Fourier Integral Operators and Applications - Lecture 3

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Microlocal Analysis: Theory and Applications

Séminaire de Mathématiques Supérieures
Summer School

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[1] Based on 2019 MSRI lectures with Raluca Felea.

Overview of Lecture 3

1. Symbol calculus of Fourier integral distributions
2. Canonical relations
3. Fourier integral operators
4. FIO calculus: Functional and composition
5. Applications: Egorov's Theorem and generalized Radon transforms
6. Some extensions of FIO theory (in online notes)
7. Readings for all three lectures

Symbol calculus of Fourier integral distributions

For $\Lambda \subset T^*X^n \setminus \mathbf{0}$ a smooth (immersed) conic Lagrangian,

$$I^m(X;\Lambda) = \text{ all locally finite sums of } u \in \mathcal{D}'(X)$$

given by oscillatory integrals

$$u = u(a,\phi) := \int_{\mathbb{R}^{N_\phi}} e^{i\phi(x,\theta)}\, a(x,\theta)\, d\theta, \quad a \in S_{1,0}^{m-\frac{N_\phi}{2}+\frac{n}{4}},$$

with $\phi(x,\theta)$ a nondeg phase on $X \times (\mathbb{R}^{N_\phi} \setminus 0)$ which **parametrizes** Λ:

$$Crit_\phi := \{(x,\theta) : d_\theta\phi(x,\theta) = 0\}$$

$$j \downarrow$$

$$\Lambda_\phi := \left\{(x, d_x\phi) : (x,\theta) \in Crit_\phi\right\} \overset{rel\, open}{\subseteq} \Lambda.$$

Symbol calculus of Fourier integral distributions

- Define n-form μ_ϕ on $Crit_\phi$ by requiring

$$\mu_\phi \wedge d(\frac{\partial \phi}{\partial \theta_1}) \cdots \wedge d(\frac{\partial \phi}{\partial \theta_N}) = dx_1 \cdots \wedge dx_n \wedge d\theta_1 \cdots \wedge \theta_N$$

- If λ_i are local coord on $Crit_\phi$ then $\mu_\phi = f d\lambda_1 \cdots \wedge d\lambda_n$, with

$$f = \frac{dx_1 \cdots \wedge dx_n \wedge d\theta_1 \cdots \wedge \theta_N}{d\lambda_1 \cdots \wedge d\lambda_n \wedge d(\frac{\partial \phi}{\partial \theta_1}) \cdots \wedge d(\frac{\partial \phi}{\partial \theta_n})}$$

- To obtain an invariantly defined principal symbol, $\sigma_{prin}(u)$, if

$$a^0 := [a|_{Crit_\phi}] \in S^{m - \frac{N}{2} + \frac{n}{4}}_{1,0} / S^{m - \frac{N}{2} + \frac{n}{4} - 1}_{1,0},$$

Def. The **principal symbol** $\sigma_{prin}(u)$ of $u(a, \phi)$ is the push-forward of the half-density $a^0 \sqrt{\mu_\phi}$ from $Crit_\phi$ to Λ. [Ignores **Maslov index**.]

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Canonical relations: Definition

Let X, Y be manifolds (not necessarily of the same dimension).

- On $T^*(X \times Y)$ the natural symplectic form is $\omega_X + \omega_Y$.

  But on $T^*X \times T^*Y$ the 'natural' form (for MLA) is $\omega_X - \omega_Y$.

- **Def.** A **canonical relation** from $T^*Y \setminus \mathbf{0}$ to $T^*X \setminus \mathbf{0}$ is a conic Lagrangian

$$C \subset \big((T^*X \setminus 0) \times (T^*Y \setminus 0), \, \omega_X - \omega_Y \big).$$

  Thus, if $\Lambda \subset T^*(X \times Y)$ is a conic Lagrangian which does not intersect either zero section, then

$$C' := \{(x, \xi; y, -\eta) : (x, y; \xi, \eta) \in \Lambda\}$$

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Canonical relations: Canonical graphs

- **Def.** If (M, ω_M) and (N, ω_N) are symplectic manifolds of the same dimension, then a C^∞ map $\Phi : M \to N$ is a **canonical transformation** (or **symplectomorphism**) if $\Phi^* \omega_N = \omega_M$

  Since ω_M nondeg, this $\implies$ Φ is a local diffeomorphism.

