Fourier Integral Operators and Applications - Lecture 1

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Microlocal Analysis: Theory and Applications

Séminaire de Mathématiques Supérieures
Summer School

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[1] Based on 2019 MSRI lectures with Raluca Felea.

Homogeneous Microlocal Analysis

- Hamid Hezari's course treated **semiclassical** microlocal analysis, and other lecturers have worked in this setting:

  Size estimates or asymptotics **as** $h \to 0^+$.

- These lectures: an introduction to **homogeneous** microlocal analysis, focussing on **Fourier integral operators (FIO)**.

- Homogeneity: various actions of $\mathbb{R}_+$ are respected: **Phase functions** $\phi(x, \theta)$ are homogeneous of degree 1 in $\theta \in \mathbb{R}^N$; **wave front sets** and **Lagrangian manifolds** are **conic**: invariant w.r.t. $(x, \xi) \in T^*X \to (x, t\xi)$, $t > 0$, etc.

- Interested in **high frequency asymptotics:** as phase variable $\to \infty$.

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1. Examples of **Fourier integral operators** (FIOs)
2. Symbol classes and pseudodifferential operators (ΨDOs)
3. Wavefront (WF) sets and the Hörmander-Sato Lemma
4. Conormal distributions
5. Wavefront relations: examples
6. Distributions defined by oscillatory integrals
7. Suggested reading

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Examples of FIOs: ΨDOs and pull-backs

1. **Pseudodifferential operators**

$$T_1 f(x) := \int \int e^{i(x-y)\cdot\theta} a(x,y,\theta) f(y)\, d\theta\, dy,$$

with $a \in S^m_{1,0}(\mathbb{R}^{2n} \times \mathbb{R}^n)$.

2. **Pull-back**: composition with a diffeomorphism

Let X, Y be open subsets of $\mathbb{R}^n$, $\chi : X \to Y$ a C^∞ diffeom.

$T_2(f)(x) = \chi^*(f)(x) := f(\chi(x)) = \int\int e^{i(\chi(x)-y)\cdot\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$

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Examples of FIOs: Radon transform

3. **Radon transform $\mathcal{R}$:** hyperplane integrals of a function on $\mathbb{R}^n$

Space of hyperplanes in $\mathbb{R}^n$ is Grassmannian $M_{n-1,n} \simeq \mathbb{S}^{n-1} \times \mathbb{R}$,

$$\mathbb{S}^{n-1} \times \mathbb{R} \ni (\omega, s) \leftrightarrow \{y \in \mathbb{R}^n : y \cdot \omega = s\} \in M_{n-1,n}$$

$$T_3 f(\omega, s) = \mathcal{R} f(\omega, s) := \int_{\{y \cdot \omega = s\}} f(y)\, d\sigma(y)$$

$$= c_n \int \int e^{i(y \cdot \omega - s)\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$$

Examples of FIOs: Spherical mean operator

4. **Spherical mean operator:** Let $d\sigma_t$ = surface measure on $t \cdot \mathbb{S}^{n-1}$.

   For $t > 0$, define convolution operator $\mathcal{A}_t : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$.

$$T_4(f)(x) = \mathcal{A}_t f(x) := f * d\sigma_t = \int_{\mathbb{S}^{n-1}} f(x - t\omega)\, d\sigma(\omega)$$
$$= c_{t,n} \int\int e^{i(|x-y|-t)\theta}\, 1(\theta)\, f(y)\, d\theta\, dy$$

   Solution operator for Cauchy problem for the wave equation on $\mathbb{R}^{n+1}$ can be expressed in terms of $\mathcal{A}_t$ and its derivatives in t.

Examples of FIOs: Canonical operators

5. A **generating function** $S(x, \eta)$: $\mathbb{R}$-valued, smooth on $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$, homogeneous of deg 1 in η, with $det \left(\partial^2_{x,\eta} S\right) \neq 0$.

