Fourier Integral Operators and Applications - Lecture 2

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Microlocal Analysis: Theory and Applications

Séminaire de Mathématiques Supérieures
Summer School

June 9, 2021

[1]Based on 2019 MSRI lectures with Raluca Felea.

Overview of Lecture 2

1. Wave front sets and relations (review)
2. Distributions defined by oscillatory integrals (review)
3. Nondegenerate phase functions
4. Symplectic geometry, Lagrangians and canonical relations
5. Fourier integral (Lagrangian) distributions
6. Invariance of phase function

Wavefront sets and relations

- **Cotangent space** $T^*\mathbb{R}^n = \{(x,\xi) : x \in \mathbb{R}^n,\, \xi \in (T_x\mathbb{R}^n)^*\}$, has **zero section**, $\mathbf{0} = \{(x,\xi) : \xi = 0\}$.
- $\Sigma \subset T^*\mathbb{R}^n \setminus \mathbf{0}$ is **conic** if $(x,\xi) \in \Sigma \implies (x,t\xi) \in \Sigma,\ \forall t > 0$
- **Def.** Let $u \in \mathcal{D}'(\mathbb{R}^n)$. We say that $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus \mathbf{0}$ is **not** in $WF(u)$ if $\exists\ \phi(x) \in C_0^\infty$, $\phi(x_0) \neq 0$, and a **conic neighborhood** $\Gamma \subset \mathbf{R}^n$ *of* ξ_0 such that

$$|\widehat{\phi u(\xi)}| \lesssim (1+|\xi|)^{-N},\ \forall \xi \in \Gamma.$$

- $WF(u)$ is a **closed, conic** set $\subset T^*\mathbb{R}^n \setminus \mathbf{0}$
- Extends to $WF(u)$ on general C^∞ manifold X: $WF(u) \subseteq T^*X \setminus \mathbf{0}$

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Wavefront sets and relations

- Let X, Y be manifolds, $T : \mathcal{E}'(Y) \to \mathcal{D}'(X)$, with **Schwartz kernel** $K \in D'(X \times Y)$: formally, $Tf(x) = \int K(x,y)\, f(y)\, dy$
- The **wavefront relation** of T is

$$WF_T := \{(x,\xi;y,\eta) : (x,y,\xi,-\eta) \in WF(K)\} \subset T^*X \times T^*Y$$

- **Thm.(Hörmander-Sato)** If $WF(K) \subseteq \{(x,y,\xi,\eta) : \xi \neq 0, \eta \neq 0\}$, so that $WF_T \subset (T^*X \setminus \mathbf{0}) \times (T^*Y \setminus \mathbf{0})$, then

$$WF(Tu) \subseteq WF_T \circ WF(u) :=$$

$$\big\{(x,\xi) : \exists (y,\eta) \text{ s.t.} (x,\xi;y,\eta) \in WF_T \text{ and } (y,\eta) \in WF(u)\big\}.$$

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Distributions defined by oscillatory integrals

- **Def.** $\phi(x,\theta)$ is a **phase function** on $X \times (\mathbb{R}^N \setminus \mathbf{0})$ if it is smooth, $\mathbb{R}$-valued, positively homogeneous of degree 1 in θ, i.e., $\phi(x,t\theta) = t\phi(x,\theta)$ for $t > 0$, and satisfies

$$(d_x\phi,\, d_\theta\phi) \neq (0,0).$$

- **Prop.** If $\phi(x,\theta)$ is a phase function and $a \in S^m_{1,0}(X \times (\mathbb{R}^N \setminus \mathbf{0}))$, then the oscillatory integral $u(x) = \int e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta$ is a well-defined distribution, $u \in \mathcal{D}'(X)$, given by

$$\langle u, f\rangle := \int\int e^{i\phi(x,\theta)}\, a(x,\theta)\, f(x)\, d\theta\, dx, \quad \forall f \in C_0^\infty(X).$$

- **Thm.** $\mathrm{WF}(u) \subseteq \left\{(x, d_x\phi) : d_\theta\phi(x,\theta) = 0,\ \theta \neq 0\right\} \subseteq T^*X \setminus \mathbf{0}$