- **Prop.** (i) The graph of a canonical transformation,

$$\{(m, \Phi(m)) : m \in M\} \subset (M \times N, \omega_M - \omega_N)$$

  is a canonical relation from N to M, called a **canonical graph**.
  (ii) $C \subset M \times N$ is a **local** can grph iff $\pi_M : C \to M$ is a loc diffeom iff $\pi_N : C \to N$ is a loc diffeom.

- **Ex.** A diffeom $\chi : X^n \to Y^n$ induces a map $\Phi : T^*X \to T^*Y$, linear in each fiber, which is a canonical transformation:

$$\Phi(x, \xi) = \left(\chi(x), \, ((D\chi)^{-1})^t(\xi)\right)$$

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$$C \subset \big((T^*X \setminus 0) \times (T^*Y \setminus 0), \omega_X - \omega_Y\big)$$
be a canonical relation, so that
$$\Lambda = C' := \{(x, y; \xi, -\eta) : (x, \xi; y, \eta) \in C\} \subset T^*(X \times Y) \setminus \mathbf{0}$$
is a conic Lagrangian. Also write $C = \Lambda'$.

- **Def.** The **Fourier integral operators** (FIO) associated with C and of order $m \in \mathbb{R}$, denoted $I^m(X, Y; C) = I^m(C)$, are those maps $\mathcal{D}(Y) \to \mathcal{D}'(X)$ whose Schwartz kernels are in $I^m(X \times Y; C')$.

- Thus, $T \in I^m(X, Y; C)$ is given weakly by a locally finite sum of
$$Tf(x) = \int_{Y \times \mathbb{R}^{N_\phi}} e^{i\phi(x,y,\theta)}\, a(x, y, \theta) f(y)\, d\theta dy,$$
with $\Lambda_\phi \subseteq C'$ and $a \in S_{1,0}^{m+\frac{dimX+dimY}{4}-\frac{N_\phi}{2}}(X \times Y \times \mathbb{R}^{N_\phi})$.

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Fourier integral operators: Examples

- **ΨDOs**: $T_1 \in \Psi^m(X)$ has form (w.r.t. loc coords)

$$T_1 f(x) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta} a(x,y;\theta)\, f(y)\, d\theta\, dy, \quad a \in S^m_{1,0},$$

and saw that

$$\phi \rightsquigarrow \Lambda_\phi = \{(x,x;\xi,-\xi) : (x,\xi) \in T^*X\} = N^*\Delta_X.$$

- Taking $Y = X$, $\Psi^m(X) = I^{m+\frac{n}{2}-\frac{2n}{4}}(X,X;C) = I^m(C)$, where

$$C = (N^*\Delta_X)' = \Delta_{T^*X} = graph\,(I_{T^*X})\,.$$

- So: pseudodifferential operators are **special cases** of FIOs, associated with the **diagonal** relation, which is a **canonical graph.**

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Radon transform: For $f \in \mathcal{D}(\mathbb{R}^n)$,

$$T_3 f(\omega, s) = \int_{\{y\cdot\omega = s\}} f(y)\, d\sigma(y), \quad \omega \in \mathbb{S}^{n-1},\ s \in \mathbb{R}.$$

- K_{T_3} is conormal for the hypersurface $Y = \{y \cdot \omega - s = 0\}$:

$$K_{T_3}(\omega, s, y) = \int_{\mathbb{R}} e^{i(y\cdot\omega - s)\theta}\, 1(\theta)\, d\theta,$$

with conormal bundle $\Lambda_3 = N^*Y \subset T^*\left(\mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n\right) \setminus \mathbf{0}$,

$$\Lambda_3 = \left\{ (\omega, y\cdot\omega, y; \theta i_\omega^*(y), -\theta, \theta\omega) : \omega \in \mathbb{S}^{n-1}, y \in \mathbb{R}^d, \theta \in \mathbb{R} \setminus 0 \right\}.$$

$\implies T_3 \in I^{0 + \frac{1}{2} - \frac{2n}{4}} = I^{-\frac{n-1}{2}}(\mathbb{S}^{n-1} \times \mathbb{R}, \mathbb{R}^n; C_3)$ with loc can grph