For $a \in S^0_{1,0}$ (defined below), let

$$T_5(f) := \int e^{iS(x,\eta)}\, a(x,\eta) \hat{f}(\eta)\, d\eta$$
$$= \int \int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, f(y)\, dy\, d\eta$$

Early version of FIOs used by Maslov, Egorov ...

Examples of FIOs: Half-wave operator

6 • (M, g) an n-dim Riemannian manifold without boundary.

• Δ_g the **Laplace-Beltrami** op. on M, symbol $\sigma(x,\xi) = -|\xi|_g^2$

$P(x,D) := (-\Delta_g)^{\frac{1}{2}} \in \Psi^1(M)$: 1^{st} order, elliptic, pos. def., w/ principal symbol $p(x,\xi) = |\xi|_g$. Can form 1-param group

$$\left\{e^{itP} : t \in \mathbb{R}\right\}, \quad \text{of interest for spectral theory of } \Delta_g.$$

• W.r.t local coordinates on M,

$$e^{itP(x,D)} f(x) := \int\int_{M \times \mathbb{R}^n} e^{i\phi_t(x,y,\theta)} a(t,x,y,\theta)\, f(y)\, d\theta\, dy$$

for appropriate **phase function** ϕ_t and **amplitude** a.

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Two symbol classes

- For $\theta \in \mathbb{R}^N$, $\langle\theta\rangle := \left(1+|\theta|^2\right)^{\frac{1}{2}}$ ("**bracket** of θ").

- **Def.1** For $m \in \mathbb{R}$, $N \in \mathbb{N}$, and X a manifold, the space of (Hörmander) **symbols** of **order** m and **type** $(1,0)$ is

$$S^m_{1,0}(X\times\mathbb{R}^N) = \Big\{ a(x,\theta) \in C^\infty(X\times\mathbb{R}^N) : \forall \alpha,\ \beta \text{ and } K \subset\subset X, \\ |\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| \le C_{\alpha\beta K}\langle\theta\rangle^{m-|\alpha|},\ x\in K \Big\}$$

- **Def. 2** The space of **classical** symbols of order $m \in \mathbb{R}$ on $X\times\mathbb{R}^N$ is

$$S^m_{cl}(X\times\mathbb{R}^N) = \Big\{ a(x,\theta) : a \sim \sum_{j=0}^\infty a_j(x,\theta),\ a_j \text{ homog deg } m-j \text{ in } \theta \Big\}$$

- Both of these make sense on an N-dim vec bundle E over X, or an open **conic** set $\Gamma \subset E$ (i.e., $(x,v)\in\Gamma \implies (x,tv)\in\Gamma,\ t>0$).

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Pseudodifferential operators (ΨDOs)

Homogeneous microlocal analysis version of ΨDOs:
If X an n-dim manifold and $m \in \mathbb{R}$,

- $\Psi^m(X) = OPS^m_{1,0}$, the **pseudodifferential operators** of **order** m on X, are those continuous lin maps $P(x,D) : \mathcal{D}'(X) \to \mathcal{D}'(X)$ of form

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  in local coordinates, with $a \in S^m_{1,0}$.

- $\Psi^m_{cl}(X) = OPS^m_{cl}$, the **classical (Kohn-Nirenberg)** ΨDOs, are given by the same expression, but with $a \in S^m_{cl}$.

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Wavefront sets: $WF(u)$

- **Cotangent space** $T^*\mathbb{R}^n = \{(x,\xi) : x \in \mathbb{R}^n,\ \xi \in (T_x\mathbb{R}^n)^*\}$, has **zero section**, $\mathbf{0} = \{(x,\xi) : \xi = 0\}$.
- $\Sigma \subset T^*\mathbb{R}^n \setminus \mathbf{0}$ is **conic** if $(x,\xi) \in \Sigma \implies (x,t\xi) \in \Sigma,\ \forall t > 0$
- **Def.** Let $u \in \mathcal{D}'(\mathbb{R}^n)$. We say that $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus \mathbf{0}$ is **not** in $WF(u)$ if $\exists\ \phi(x) \in C_0^\infty$, $\phi(x_0) \neq 0$, and a conic neighborhood $\Gamma \subset \mathbb{R}^n$ *of* ξ_0 such that

$$|\ \widehat{\phi u(\xi)}\ | \lesssim (1+ |\ \xi\ |)^{-N},\ \forall \xi \in \Gamma.$$

