Minicourse on fractal uncertainty principle
Lecture 3: Fractal Uncertainty Principle

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Definition

Fix $\nu > 0$. A set $X \subset \mathbb{R}$ is ν-porous up to scale h if for each interval $I \subset R$ of length $h \leq |I| \leq 1$, there is an interval $J \subset I$, $|J| = \nu|I|$, $J \cap X = \emptyset$

Theorem 2 (Fractal Uncertainty Principle)

Assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale h. Then $\exists \beta = \beta(\nu) > 0$:

$$\|\mathbb{1}_X(\tfrac{h}{i}\partial_x)\mathbb{1}_Y(x)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0$$

We can rewrite this uncertainty principle as

$$\|\mathbb{1}_X\,\mathcal{F}_h\,\mathbb{1}_Y\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as} \quad h \to 0$$

where $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary semiclassical Fourier transform:

$$\mathcal{F}_h f(x) = (2\pi h)^{-\frac{1}{2}}\,\widehat{f}\Big(\frac{x}{h}\Big) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\frac{i}{h}xy} f(y)\,dy$$

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Basic uncertainty principles

- Looking for

$$\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as} \quad h \to 0$$

- Trivial bound: $\beta = 0$ as $\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2\to L^2} \leq 1$
- Volume bound: if $|X|, |Y| = \mathcal{O}(h^{1-\delta})$ then get $\beta = \frac{1}{2} - \delta$:

$$\|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y\|_{L^2\to L^2} \leq \|\mathbb{1}_X\|_{L^\infty\to L^2} \|\mathcal{F}_h\|_{L^1\to L^\infty} \|\mathbb{1}_Y\|_{L^2\to L^1}$$
$$\leq \sqrt{\frac{|X|\cdot|Y|}{2\pi h}} = \mathcal{O}(h^{\frac{1}{2}-\delta})$$

- Cannot be improved if we only know the volume, e.g.

$$X = Y = [-\sqrt{h}, \sqrt{h}] \quad \Longrightarrow \quad \text{cannot get} \quad \beta > 0$$

So we need to know more about the structure of X, Y (e.g. porosity)

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A bit on the proof of FUP for Fourier transform

Theorem 2' (a restatement of Theorem 2)

Let X, Y be ν-porous up to scale h. Then there exists $\beta = \beta(\nu) > 0$:

$$f \in L^2(\mathbb{R}), \quad \operatorname{supp} \hat{f} \subset h^{-1} \cdot Y \quad \Longrightarrow \quad \|\mathbf{1}_X f\|_{L^2(\mathbb{R})} \leq C h^\beta \|f\|_{L^2(\mathbb{R})}$$

- Write $X \subset \bigcap_j X_j$ where each $X_j \subset X_{j-1}$ has holes on scale $2^{-j} \geq h$
- Will show: for each j, $\|\mathbf{1}_{X_j} f\|_{L^2} \leq (1 - \epsilon)\|1_{X_{j-1}} f\|_{L^2}$
- This requires a lower bound on the mass of f on the 'holes' in $\mathbb{R} \setminus X_j$
- Such bounds exist if we know about decay of $\hat{f}$, e.g.

$$|\hat{f}(\xi)| \leq Ce^{-w(\xi)} \quad \text{where} \quad \int_{\mathbb{R}} \frac{w(\xi)}{1+\xi^2}\, d\xi = \infty$$

- To pass from $\operatorname{supp} \hat{f} \subset h^{-1} \cdot Y$ to Fourier decay bounds, take the convolution $f * g$, $\widehat{f * g} = \hat{f}\hat{g}$, where g is compactly supported and $\hat{g}$ has the right decay but only on $h^{-1} \cdot Y$
- Existence of g follows from Beurling–Malliavin theorem, porosity of Y

