Toeplitz operators and Bergman kernel asymptotics

Michael Hitrik

Department of Mathematics, UCLA

Séminaire de Mathématiques Supérieures, June 2, 2021

Joint work with L. Coburn, J. Sjöstrand, and F. White (Part I) and with A. Deleporte, J. Sjöstrand, and M. Stone (Part II)

Part I : Toeplitz operators

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi_0(x)}{\partial x_j \partial \overline{x}_k} \zeta_j \overline{\zeta}_k > 0, \quad x \in \mathbb{C}^n, \quad 0 \neq \zeta \in \mathbb{C}^n.$$

Example $\Phi(x) = \dfrac{|x|^2}{2} = \dfrac{x \cdot \overline{x}}{2}, \quad x \in \mathbb{C}^n.$

Associated to Φ_0 we introduce the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)).$$

Here $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$.

Original idea of V. Bargmann (1961) : express Quantum Mechanics directly in phase space $T^*\mathbb{R}^n \simeq \mathbb{C}^n$.

Toeplitz quantization

Given a measurable function $p : \mathbb{C}^n \to \mathbb{C}$, let us consider the Toeplitz operator with symbol p,

$$\mathrm{Top}(p) = \Pi_{\Phi_0} \circ p \circ \Pi_{\Phi_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n),$$

equipped with the natural domain

$$\mathcal{D} = \{u \in H_{\Phi_0}(\mathbb{C}^n);\ pu \in L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx))\}.$$

Here

$$\Pi_{\Phi_0} : L^2(\mathbb{C}^n, e^{-2\Phi_0} L(dx)) \to H_{\Phi_0}(\mathbb{C}^n)$$

is the orthogonal (Bergman) projection.

Toeplitz vs Weyl quantization

Let $p \in L^\infty(\mathbb{C}^n)$, say. We have

$$\mathrm{Top}(p) = a^w(x, D_x),$$

where $a \in C^\infty(\mathbb{C}^n)$ is given by

$$a(x) = \left(\exp\left(\frac{1}{4}(\Phi''_{0,x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}\right) p\right)(x), \quad x \in \mathbb{C}^n.$$

The symbol of $(\Phi''_{0,x\overline{x}})^{-1}\partial_x \cdot \partial_{\overline{x}}$ is

$$-\frac{1}{4}(\Phi''_{0,x\overline{x}})^{-1}\overline{\zeta} \cdot \zeta < 0, \quad 0 \neq \zeta \in \mathbb{C}^n \simeq \mathbb{R}^{2n} \Longrightarrow$$

the Weyl symbol a is given by the forward heat flow acting on p.

V. Guillemin (1985),... J. Sjöstrand (1994),... M. Zworski (2012).

When is a Toeplitz operator bounded on $H_{\Phi_0}(\mathbb{C}^n)$?

Example (C. Berger – L. Coburn, 1994). Let $\Phi_0(x) = |x|^2/2$ and let

$$p(x) = \exp(\lambda |x|^2), \quad \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda < 1/2.$$

Explicit computations show that

$$\mathrm{Top}(p) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)) \Longleftrightarrow |1-\lambda| \geq 1.$$

The Weyl symbol a can be computed by exact stationary phase and we see that

$$|1-\lambda| \geq 1 \Longleftrightarrow a \in L^\infty(\mathbb{C}^n).$$

Conjecture (C. Berger – L. Coburn, 1994) For any "reasonable" Toeplitz symbol p, we have

$$\mathrm{Top}(p) \in \mathcal{L}(H_{\Phi_0}(\mathbb{C}^n), H_{\Phi_0}(\mathbb{C}^n)) \Longleftrightarrow \text{the Weyl symbol } a \in L^\infty(\mathbb{C}^n).$$

The conjecture still stands.

C. Berger – L. Coburn, 1994 : some partial results towards the conjecture.

Theorem (L. Coburn – J. Sjöstrand — F. White – M. H., 2019)

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let Q be a quadratic polynomial on $\mathbb{C}^n$ with the principal part q. Assume that

$$\mathrm{Re}\, q(x) < \Phi_{\mathrm{herm}}(x) := \frac{1}{2}\left(\Phi_0(x) + \Phi_0(ix)\right), \quad 0 \neq x \in \mathbb{C}^n,$$

and that

$$\det \partial_{\overline{x}}\partial_x \left(2\Phi_0 - q\right) \neq 0.$$

Let $a \in C^\infty(\mathbb{C}^n)$ be the Weyl symbol of $\mathrm{Top}(e^Q)$ and assume that $a \in L^\infty(\mathbb{C}^n)$. Then the Toeplitz operator

$$\mathrm{Top}(e^Q) : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_0}(\mathbb{C}^n)$$

is bounded.