- $WF(u)$ is a **closed, conic** set $\subset T^*\mathbb{R}^n \setminus \mathbf{0}$
- **Ex:** $WF(\delta) = \{(0,\xi), \xi \neq 0\} = T_0^*\mathbb{R}^n \setminus 0$

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- If π = the projection from $T^*\mathbb{R}^n$ to $\mathbb{R}^n$, then

$$\pi(WF(u)) = \text{ singular support of } u$$

- If $P(x,D)$ is a ΨDO, then P is **pseudolocal:**

$$\text{sing supp}(Pu) \subseteq \text{sing supp}(u)),$$

and more precisely is **microlocal:**

$$WF(Pu) \subseteq WF(u)$$

- All of this extends to $WF(u)$ on general C^∞ manifolds X: $WF(u) \subseteq T^*X \setminus \mathbf{0}$ ....

- **Q.** What do other operators do to WF of functions they act on?

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Hörmander-Sato Lemma

- Let X, Y be manifolds, $T : \mathcal{D}(Y) \to \mathcal{D}'(X)$, with **Schwartz kernel** $K \in \mathcal{D}'(X \times Y)$: weakly, $Tf(x) = \int K(x,y)\, f(y)\, dy$
- The **wavefront relation** of T is

$$WF_T := \{(x,\xi; y,\eta) : (x,y,\xi,-\eta) \in WF(K)\} \subset T^*X \times T^*Y$$

- **Thm.** Suppose $WF(K) \subseteq \{(x,y,\xi,\eta) : \xi \neq 0 \text{ and } \eta \neq 0\}$, so that $WF_T \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$. Then

$$WF(Tu) \subseteq WF_T \circ WF(u) :=$$

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- Let X, Y Z be manifolds, $T_1 : \mathcal{E}'(Y) \to \mathcal{E}'(X)$, $T_2 : \mathcal{E}'(Z) \to \mathcal{E}'(Y)$, with Schwartz kernels $K_1 \in \mathcal{D}'(X \times Y)$, $K_2 \in \mathcal{D}'(Y \times Z)$

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Conormal distributions: examples

1. $\mathbb{R}^1$: Dirac delta $\delta(x)$ is conormal for $Y = \{0\}$, as are the families x_+^z, x_-^z, $(x+i0)^z$, $(x-i0)^z$, which are are meromorphic, conormal-for-$\{0\}$-distribution-valued functions of z.

2. $\mathbb{R}^n$: Dirac delta is again conormal for $\{0\}$, as is

   Newtonian potential, $N(x) = c_n|x|^{2-n}$, $n \geq 3$.

3. $\Phi : \mathbb{R}^n \to \mathbb{R}^k$ defining functions for $Y^{n-k} = \{x : \phi(x) = 0\} \subset \mathbb{R}^n$, $u(x)$ a conormal distn on $\mathbb{R}^k$ for $\{t = 0\}$, then $u(\phi) \in \mathcal{D}'(\mathbb{R}^n)$, is conormal for Y. Many examples in Gelfand-Shilov.

   Ex. Using $u = \delta(t) \implies$ a smooth meas μ on Y is conormal for Y.

   Ex. If (X, g) is Riemannian, $s > 0$, $dist(x, Y)^{-k+s}$ is (near Y) conormal for Y.