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FIO calculus: Adjoints

Suppose $A \in I^m(X, Y; C)$. What about (formal) A^*? If

$$K_A(x,y) = \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)}\, a(x,y,\theta)\, d\theta, \quad a \in S^{m-\frac{N}{2}+\frac{n_X+n_Y}{4}},$$

then

$$K_{A^*}(y,x) = \overline{K_A(x,y)}$$

$$= \int_{\mathbb{R}^N} e^{-i\phi(x,y,\theta)}\, \overline{a}(x,y,\theta)\, d\theta, \quad \overline{a} \in S^{m-\frac{N}{2}+\frac{n_X+n_Y}{4}}$$

$\implies$ $A^* \in I^m(Y, X; C^t)$, where C^t is the **transpose** relation,

$$C^t = \{(y,\eta;x,\xi) : (x,\xi;y,\eta) \in C\} \subset T^*Y \times T^*X.$$

FIO calculus: Composition

Suppose $A_1 \in I^{m_1}(X, Y; C_1)$, $A_2 \in I^{m_2}(Y, Z; C_2)$ are properly supported.

- **Q.** Is $A_1 A_2$ an FIO? **No** in general, but **yes** if we impose some geometric conditions.
- Note

$$WF_{A_1 A_2} \subseteq WF_{A_1} \circ WF_{A_2} = WF(K_{A_1})' \circ WF(K_{A_2})'$$

$$\subseteq C_1 \circ C_2 \subset (T^*X \setminus \mathbf{0}) \times (T^*Z \setminus \mathbf{0})$$

  Basic examples show $C_1 \circ C_2$ need not be a smooth canonical relation. However, under a **transversality** or **clean intersection** condition, it is: *The analysis follows the geometry.*

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FIO calculus: Transverse intersection

- **Def.** S_1, $S_2 \subset M$ intersect **transversally** if $T_m S_1 + T_m S_2 = T_m M$ for all $m \in S_1 \cap S_2$. (Holds $\iff$ $N_m^* S_1 \cap N_m^* S_2 = (0)$.) Write $S_1 \overline{\pitchfork} S_2$.

- **Prop.** If $S_1 \overline{\pitchfork} S_2$, then

  (i) $S_3 := S_1 \cap S_2$ is smooth;

  (ii) codim(S_3) =codim(S_1)+codim(S_2); and

  (iii) $TS_3 = TS_1 \cap TS_2$ at all points.

- **Ex.** In $\mathbb{R}^3$: $\{z = 0\} \overline{\pitchfork} \{z = x\}$, but $\{z = 0\} \not\overline{\pitchfork} \{z = xy\}$.

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$$C_1 \circ C_2 = \{(x,\xi;\, z,\zeta) : \exists (y,\eta)\, s.t.\, (x,\xi,y,\eta) \in C_1,\ (y,\eta,z,\zeta) \in C_2\}$$

$$= (\pi_1 \times \pi_4)\big((C_1 \times C_2) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z)\big)$$

- To have a chance of $A_1 A_2$ being an FIO associated with a smooth canonical relation, need that the intersection set be **smooth.**

- One way to get this is to demand that the intersection be **transverse.**

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FIO calculus: Transverse intersection calculus

- **Thm. (Hörmander)** If $A_1 \in I^{m_1}(X, Y; C_1)$ and $A_2 \in I^{m_2}(Y, Z; C_2)$ are properly supported, and $(C_1 \times C_2) \,\overline{\pitchfork}\, (T^*X \times \Delta_{T^*Y} \times T^*Z)$, then $C_1 \circ C_2$ is a **smooth canonical relation** and

$$A_1 A_2 \in I^{m_1+m_2}(X, Z; C_1 \circ C_2)$$

- If either C_1 **or** C_2 is a local **canonical graph**, then $A_1 A_2$ is covered by the $\overline{\pitchfork}$ calculus. In particular, any $I^m(X, Y; C)$ is closed under composition on the right with $\Psi^0(Y)$ and on the left with $\Psi^0(X)$.