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Hyperbolic FUP

For applications to hyperbolic surfaces, we replace the phase xy in $\mathcal{F}_h$ by $2\log|x-y|$ and introduce a cutoff $\chi \in C_c^\infty(\mathbb{R}^2)$, $\operatorname{supp}\chi \cap \{x=y\} = \emptyset$:

$$\mathcal{B}_{\chi,h} f(x) = (2\pi h)^{-\frac{1}{2}} \int_{\mathbb{R}} |x-y|^{-\frac{2i}{h}} \chi(x,y) f(y)\, dy$$

The operator $\mathcal{B}_{\chi,h}$ appears naturally in the composition $B_-^{-1} B_+$ where $B_\pm : L^2(M) \to L^2(\mathbb{R}^2)$ are FIOs straightening out L_s, L_u locally

One can deduce from FUP for $\mathcal{F}_h$ a similar statement for $\mathcal{B}_{\chi,h}$:

Theorem 2" (Hyperbolic FUP)

Assume that $X, Y \subset \mathbb{R}$ are ν-porous up to scale h. Then there exist $\beta = \beta(\nu) > 0$ and $C = C(\nu, \chi)$ such that

$$\|\mathbb{1}_X \mathcal{B}_{\chi,h} \mathbb{1}_Y\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \le C h^\beta.$$

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A bit on reducing hyperbolic FUP to Fourier FUP

- Replace Y by its $h^{1/2-}$-neighborhood $\widetilde{Y}$: $\|\mathbb{1}_X \mathcal{B}_h \mathbb{1}_Y\| \leq \|\mathbb{1}_X \mathcal{B}_h \mathbb{1}_{\widetilde{Y}}\|$
- Split $X = \bigsqcup_j X_j$, each X_j lies in an $h^{1/2}$-sized interval $[x_j, x_j + h^{1/2}]$
- Show $B_j := \mathbb{1}_{X_j} \mathcal{B}_h \mathbb{1}_{\widetilde{Y}}$ almost orthogonal: for $|j - \ell| \gg 1$

$$\|B_j^* B_\ell\| = \mathcal{O}(h^\infty), \quad \|B_j B_\ell^*\| = \mathcal{O}(h^\infty)$$

  so by Cotlar–Stein $\|\mathbb{1}_X \mathcal{B}_h \mathbb{1}_{\widetilde{Y}}\| \lesssim \max_j \|\mathbb{1}_{X_j} \mathcal{B}_h \mathbb{1}_{\widetilde{Y}}\|$
- Use a change of variables to bound $\|\mathbb{1}_{X_j} \mathcal{B}_h \mathbb{1}_{\widetilde{Y}}\|$ using the Fourier FUP: if $\Phi(x, y) = -2 \log |x - y|$ and $|x - x_j| \leq h^{1/2}$ then on $\operatorname{supp} \chi$

$$e^{\frac{i}{h}\Phi(x,y)} \approx e^{\frac{i}{h}\Phi(x_j,y)} e^{\frac{i}{h}(x-x_j)\kappa_j(y)}, \quad \kappa_j(y) := \partial_x \Phi(x_j, y)$$

- The β for hyperbolic FUP is $\frac{1}{2}$ of the β for the Fourier FUP

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Discrete Cantor sets

We now present a proof of FUP in the special setting of Cantor sets. This is much simpler than the general case but keeps some key features. We follow D–Jin '17, with the exposition from [arXiv:1903.02599]

- Discrete unitary Fourier transform $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-\frac{2\pi i j \ell}{N}} u(\ell)$$

- Fix $M \geq 3$, $\mathscr{A} \subset \{0, \dots, M-1\}$. Put $N := M^k$, $k \gg 1$ and define

$$\mathcal{C}_k := \{a_0 + a_1 M + \dots + a_{k-1} M^{k-1} \mid a_0, \dots, a_{k-1} \in \mathscr{A}\}$$