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If $Y^{n-k} \subset X^n$, the **conormal bundle** of Y is

$$N^*Y := \{(x,\xi) \in T^*X : x \in Y, \, \xi|_{T_xY} = 0\} \subset T^*X$$

- **Ex 1:** In $X = \mathbb{R}^n$, write $x = (x', x'')$ with $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$

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Ex. On $\mathbb{R}^k$: $\hat{\delta}(\theta) \equiv 1 \implies \delta(t) = (2\pi)^{-\frac{k}{2}} \int_{\mathbb{R}^k} e^{it\cdot\theta}\, 1(\theta)\, d\theta$

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- $I^m(X;Y) :=$ class of distributions on X **conormal to** Y of order m and $I(X;Y) = \bigcup_{m\in\mathbb{R}} I^m(X;Y)$.

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- **Prop.** If $u \in I(X;Y)$, then $WF(u) \subseteq N^*Y \setminus \mathbf{0}$.

  Thus, sing supp$(u) \subseteq Y$.

- **N.B.** In general $WF(u) \subseteq N^*Y \setminus \mathbf{0} \not\Longrightarrow u$ is conormal for Y.

- **Ex.** If $\mu \in \mathcal{D}'(X)$ is a smooth density on Y, can write

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- $T_1 f(x) = \int e^{i(x-y)\cdot\theta}\, a(x,y,\theta)\, f(y)\, d\theta dy$ has

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- WF$(K_{T_1}) \subseteq N^*\{x-y=0\} = \{(x,y,\theta,-\theta); x=y,\ \theta\neq 0\}$

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  the **diagonal** of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$, a refinement of

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Wave front relations: Pull-backs

- Pull-back/composition with a diffeomorphism: on $\mathbb{R}^n$,

  $T_2 f(x) = \int e^{i(\chi(x)-y)\cdot\theta} f(y) d\theta dy,$

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Wave front relations: Radon Transform

- $T_3 f(\omega, s) = \int e^{i(y\cdot\omega - s)\theta} 1(\theta) f(y) d\theta dy, \; K_{T_3} = \int e^{i(y\cdot\omega - s)\theta} 1(x, y, \theta) d\theta$
- $\mathrm{WF}(K_{T_3}) = N^*\{s - y\cdot\omega = 0\}$

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where $i^* : T^*_\omega \mathbb{R}^n \hookrightarrow T^*_\omega \mathbb{S}^{n-1}$ = restriction, $\implies$

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Distributions defined by oscillatory Integrals

- **Def.** $\phi(x,\theta)$ is a **phase function** on $X \times (\mathbb{R}^N \setminus \mathbf{0})$ if it is smooth, $\mathbb{R}$-valued, positively homogeneous of degree 1 in θ ($\phi(x,t\theta) = t\phi(x,\theta)$ for $t>0$) and satisfies

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- **Prop.** If $\phi(x,\theta)$ is a phase function and $a \in S^m_{1,0}(X \times (\mathbb{R}^N \setminus \mathbf{0}))$, then the oscillatory integral $u(x) = \int e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta$ is a well-defined distribution, $u \in \mathcal{D}'(X)$, given by

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Prop: Oscillatory integrals $\in \mathcal{D}'(X)$

- $(d_x\phi, d_\theta\phi) \neq (0,0) \implies$ can form

$$L = \sum_j b_j(x,\theta)\frac{\partial}{\partial x_j} + \sum_k c_k(x,\theta)\frac{\partial}{\partial \theta_k} + b_0(x,\theta) \text{ s.t. } L^t(e^{i\phi}) = e^{i\phi},$$

with $b_0,\, b_j \in S^{-1}_{1,0},\, c_k \in S^0_{1,0}$.

- For $f \in \mathcal{D}(X)$ and $r \in \mathbb{N}$ large, define $\langle u, f\rangle$ as

$$\int\int e^{i\phi(x,\theta)}a(x,\theta)\,f(x)\,d\theta\,dx := \int\int (L^t)^r(e^{i\phi})a(x,\theta)\,f(x)\,d\theta\,dx$$

$$:= \int\int e^{i\phi(x,\theta)}L^r\big(a(x,\theta)f(x)\big)\,d\theta\,dx$$

But $L^r : S^m_{1,0} \to S^{m-r}_{1,0}$, integral converges for $m - r < -N$.