- If C is a canonical graph and $A \in I^m(X, Y; C)$ is properly supported, then $A^*A \in \Psi^{2m}(Y)$, and A elliptic at $(x_0, \xi_0, y_0, \eta_0) \implies$ A^*A elliptic at (y_0, η_0).

- If C is a only a **local** can grph, then $A^*A \in \Psi^{2m}(Y) +$ a sum of other FIOs. (See exercise #3.)

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FIO calculus: Transverse intersection calculus

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FIO calculus: Clean intersection calculus

- **Def.** $S_1, S_2 \subset M$ intersect **cleanly** if (i) $S_3 := S_1 \cap S_2$ is smooth; and (ii) $TS_3 = TS_1 \cap TS_2$ at all points. The **excess** of the intersection is $e :=$ codim(S_1)+codim(S_2)−codim$(S_3) \geq 0$.

  Ex. $S_1 = x$-axis and $S_2 = y$-axis in $\mathbb{R}^3$, with excess $e = 2+2-3 = 1$.

  Ex. $S_1 = x$-axis and $S_2 = \{y = x^2\}$ do not intersect cleanly in $\mathbb{R}^2$.

- **Thm. (Duistermaat-Guillemin; Weinstein)** If $C_1 \times C_2$ intersects $T^*X \times \Delta_{T^*Y} \times T^*Z$ cleanly with excess e, then

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Clean intersection calculus: flowouts

Ex. Let $\Sigma \subset T^*X^n \setminus \mathbf{0}$ be a conic hypersurface.

- Σ is automatically **co-isotropic:** $(T\Sigma)^\omega \subset T\Sigma$ at all pts.
- Microlocally, can write $\Sigma = \{p(x,\xi) = 0\}$, $p \in C^\infty_\mathbb{R}$, homog of deg 1.
- $(T\Sigma)^\omega = \mathbb{R} \cdot H_p$, where H_p is the **Hamiltonian vector field** of p,

$$H_p(x,\xi) = (dp(x,\xi))^\omega = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}$$

- But $H_p \in T\Sigma$, since $\langle dp, H_p \rangle = \omega(H_p, H_p) = 0$ by skew-symmetry.
- Thus, Σ is foliated by the integral curves of H_p, called the **bicharacteristic curves** of Σ, which are **nonradial** if $H_p \not\parallel \xi \cdot \partial_\xi$. The curve passing through $(x,\xi) \in \Sigma$ is denoted $\Xi_{x,\xi}$.

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- **Def.** The **flowout relation** of Σ,

  $$C_\Sigma = \left\{(x,\xi,y,\eta) : (x,\xi) \in \Sigma,\, (y,\eta) \in \Xi_{x,\xi}\right\} \subset (T^*X\setminus\mathbf{0})\times(T^*X\setminus\mathbf{0})$$

  is a smooth, conic canonical relation, with $C^t = C$.

- Note: C_Σ is **not** a canonical graph: $D\pi_L$, $D\pi_R$ drop rank by 1 **everywhere**.

- $C_\Sigma \circ C_\Sigma$ covered by the clean intersection calc, with excess $e = 1$:

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- Flowout relations C_Σ describe the propagation of singularities of solutions to $Pu = f$, for some $P(x, D) \in \Psi^m_{cl}(X)$.

- **Def.** $P(x, D) \in \Psi_{cl}$ is of **real principal type** if $p(x, \xi) := \sigma_{prin}(P)$ is $\mathbb{R}$-valued, $d_{x,\xi} p \neq (0, 0)$ at $\Sigma = p^{-1}(0)$, and no bicharacteristic $\Xi_{x,\xi}$ of p is trapped over a compact set $K \subset\subset X$.
  (In particular, there are no radial points.)

- **Thm. (Duistermaat-Hörmander)** If $P(x, D)$ is RPT and $Pu = f$, then $WF(u) \setminus WF(f)$ is a union of maximally extended $\Xi_{x,\xi}$. Futhermore, there exists a two-sided parametrix Q, $QP = I - R_1$ and $PQ = I - R_2$ with $R_1, R_2 \in \Psi^{-\infty}(X)$, with $Q \in I^{\frac{1}{2}-m}(C_\Sigma)$ away from Δ_{T^*X}.