- Example: if $M = 3$, $\mathscr{A} = \{0, 2\}$, then $\mathcal{C}_k \subset \{0, \dots, N-1\}$, $N = 3^k$, is the discrete mid-3rd Cantor set $\{0, 2, 6, 8, 18, 20, 24, 26, \dots\}$
- The number of elements of $\mathcal{C}_k$ is $|\mathcal{C}_k| = N^\delta$ where $\delta = \log_M |\mathscr{A}|$

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Uncertainty principle for discrete Cantor sets

Theorem

Assume that $0 < \delta < 1$, i.e. $1 < |\mathscr{A}| < M$. Then there exists $\beta = \beta(M, \mathscr{A}) > \max(0, \frac{1}{2} - \delta)$ such that as $N = M^k \to \infty$,

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} = \mathcal{O}(N^{-\beta}).$$

- Trivial bound $\beta = 0$: since $\mathcal{F}_N$ is unitary, $\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq 1$
- Volume bound $\beta = \frac{1}{2} - \delta$: defining the Hilbert–Schmidt norm

$$\|A\|_{\mathrm{HS}}^2 = \sum_{j,k} |a_{jk}|^2 \quad \text{where} \quad A = (a_{jk})_{j,k=1}^N$$

we have

$$\| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N} \leq \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathrm{HS}} = N^{\delta - \frac{1}{2}}.$$

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Submultiplicativity

The proof of FUP for Cantor sets is greatly simplified by the

Submultiplicativity Lemma

Define $r_k := \| \mathbb{1}_{\mathcal{C}^k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$. Then $r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}$ for all k_1, k_2.

To prove it, we employ the following decomposition also used in FFT:

- Write $k = k_1 + k_2, \quad N = N^k = N_1 \cdot N_2, \quad N_j := M^{k_j}$
- Identify $u \in \mathbb{C}^N$ with an $N_1 \times N_2$ matrix $U_{ab} = u(N_1 b + a)$
- Apply the Fourier transform $\mathcal{F}_{N_2}$ to each row of U
- Multiply the entries of U by the twist factors $e^{-\frac{2\pi iab}{N}}$
- Apply the Fourier transform $\mathcal{F}_{N_1}$ to each column of U
- The resulting matrix V gives $v = \mathcal{F}_N u$ by $V_{ab} = v(N_2 a + b)$
- Using that $\mathcal{C}_k = N_1 \mathcal{C}_{k_2} + \mathcal{C}_{k_1} = N_2 \mathcal{C}_{k_1} + \mathcal{C}_{k_2}$, we get $r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}$

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An example of the 'Fast Fourier Transform' decomposition

Let's say $N = 4 = N_1 N_2$ where $N_1 = N_2 = 2$.

Take $u = (u_0, u_1, u_2, u_3) \in \mathbb{C}^4$. Follow the instructions on the last slide:

- Take $U = \begin{pmatrix} u_0 & u_2 \\ u_1 & u_3 \end{pmatrix}$, $\mathcal{F}_2$ each row to get $\frac{1}{\sqrt{2}} \begin{pmatrix} u_0 + u_2 & u_0 - u_2 \\ u_1 + u_3 & u_1 - u_3 \end{pmatrix}$
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$$V = \begin{pmatrix} \mathcal{F}_4 u(0) & \mathcal{F}_4 u(1) \\ \mathcal{F}_4 u(2) & \mathcal{F}_4 u(3) \end{pmatrix}$$

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FUP with $\beta > 0$

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$$\mathcal{F}_N u(j) = N^{-1/2} P(\omega^j), \quad \omega := e^{-\frac{2\pi i}{N}}$$

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FUP with $\beta > \frac{1}{2} - \delta$ ('baby Dolgopyat')

- Similarly to the previous slide, enough to show that $\exists k : r_k < N^{\delta - \frac{1}{2}}$ where $r_k := \| \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} \|_{\mathbb{C}^N \to \mathbb{C}^N}$, $N = M^k$
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$$(j - j')(\ell - \ell') \in N\mathbb{Z} \quad \text{for all} \quad j, j', \ell, \ell' \in \mathcal{C}_k$$