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Nondegenerate phase functions

- Recall: **Def.** $\phi(x, \theta)$ is a **phase function** on $X \times (\mathbb{R}^N \setminus 0)$ if it is smooth, $\mathbb{R}$-valued, homog of degree 1 in θ and

$$(d_x\phi,\, d_\theta\phi) \neq (0, 0).$$

- **Def.** A phase function ϕ is **nondegenerate** if in addition, $d_{x,\theta}(\frac{\partial\phi}{\partial\theta_1}), \ldots., d_{x,\theta}(\frac{\partial\phi}{\partial\theta_N})$ are linearly independent on

$$Crit_\phi := \{(x, \theta) :\, d_\theta\phi = 0\} \subset X \times (\mathbb{R}^N \setminus 0).$$

- $\phi(x, \theta)$ nondegenerate $\iff$

$$rank\Big[d^2_{x\theta}\phi,\, d^2_{\theta\theta}\phi\Big] = N, \quad \forall (x, \theta) \in Crit_\phi$$

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- **Prop.** ϕ nondeg $\implies$ $Crit_\phi$ is a closed, conic n-dim submfld of $X \times \left(R^N \setminus \mathbf{0}\right)$. Furthermore, the map $j : Crit_\phi \to T^*X \setminus \mathbf{0}$,

$$j(x,\theta) = (x, d_x\phi(x,\theta)),$$

  is an **immersion**, and $j\,(Crit_\phi) =: \Lambda_\phi =$ a conic **Lagrangian** submfld of $T^*X \setminus \mathbf{0}$ (to be defined).

- We say that ϕ **parametrizes** the Lagrangian Λ_ϕ.

- We need to discuss **symplectic** manifolds and their **Lagrangian submanifolds.** But first ...

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Symplectic geometry: Linear algebra

- **Def.** A **symplectic vector space** is a pair (V, ω), with ω a bilinear, nondegenerate, skew-symmetric form on V.
- If V is finite dim, dim(V) is necessarily even, say dim$(V) = 2n$.
- **Ex.** $V = \mathbb{R}^2$ with the area form $dx \wedge dy$

$$V = \mathbb{R}^{2n} = \{(x_1, \dots, x_n, y_1, \dots, y_n)\}, \quad \omega = \sum dy_j \wedge dx_j$$

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- **Def.** Let (V, ω) be symplectic and $L \subseteq V$ a linear subsp. Then
- (i) $L^\omega := \{v \in V : \omega(u, v) = 0, \forall u \in L\}$ (**annihilator** of L)
  $\dim(L^\omega) = \dim(V) - \dim(L)$ since ω nondegenerate
- (ii) L is **isotropic** if $L \subseteq L^\omega$, i.e., $\omega|_{L \times L} \equiv 0$
  ($\implies \dim(L) \leq \frac{1}{2}\dim(V)$)
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- **Ex.** $\dim(L) = 1 \implies$ isotropic, $\text{codim}(L) = 1 \implies$ co-isotropic.
- **Ex.** $\dim(V) = 2 \implies$ any 1-dim subspace is Lagrangian
- **Ex.** In $\mathbb{R}^{2n}$, $L = \mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ are Lagrangian

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- **Ex.** In $\mathbb{R}^{2n}$, $L = \mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$ are Lagrangian

Symplectic geometry: Linear algebra

- **Def.** Let (V, ω) be symplectic and $L \subseteq V$ a linear subsp. Then
- (i) $L^\omega := \{v \in V : \omega(u, v) = 0, \forall u \in L\}$ (**annihilator** of L)
  $\dim(L^\omega) = \dim(V) - \dim(L)$ since ω nondegenerate
- (ii) L is **isotropic** if $L \subseteq L^\omega$, i.e., $\omega|_{L \times L} \equiv 0$
  ($\implies \dim(L) \leq \frac{1}{2}\dim(V)$)
- (iii) L is **co-isotropic (involutive)** if $L^\omega \subseteq L$
  ($\implies \dim(L) \geq \frac{1}{2}\dim(V)$)
- (iv) L is **Lagrangian** if $L = L^\omega$ ($\implies \dim(L) = \frac{1}{2}\dim(V)$)
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Symplectic geometry: Manifolds

- **Def.** (M, ω) is a **symplectic** manifold if ω is a **closed** diff 2-form on M and $\omega|_{T_xM}$ is **symplectic** for all $x \in M$. Thus, $\dim(M) = 2n$ and ω^n is a volume $2n$-form orienting M.