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Applications: Egorov's Theorem

- Let $\Phi : T^*Y \setminus \mathbf{0} \to T^*X \setminus \mathbf{0}$ be a canonical transformation defined on a conic nhood of (y_0, η_0). Then $C := graph(\Phi)$ is a canonical graph, and can assume that it is generated by a gen func $S(x, \eta)$.
- Let $F \in I^0(C; X, Y)$ be an elliptic FIO (which we can assume is a **canonical operator** $\leftrightarrow S$), and $G \in I^0(C^t; Y, X)$ a parametrix (microlocal inverse mod C^∞), with $C^t = graph(\Phi^{-1})$:

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- $\Longrightarrow$ Large literature on reducing ΨDO to normal forms, proving propagation of singularities or local solvability.

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Suppose $Z \subset X^{n_X} \times Y^{n_Y}$, codim k. Consider

$$\begin{array}{ccc} & Z & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_Y} \\ X & & Y \end{array}$$

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- Choice of smooth densities on X, Y, Z induces pair of **generalized Radon transforms**, $\mathcal{R} : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ and $\mathcal{R}^t : \mathcal{E}'(X) \to \mathcal{D}'(Y)$,

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Z is the **incidence relation** of a **generalized Radon transform,** $\mathcal{R}$.

- **Guillemin-Sternberg:** Schwartz kernel of $\mathcal{R} = \delta_Z$, which is a conormal, hence Fourier integral distribution: Locally describe Z as

$$Z = \{(x, y) : \Phi_1(x, y) = \cdots = \Phi_k(x, y) = 0\}.$$

- Writing δ_Z as shorthand for a smooth multiple of $\delta_{\mathbb{R}^k}(\Phi)$,

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$\Longrightarrow$

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- **Ex.** Radon transform: $Y = \mathbb{R}^n$, $X = \mathbb{S}^{n-1} \times \mathbb{R}$,

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Classical: $\mathcal{R}^*\mathcal{R}f = c_n f * |y|^{1-n}$, which has inverse $c_n(-\Delta)^{\frac{n-1}{2}}$.

The **filtered backprojection** inversion formulae for the Radon transform,

$$f = c_n\big((-\Delta)^{\frac{n-1}{2}}\mathcal{R}^*\big)\mathcal{R}f = c_n\mathcal{R}^*(|\partial_s|^{n-1})\mathcal{R}f,$$

thus generalize ($\mod C^\infty$) to a wide variety of GRTs.

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thus generalize ($\mod C^\infty$) to a wide variety of GRTs.

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- If C is a canonical graph, then $\mathcal{R}^*\mathcal{R} \in \Psi^{\frac{n_X+n_Y-2k}{2}}(Y)$, elliptic if $\mathcal{R}$ is.
- $\exists$ parametrix $Q \in \Psi^{-\frac{n_X+n_Y-2k}{2}}(Y)$, $\quad Q\mathcal{R}^*\mathcal{R} = I \mod C^\infty$, and thus $\mathcal{R}f$ determines $f \mod C^\infty, \forall f \in \mathcal{E}'(Y)$.
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  is elliptic and properly supported. Then $F^*F \in \Psi^{2m}(Y)$.

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Some extensions of FIO theory

1. **Paired Lagrangian distributions:**

- ∃ need for distributions [operators] whose wavefront sets [relations] are not a smooth Lagrangian [canonical relation]:

- Duistermaat-Hörmander constructed parametrices Q for **real principal type** operators $P(x, D)$, with

$$WF_Q \subseteq \Delta_{T^*X} \cup C_\Sigma$$

where C_Σ is the flowout of Σ. $\Delta \cap C_\Sigma$ cleanly with excess $e = n - 1$.

- Each of Δ_{T^*X}, C_Σ is smooth, but their union is not, and K_Q is **not** simply a sum in $I^{m_1}(\Delta) + I^{m_2}(C_\Sigma)$.

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- $I^{p,l}$-operators arise in several applications:

  (i) **Parametrices** for RPT: $Q \in I^{\frac{1}{2}-m,-\frac{1}{2}}(\Delta, C_\Sigma)$ [Melrose-Uhlmann]

  (ii) Parametrices for GRT failing the Bolker condition, e.g., **restricted X-ray transforms** [G.-Uhlmann];

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