- This cannot happen already when $k = 2$ (and $|\mathscr{A}| > 1$): just take two different $a, b \in \mathscr{A}$ and put

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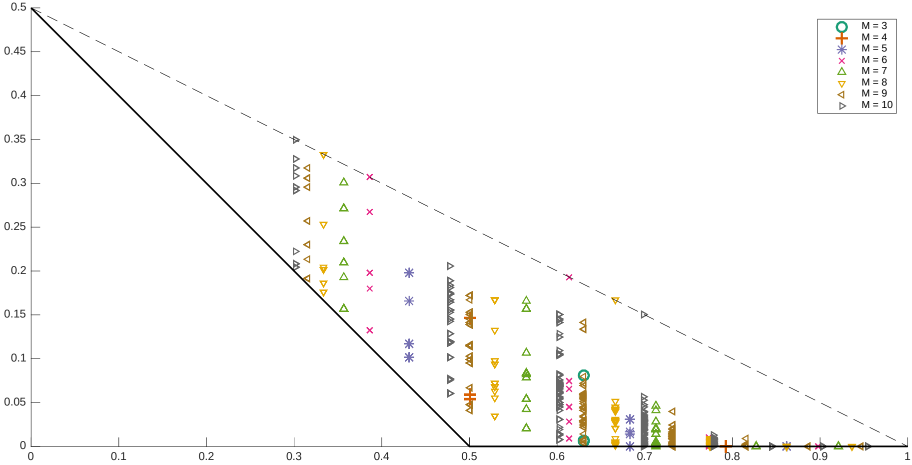
A picture of FUP exponents for all alphabets with $M \le 10$



Horizontal axis: δ, vertical axis: β, solid line: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed line: $\beta = \frac{1-\delta}{2}$ (corresponding to the gap conjectured by Jakobson–Naud)

A higher dimensional FUP?

- Open problem: get FUP with $\beta > 0$ on $\mathbb{R}^n$, $n > 1$. Let's take $n = 2$
- $\mathcal{F}_h f(x) = (2\pi h)^{-1}\widehat{f}(\frac{x}{h})$ semiclassical Fourier transform
- Want $\|\,\mathbb{1}_X\,\mathcal{F}_h\,\mathbb{1}_Y\,\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} = \mathcal{O}(h^\beta)$ where $X, Y \subset \mathbb{R}^2$ are δ-regular up to scale h and $\delta < 2$
- This is false: take $\delta = 1$, $X = [0,h]\times[0,1]$, $Y = [0,1]\times[0,h]$
- Han–Schlag '20: FUP holds with $\beta > 0$ if one of X, Y is contained in the product of 2 fractal sets
- It could be that the hyperbolic FUP (with $e^{-\frac{i}{h}\langle x,y\rangle}$ replaced by $|x-y|^{-\frac{2i}{h}}$) still holds.
  Partial result by D–Zhang WIP, when one of X, Y is a curve

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- Han–Schlag '20: FUP holds with $\beta > 0$ if one of X, Y is contained in the product of 2 fractal sets
- It could be that the hyperbolic FUP (with $e^{-\frac{i}{h}\langle x,y\rangle}$ replaced by $|x - y|^{-\frac{2i}{h}}$) still holds.
  Partial result by D–Zhang WIP, when one of X, Y is a curve

A higher dimensional FUP?

- Open problem: get FUP with $\beta > 0$ on $\mathbb{R}^n$, $n > 1$. Let's take $n = 2$
- $\mathcal{F}_h f(x) = (2\pi h)^{-1} \widehat{f}(\frac{x}{h})$ semiclassical Fourier transform
- Want $\| \mathbb{1}_X \mathcal{F}_h \mathbb{1}_Y \|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} = \mathcal{O}(h^\beta)$ where $X, Y \subset \mathbb{R}^2$ are δ-regular up to scale h and $\delta < 2$
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Thank you for your attention!