  Ex. $(\mathbb{R}^n, \sum dy_j \wedge dx_j)$

- **Ex.** Cotangent bundle T^*X of a smooth X^n:
- Local coordinates $(x_1, \ldots, x_n)$ on X, induce local coords $(x_1, \ldots x_n, u_1, \ldots u_n)$ on TX, $(x_1, x_2, \ldots x_n, \xi_1, \xi_2, \ldots \xi_n)$ on T^*X
- If $u \in T_xX \implies u = \sum u_j \frac{\partial}{\partial x_j}$, $\xi \in T^*_xX \implies \xi = \sum \xi_j dx_j$
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  If $t = (t_x, t_\xi)$, $s = (s_x, s_\xi)$ then

$$\omega(s,t) = \frac{1}{2}\left(\langle t_\xi, s_x\rangle - \langle s_\xi, t_x\rangle\right)$$

- ω is bilinear, antisymmetric, nondegenerate, **closed.**

- $\omega^n = d\xi_1 \wedge \ldots d\xi_n \wedge dx_1 \cdots \wedge dx_n =$ **Liouville measure** on T^*X.

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Symplectic geometry: Conic manifolds

- (T^*X, ω) has more structure: than a general symplectic manifold: it is **conic**, since $\mathbb{R}_+$ acts on $T^*X \setminus \mathbf{0}$, $(x, \xi) \to (x, t\xi)$.

- **Def.** $\Gamma \subset T^*X \setminus \mathbf{0}$ is **conic** if $(x, \xi) \in \Gamma$ then $(x, t\xi) \in \Gamma, \forall t$.

  Ex. If $P(x, D)$ is an mth order partial differential operator on X, or more generally $P \in \Psi^m_{cl}(X)$ with principal symbol $p_m(x, \xi)$, then the **characteristic variety** of P,

$$\Sigma_P := \left\{(x, \xi) \in T^*X \setminus \mathbf{0} : p_m(x, \xi) = 0\right\} \text{ is closed, conic.}$$

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- **Def.** Let (M, ω) be a symplectic manifold and $L \subset M$ a smooth submanifold. Then L is **isotropic/co-isotropic/Lagrangian**, resp., if $T_x L \leq T_x M$ is isotropic/co-isotropic/Lagrangian, $\forall x \in L$, resp.

  { Lagrangian submanflds } = { co-isotropic } ∩ { isotropic }

- **Prop.** L is Lagrangian iff $\omega|_L = 0$ and $\dim(L) = \frac{1}{2}\dim(M)$.

- **Ex.** If $f \in C^\infty_{\mathbb{R}}(X)$, then $\Lambda_f := \{(x, df(x)) : x \in X\} \subset (T^*X, \omega)$ is Lagrangian (but not conic).
  E.g., $f = 0 \rightsquigarrow$ zero-section $\{(x, 0) : x \in X\} = \mathbf{0}$.

- In **homogeneous** microlocal analysis, study conic objects (e.g., sets, functions) in $T^*X \setminus \mathbf{0}$.

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Conic Lagrangian manifolds

- **Prop.** If $Y \subset X$ is smooth, then its conormal bundle $N^*Y \setminus \mathbf{0}$ is a conic Lagrangian in $T^*X \setminus \mathbf{0}$.

- **Thm.** Any conic lagrangian Λ can be microlocally parametrized by a nondegenerate phase function ϕ. I.e., $\forall\, \lambda_0 = (x_0, \xi_0) \in \Lambda$, $\exists\, \phi$, a nondeg phase on a conic nhood of $(x_0, \theta_0) \in X \times (\mathbb{R}^{N_0} \setminus \mathbf{0})$, s.t. $\Lambda = \Lambda_\phi$ near λ_0.