Some ideas of the proof

Let Ψ_0 be the holomorphic quadratic form on $\mathbb{C}^{2n}_{x,y}$ such that $\Psi_0(x,\overline{x}) = \Phi_0(x)$ (the polarization of Φ_0). The Bergman projection is given by

$$\Pi_{\Phi_0} u(x) = a_0 \int e^{2\Psi_0(x,\overline{y})} u(y) e^{-2\Phi_0(y)} dy\, d\overline{y}, \quad a_0 \neq 0,$$

and let us write using the polarized form,

$$\begin{aligned} \mathrm{Top}(e^{Q}) u(x) &= a_0 \int e^{2\Psi_0(x,\overline{y})} e^{Q(y,\overline{y})} u(y) e^{-2\Phi_0(y)} dy\, d\overline{y} \\ &= a_0 \iint_{\Gamma} e^{2(\Psi_0(x,\theta)-\Psi_0(y,\theta))+Q(y,\theta)} u(y)\, dy\, d\theta. \end{aligned}$$

Here $u \in \mathcal{D}$ and Γ is the contour in $\mathbb{C}^{2n}_{y,\theta}$ given by $\theta = \overline{y}$ (the anti-diagonal).

Viewing Toeplitz operators as an FIO

The holomorphic quadratic polynomial

$$F(x,y,\theta) = \frac{2}{i}\left(\Psi_0(x,\theta) - \Psi_0(y,\theta) + \frac{1}{2}Q(y,\theta)\right)$$

is a non-degenerate phase function in the sense of Hörmander,

$$\mathrm{rank}\, F''_{\theta,(x\,y\,\theta)} = n,$$

$\Longrightarrow$ the operator $\mathrm{Top}(e^Q)$ can be regarded as a Fourier integral operator associated to the complex affine canonical transformation

$$\kappa : \mathbb{C}^{2n} \ni (y, -F'_y(x,y,\theta)) \mapsto (x, F'_x(x,y,\theta)) \in \mathbb{C}^{2n}, \quad F'_\theta(x,y,\theta) = 0.$$

Positivity

Assume first that $Q = q$ is a quadratic form on $\mathbb{C}^n$.

Main observation : We have :

$$\text{the Weyl symbol } a \in L^\infty \iff \kappa \text{ is positive relative to } \Phi_0.$$

L. Hörmander, 1971, 1983, 1995, ... A. Melin – J. Sjöstrand (1974–1977).

Let

$$\Lambda_{\Phi_0} = \left\{ \left(x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) ; x \in \mathbb{C}^n \right\} \subset \mathbb{C}^{2n}.$$

Λ_{Φ_0} is maximally totally real $\Longrightarrow$ let

$$\iota_{\Phi_0} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$$

be the unique antilinear involution such that $\iota_{\Phi_0} = 1$ along Λ_{Φ_0} (the complex conjugation with respect to Λ_{Φ_0}).

We say that a $\mathbb{C}$-Lagrangian space $\Lambda \subset \mathbb{C}^{2n}$ is positive relative to Λ_{Φ_0} if

$$\frac{1}{i}\sigma(\rho, \iota_{\Phi_0}(\rho)) \geq 0, \quad \rho \in \Lambda.$$

A complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is said to be positive relative to Λ_{Φ_0} provided that

$$\frac{1}{i}\Big(\sigma(\kappa(\rho), \iota_{\Phi_0}\kappa(\rho)) - \sigma(\rho, \iota_{\Phi_0}(\rho))\Big) \geq 0, \quad \rho \in \mathbb{C}^{2n}.$$

Here

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j$$

is the complex symplectic form on $\mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$.

Example. Let q be a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,\xi}$ such that

$$\mathrm{Re}\, q|_{\Lambda_{\Phi_0}} \geq 0.$$

Then the canonical transformation

$$\kappa = \exp\left(\frac{1}{i} H_q\right)$$

is positive relative to Λ_{Φ_0}. Here

$$H_q = \partial_\xi q \cdot \partial_x - \partial_x q \cdot \partial_\xi$$

is the Hamilton field of q.

Characterizing positive canonical transformations

Proposition (L. Coburn – J. Sjöstrand — M. H., 2019)

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$. A complex linear canonical transformation $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is positive relative to Λ_{Φ_0} precisely when we have

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi},$$

where Φ is a strictly plurisubharmonic quadratic form such that $\Phi \leq \Phi_0$.

