exercise 5.5. *Show that the sequence $W_n \searrow x$ is v-regular. Hint: use exercise 4.10*

Exercise 5.6. *Let $f_\lambda : \mathbf{T}^4 \to \mathbf{T}^4$ be the example given in Exercise 1.9. Let*

$$f_{\lambda,\epsilon}(x, y) = (A(x), g_\lambda(y) + \epsilon\varphi(x)),$$

where $\varphi : \mathbf{T}^4 \to \mathbf{T}^4$. Show that $f_{\lambda,\varepsilon}$ is center bunched if $|\lambda| < 1$. Hence, if $f_{\lambda,\varepsilon}$ is stably accessible (cf. Theorem 1.1 in Lecture 1), then it is stable ergodic.

Problem: Is it possible to remove the center bunching hypothesis in Theorem 1.3? Can it be removed even in a special case, such as $f_{\lambda,\epsilon}$ in Exercise 1.9, with $\lambda \gg 1$ (where we have dynamical coherence, smooth center bundle, etc.)

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