  Sketch of pf. (i) If projection $(x, \xi) \to \xi$, is a submersion near λ_0, then microlocally Λ has form $\left\{(x, \xi) : x = \frac{\partial H}{\partial \xi}\right\}$, with $H(\xi)$ homog degree 1 and then $\phi = x \cdot \xi - H(\xi) \rightsquigarrow \Lambda$.
  (ii) Show (i) holds after a suitable quadratic change of coordinates.

- **Note:** A conic Lagrangian need not be of the form N^*Y for some smooth Y. For $H(\xi) = \frac{\xi_1^3}{\xi_2^2}$ above, Λ_ϕ is the closure of the conormal bundle of the smooth pts of the curve: $(\frac{x_1}{3})^3 = (\frac{x_2}{2})^2$.

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Fourier integral distributions: Definition

- For $\phi(x,\theta)$ a nondeg phase, $Crit_\phi := \{(x,\theta); d_\theta\phi = 0\}$ and $j : Crit_\phi \to T^*X \setminus \mathbf{0}$; $j(x,\theta) := (x, d_x\phi(x,\theta))$, $j(Crit_\phi) = \Lambda_\phi = \{(x, d_x\phi); (x,\theta) \in C_\phi\}$
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- **Def.** For $m \in \mathbb{R}$, the class $I^m(X;\Lambda) \subset \mathcal{D}'(X)$ of **Fourier integral** (or **Lagrangian**) distributions **associated with** Λ and of **order** m consists of all **locally finite sums** of terms

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- **Ex. Conormal** distributions are Fourier integral distributions: For

$$u(x) = \int e^{i\Sigma_{j=1}^{k}\theta_j\phi_j(x)}\, a(x,\theta)\, d\theta \in I^m(X;Y),$$

  with $\{\phi_j\}_{j=1}^{k}$ defining functions for $Y^{n-k} \subset X^n$:

- $\phi(x,\theta) := \Sigma_{j=1}^{k}\theta_j\phi_j(x)$ is homog of deg 1

  $d\phi = \left(\left(\phi_1\left(x\right), \ldots, \phi_k\left(x\right)\right), \sum \theta_j d\phi_j(x) \right) \neq (0,0)$

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- **Surface measures:** In $\mathbb{R}^k$,

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Fourier integral distributions: Examples

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Fourier integral distributions: Examples

Phase functions for T_1, T_2, T_3, T_4 are **linear** in θ, so their Schwartz kernels are **conormal** distributions. E.g., for ΨDOs:

$\phi(x, y, \theta) = (x - y) \cdot \theta$ on $\mathbb{R}^{2n}_{x,y} \times \mathbb{R}^n_\theta$, with

$\{x_j - y_j\}_{j=1}^n$ def funcs for **diagonal**, $Y = \Delta_{\mathbb{R}^n} = \{x = y\} \subset \mathbb{R}^{2n}$.

$$d\phi(x, y, \theta) = (d_x\phi,\, d_y\phi,\, d_\theta\phi) = (\theta,\, -\theta,\, x - y) \implies$$

$$K_{T_1}(x, y) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta}\, a(x, y, \theta)\, d\theta \in I^{m+\frac{n}{2}-\frac{2n}{4}} = I^m\left(\mathbb{R}^{2n}; \Lambda_1\right),$$

where

$$\Lambda_1 = N^*\Delta_{\mathbb{R}^n} \setminus \mathbf{0} = \{(x, x; \theta, -\theta) : x \in \mathbb{R}^n,\, \theta \in \mathbb{R}^n \setminus \mathbf{0}\}.$$

Fourier integral distributions: Examples

Similarly, for $T_3 =$ **Radon transform:**

$$K_{T_3}(\omega, s, y) = \int_{\mathbb{R}} e^{i(y\cdot\omega - s)\theta}\, 1(\theta)\, d\theta$$

$$\phi(\omega, s, y, \theta) = (y\cdot\omega - s)\theta \implies$$

$$d\phi = (d_\omega\phi, d_s\phi, d_y\phi, d_\theta\phi) = (i^*\omega(y), -1, \omega, y\cdot\omega - s) \neq (0,0,0,0)$$

where i_ω^* is the dual of the inclusion $i_\omega : T_\omega\mathbb{S}^{d-1} \hookrightarrow T_\omega\mathbb{R}^d$.