Quantizing complex canonical transformations

The operator theory follows the geometry :

Proposition (J. Sjöstrand, 1982)

Let $\kappa : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ be a complex linear canonical transformation such that

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_{\Phi},$$

where Φ_0, Φ are strictly plurisubharmonic quadratic forms. Then, if U is a Fourier integral operator quantizing κ, we have

$$U : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi}(\mathbb{C}^n)$$

is bounded.

Example. Let q be a holomorphic quadratic form on $\mathbb{C}^{2n}_{x,\xi}$ such that

$$\mathrm{Re}\, q|_{\Lambda_{\Phi_0}} \geq 0.$$

Then the heat evolution semigroup e^{-tq^w}, $t \geq 0$, is an FIO quantizing the positive canonical transformation

$$\exp\left(\frac{t}{i} H_q\right), \quad t \geq 0,$$

and therefore for all $t \geq 0$ we have

$$\exp\left(\frac{t}{i} H_q\right)(\Lambda_{\Phi_0}) = \Lambda_{\Phi_t},$$

and

$$e^{-tq^w} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_t}(\mathbb{C}^n).$$

Here $t \mapsto \Phi_t(x)$ is decreasing. F. Hérau – J. Sjöstrand – C. Stolk (2004), ... F. White (2021) : propagation of global analytic singularities for quadratic non-selfadjoint evolution equations.

Back to Toeplitz operators

If $Q = q$ is a quadratic form, we have : the Weyl symbol a of $\mathrm{Top}(e^Q)$ satisfies $a \in L^\infty(\Lambda_{\Phi_0}) \Longrightarrow$ the canonical transformation κ is positive relative to $\Lambda_{\Phi_0} \Longrightarrow$

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_\Phi,$$

where Φ is quadratic strictly plurisubharmonic, $\Phi \leq \Phi_0$.

In the general case, when Q is an inhomogeneous quadratic polynomial on $\mathbb{C}^n$, we factorize

$$\kappa = \kappa_\ell \circ \kappa_q,$$

where κ_q is linear canonical positive relative to Λ_{Φ_0}, and

$$\kappa_\ell(\rho) = \exp(H_\ell)(\rho) = \rho + H_\ell$$

is a complex phase space translation. Here $\ell(x, \xi)$ is a complex linear form on $\mathbb{C}^{2n}$.

Incorporating linear terms

It follows that

$$\kappa(\Lambda_{\Phi_0}) = \Lambda_\Psi,$$

where Ψ is a strictly plurisubharmonic inhomogeneous quadratic polynomial on $\mathbb{C}^n$, given by

$$\Psi(x) = \Phi(x) + \mathrm{Im}\,\left(\ell\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)\right)\right), \quad \Phi \leq \Phi_0.$$

Now the Weyl symbol a of $\mathrm{Top}(e^Q)$ satisfies $a \in L^\infty \Rightarrow$

$$\ell \quad \text{is real along } \Lambda_\Phi \cap \Lambda_{\Phi_0}.$$

We can then conclude that

$$\Phi_0(x) - \Psi(x) \geq -C, \quad x \in \mathbb{C}^n.$$

Since the operator

$$\mathrm{Top}(e^Q) : H_{\Phi_0}(\mathbb{C}^n) \to H_\Psi(\mathbb{C}^n) \hookrightarrow H_{\Phi_0}(\mathbb{C}^n)$$

is bounded, the result follows.

Part II : Bergman kernel asymptotics

Let $\Omega \subset \mathbb{C}^n$ be open pseudoconvex and let $\Phi \in C^\infty(\Omega; \mathbb{R})$ be strictly plurisubharmonic,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \Phi(x)}{\partial x_j \partial \overline{x}_k} \zeta_j \overline{\zeta}_k > 0, \quad x \in \Omega, \quad 0 \neq \zeta \in \mathbb{C}^n.$$

Associated to Φ we introduce the Bergman space

$$H_\Phi(\Omega) = \mathrm{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi/h} L(dx)).$$

Here $h \to 0^+$ is the semiclassical parameter and $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$.

We would like to understand the orthogonal (Bergman) projection

$$\Pi_\Phi : L^2(\Omega, e^{-2\Phi/h} L(dx)) \to H_\Phi(\Omega)$$

in the semiclassical limit $h \to 0^+$.

Weighted L^2 spaces of holomorphic functions I

Complex geometry/Toeplitz quantization : spaces of the form $H_\Phi(\Omega)$ serve as local models for the space of holomorphic sections of a high power of a holomorphic line bundle over a complex manifold.

F. Berezin (1975), ..., G. Tian (1990), T. Bouche (1990), D. Catlin (1999), S. Zelditch (1998), ... R. Berman – B. Berndtsson – J. Sjöstrand (2008).