Prop: Oscillatory integrals $\in \mathcal{D}'(X)$

- $(d_x\phi, d_\theta\phi) \neq (0,0) \implies$ can form

$$L = \sum_j b_j(x,\theta)\frac{\partial}{\partial x_j} + \sum_k c_k(x,\theta)\frac{\partial}{\partial \theta_k} + b_0(x,\theta) \text{ s.t. } L^t(e^{i\phi}) = e^{i\phi},$$

with $b_0,\ b_j \in S^{-1}_{1,0},\ c_k \in S^0_{1,0}$.

- For $f \in \mathcal{D}(X)$ and $r \in \mathbb{N}$ large, define $\langle u, f\rangle$ as

$$\int\int e^{i\phi(x,\theta)}a(x,\theta)\, f(x)\, d\theta\, dx := \int\int (L^t)^r(e^{i\phi})a(x,\theta)\, f(x)\, d\theta\, dx$$

$$:= \int\int e^{i\phi(x,\theta)} L^r\big(a(x,\theta)f(x)\big)\, d\theta\, dx$$

But $L^r : S^m_{1,0} \to S^{m-r}_{1,0}$, integral converges for $m - r < -N$.

Thm: WF sets of oscillatory integrals

Let $(x_0, \xi_0) \in T^*X \setminus \Lambda_\phi$, $\psi(x) \in \mathcal{D}(X)$ supported in nhood of x_0

- $\widehat{\psi u}(\xi) = \int \int e^{i(\phi(x,\theta) - x\cdot\xi)} a(x,\theta)\, \psi(x)\, d\theta\, dx$,
- Form vec fld near (x_0, ξ_0): $L = \frac{1}{|d_x\phi - \xi|^2} \sum_j (d_{x_j}\phi - \xi_j)\partial_{x_j}$

  $\implies L(e^{i(\phi(x,\theta)-x\cdot\xi)}) = e^{i(\phi(x,\theta)-x\cdot\xi)}$
- $|d_x\phi(x,\theta) - \xi| \geq c(|\xi| + |\theta|)$ on supp$(a \cdot \psi)$, can integrate by parts

  $\implies \widehat{\psi u}$ rapidly decreasing on conic nhood of ξ_0
- Thus, $(x_0, \xi_0) \notin$ WF(u).

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Wave front relation of canonical operators

- $T_5 f(x) = \int \int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, f(y)\, dy\, d\eta, \quad det\left(\partial^2_{x,\eta} S\right) \neq 0$

  $K_{T_5}(x,y) = \int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, d\eta = \int e^{i\phi}\, a\, d\eta$

- $\{d_\eta \phi = 0\} = \{(x,y,\eta) : d_\eta S(x,\eta) = y\}$

  $\overset{Thm.}{\Longrightarrow} WF(K_{T_5}) \subseteq \{\, (x, d_\eta S;\, d_x S, \eta) \,:\, (x,\eta) \in supp(a)\,\}$

  $\Longrightarrow WF_{T_5} \subseteq \{(x, d_x S;\, d_\eta S, -\eta) : (x,\eta) \in supp(a)\}$

  $=$ graph of a canonical transformation of $T^*\mathbb{R}^n$.

- **Lecture 2:** Impose 2nd order conditions on ϕ:
  $\Longrightarrow$ Fourier integral **distributions** and **operators**

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Basic references: Books

- J. J. Duistermaat, *Fourier integral operators*, Progress in Mathematics **130**, Birkhäuser Boston, Boston, MA, 1996.
- A. Grigis and J. Sjöstrand, *Microlocal Analysis for Differential Operators: An Introduction*, London Mathematical Society Lecture Notes **196**, Cambridge Univ. Press, 1994.
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Basic references: Classic article

- L. Hörmander, Fourier integral operators, I. *Acta Math.* **127** (1971), 79–183.
- Older papers of historical interest for introducing important ideas: V. Maslov, Y. Egorov, ...

More references at end of Lecture 3.