Thus, $Crit_\phi = \{(\omega, s, y, \theta) :\ d_\theta\phi = 0\} = \{s = y\cdot\omega\}$ and

$$\begin{aligned}\Lambda_3 &= \left\{(\omega, y\cdot\omega, y; \theta i_\omega^*(y), -\theta, \theta\omega) : \omega\in\mathbb{S}^{d-1}, y\in\mathbb{R}^d, \theta\in\mathbb{R}\setminus 0\right\} \\ &\subset T^*\left(\mathbb{S}^{d-1}\times\mathbb{R}\times\mathbb{R}^d\right)\setminus\mathbf{0}.\end{aligned}$$

Fourier integral distributions: Examples

On the other hand, the phase of a **canonical operator** T_5 is nonlinear:

- $T_5 f(x) = \int\int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, f(y)\, dy\, d\eta, \quad det\left(\partial^2_{x,\eta} S\right) \neq 0$

  $K_{T_5}(x,y) = \int e^{i\left(S(x,\eta) - y\cdot\eta\right)}\, a(x,\eta)\, d\eta = \int e^{i\phi}\, a\, d\eta$

- $d_{x,y,\eta}\phi = (d_x S(x,\eta), -\eta, d_\eta S(x,\eta) - y)$

- $Crit_\phi = \{d_\eta \phi = 0\} = \{(x,y,\eta) :\ d_\eta S(x,\eta) = y\} \implies$

  $\implies \Lambda_\phi = j\,(Crit_\phi) = \{(x, d_x S;\ d_\eta S, -\eta) :\ (x,\eta) \in supp(a)\}$

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Invariance of phase function

- **Prop.** For any two nondegenerate phase functions parametrizing the same Lagrangian, $\Lambda_\phi = \Lambda_{\tilde{\phi}} \subset T^*X \setminus \mathbf{0}$, there is a chain of operations of the following types which transforms ϕ to $\tilde{\phi}$.
- 1) **Adding variables:** $\hat{\phi}(x, \theta, \sigma) := \phi + \frac{1}{2}\frac{\sigma^2}{|\theta|}$ a nondeg phase function
- 2) **Reducing variables:** stationary phase.
- 3) **Conic change of variables:** $\tilde{\theta}(x, \theta)$ and $\tilde{\phi} = \phi(x, \tilde{\theta}(x, \theta))$

$$Crit_{\tilde{\phi}} = \{(x, \theta) : d_\theta \tilde{\theta} = 0\} = \{(x, \theta) : d_\theta \phi \cdot d_\theta \tilde{\theta} = 0\} = Crit_\phi$$

and $\Lambda_{\tilde{\phi}} = \Lambda_\phi$ (check).

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Invariance of phase function

Thm. (Invariance of phase function) If ϕ, $\tilde{\phi}$ are nondegenerate phase functions parametrizing the same Lagrangian, $\Lambda_\phi = \Lambda_{\tilde{\phi}} \subset T^*X^n \setminus \mathbf{0}$, then for any $a \in S^m_{1,0}\left(X \times \mathbb{R}^N\right)$, there exists an $\tilde{a} \in S^{\tilde{m}}_{1,0}\left(X \times \mathbb{R}^{\tilde{N}}\right)$ with

$$m + \frac{N}{2} - \frac{n}{4} = \tilde{m} + \frac{\tilde{N}}{2} - \frac{n}{4}$$

such that

$$\int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)\, d\theta = \int_{\mathbb{R}^{\tilde{N}}} e^{i\tilde{\phi}(x,\tilde{\theta})} \tilde{a}(x,\tilde{\theta})\, d\tilde{\theta} \mod C^\infty.$$

Next time: Symbol calculus of Fourier integral distributions;
Canonical relations (Lagrangians in $T^*X \times T^*Y$);
Fourier integral operators and some **applications**.