C. Fefferman (1974), L. Boutet de Monvel – J. Sjöstrand (1975) (asymptotics of the Bergman and Szegő kernels for strictly pseudoconvex smooth domains $\Subset \mathbb{C}^n$).

Weighted L^2 spaces of holomorphic functions II

Let $\mathcal{L}$ be a complex line bundle over a complex compact n-dimensional manifold X, and assume that $\mathcal{L}$ is equipped with a C^∞ metric. The curvature is given by the $(1,1)$–form $\partial\overline{\partial}\Phi$, where locally $|s| = e^{-\Phi}$, for some local non-vanishing holomorphic section s of $\mathcal{L}$. We assume that the curvature form is strictly positive :

$$i\partial\overline{\partial}\Phi = i\sum_{j,k=1}^{n} \frac{\partial^2\Phi(x)}{\partial x_j \partial \overline{x}_k} dx_j \wedge d\overline{x}_k > 0,$$

so that the local weight Φ is strictly plurisubharmonic.

Weighted L^2 spaces of holomorphic functions III

Let $\mathcal{L}^k = \overbrace{\mathcal{L} \otimes ... \otimes \mathcal{L}}^{k \text{ times}}$. The Bergman projection is the orthogonal projection

$$\Pi_k : L^2(X; \mathcal{L}^k) \to \left(L^2 \cap \mathrm{Hol}\right)(X; \mathcal{L}^k).$$

Locally, we take a non-vanishing section s as before and represent general sections of $\mathcal{L}^k$ as us^k. The asymptotic analysis of Π_k is therefore locally equivalent to the study of the orthogonal projection

$$\Pi_\Phi : L^2(\Omega, e^{-2\Phi/h} L(dx)) \to H_\Phi(\Omega).$$

Here the semiclassical parameter $h \to 0^+$ is the inverse of a high power $k \to \infty$ of the line bundle $\mathcal{L}$, $h = \dfrac{1}{k}$.

Weighted L^2 spaces of holomorphic functions IV

Exponentially weighted spaces of holomorphic functions occur naturally also in analytic microlocal analysis, in connection with FBI transforms.

Let $\varphi \in \mathrm{Hol}\left(\mathrm{neigh}\left((x_0, y_0), \mathbb{C}^{2n}\right)\right)$, $y_0 \in \mathbb{R}^n$, be such that

$$-\varphi'_y(x_0, y_0) = \eta_0 \in \mathbb{R}^n, \quad \mathrm{Im}\, \varphi''_{yy}(x_0, y_0) > 0, \quad \mathrm{det}\varphi''_{xy}(x_0, y_0) \neq 0.$$

Associated to φ is the FBI transform

$$Tu(x; h) = h^{-3n/4} \int e^{i\varphi(x,y)/h} \chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n),$$

where $u \in L^2(\mathbb{R}^n)$, $\chi \in C_0^\infty(\mathrm{neigh}(y_0, \mathbb{R}^n))$, $\chi = 1$ near y_0.

D. Iagolnitzer, H. Stapp (1969), J. Bros, D. Iagolnitzer (1975), . . . , J. Sjöstrand (1982), . . . , A. Martinez (2002).

Weighted L^2 spaces of holomorphic functions V

The FBI transform

$$Tu(x;h) = h^{-3n/4} \int e^{i\varphi(x,y)/h} \chi(y)\, u(y),\, dy, \quad x \in \mathrm{neigh}(x_0, \mathbb{C}^n),$$

satisfies

$$T = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to H_\Phi(\Omega),$$

where $\Omega \subset \mathbb{C}^n$ is a small neighborhood of x_0 and the weight

$$\Phi(x) = \sup_{y \in \mathrm{neigh}(y_0, \mathbb{R}^n)} (-\mathrm{Im}\, \varphi(x,y))$$

is strictly plurisubharmonic.

The Catlin-Zelditch expansion

Introducing the Schwartz kernel of Π_Φ, let us write

$$\Pi_\Phi u(x) = \int_\Omega K(x,\overline{y})u(y)e^{-2\Phi(y)/h}\,L(dy),$$

where $K(x,\widetilde{y}) \in \mathrm{Hol}(\Omega \times \overline{\Omega})$. The existence of a complete asymptotic expansion for the Bergman kernel K, as $h \to 0^+$, has been established by D. Catlin and S. Zelditch.

Let $x_0 \in \Omega$ and let $\Psi \in C^\infty\left(\mathrm{neigh}\left((x_0,\overline{x_0}),\mathbb{C}^{2n}\right)\right)$ be a polarization of Φ, i.e.

$$\Psi(x,\overline{x}) = \Phi(x),$$

and $\forall N$,

$$\left(\partial_{\overline{x}}\Psi\right)(x,y) = \mathcal{O}_N(|x-\overline{y}|^N), \quad \left(\partial_{\overline{y}}\Psi\right)(x,y) = \mathcal{O}_N(|x-\overline{y}|^N).$$

In other words, Ψ is an almost holomorphic extension of Φ. L. Hörmander (1969).

Theorem (D. Catlin 1999, S. Zelditch 1998)

There exists a classical elliptic symbol of the form

$$a(x,\widetilde{y};h) \sim \sum_{j=0}^{\infty} a_j(x,\widetilde{y})h^j, \quad (x,\widetilde{y}) \in \mathrm{neigh}\left((x_0,\overline{x_0}),\mathbb{C}^{2n}\right),$$

with $a_j \in C^\infty$ *holomorphic to* ∞*-order along the anti-diagonal, such that on the level of effective kernels, i.e. for the kernel of the operator* $e^{-\Phi/h} \circ \Pi_\Phi \circ e^{\Phi/h}$*, we have*

$$e^{-\Phi(x)/h}\left(K(x,\overline{y}) - \frac{1}{h^n}e^{2\Psi(x,\overline{y})/h}a(x,\overline{y};h)\right)e^{-\Phi(y)/h} = \mathcal{O}(h^\infty).$$

Remark. The original proofs of Catlin and Zelditch rely on a reduction to the main result of Boutet – Sjöstrand.

Direct approach by Berman-Berndtsson-Sjöstrand (2008)

Main idea : Express the identity operator on $H_\Phi(\Omega)$ in a nice way so that it automatically becomes the (asymptotic) Bergman projection.

Starting point : write the identity as a semiclassical pseudodifferential operator on H_Φ : let $U \Subset V \Subset \Omega$ be small open neighborhoods of $x_0 \in \Omega$. We have

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\eta} u(y)\, dy\, d\eta + \mathcal{O}(1) || \, u \, ||_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad x \in U, \quad u \in H_\Phi(V), \quad (1)$$

for some $\eta > 0$.

How do we choose the contour of integration in (1) ?

Good contours

We say that a $2n$-dimensional contour $\Gamma(x) \subset \mathbb{C}^{2n}_{y,\eta}$ is a good contour for the plurisubharmonic function

$$(y, \eta) \mapsto -\mathrm{Im}\,((x-y)\cdot\eta) + \Phi(y)$$

if it passes through the critical point $(y, \eta) = (x, \frac{2}{i}\partial_x \Phi(x))$ and we have

$$-\mathrm{Im}\,((x-y)\cdot\eta) + \Phi(y) \leq \Phi(x) - \frac{1}{C}\mathrm{dist}\left((y,\eta), \left(x, \frac{2}{i}\partial_x\Phi(x)\right)\right)^2,$$

along $\Gamma(x)$. J. Sjöstrand (1982). Any good contour works in (1).

Example. The contour

$$\eta = \frac{2}{i}\frac{\partial \Phi}{\partial x}\left(\frac{x+y}{2}\right) + iC\overline{(x-y)}, \quad C > 0,$$

is good.

Non-standard phase

We would like to express the identity operator with a non-standard phase,

$$u(x) = \frac{1}{h^n} \iint_{\widetilde{\Gamma}(x)} e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))} a(x,y,\theta;h) u(y)\, dy\, d\theta \\ + \mathcal{O}(1) || \, u \, ||_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad x \in U, \quad u \in H_\Phi(V), \quad (2)$$

observing that for this new representation, the contour $\theta = \overline{y}$ is good,

$$u(x) = \frac{1}{h^n} \int e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,y,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)} dy\, d\overline{y} \\ + \mathcal{O}(1) || \, u \, ||_{H_\Phi(V)} e^{\frac{1}{h}(\Phi(x)-\eta)}, \quad (3)$$

in view of

$$2\mathrm{Re}\, \Psi(x,\overline{y}) - \Phi(x) - \Phi(y) \asymp -|x-y|^2 .$$

Kuranishi trick

To see that the representations (1) and (2) are equivalent, we use the Kuranishi trick. Assume for simplicity that Φ is real analytic and write, by Taylor's formula,

$$\frac{2}{i}\left(\Psi(x,\theta)-\Psi(y,\theta)\right)=(x-y)\cdot\eta(x,y,\theta).$$

We can therefore pass from (x,y,θ) to (x,y,η) by a change of variables, obtaining the representation (3) for the identity operator,

$$u(x)=\frac{1}{h^n}\int e^{\frac{2}{h}\Psi(x,\overline{y})}a(x,y,\overline{y};h)u(y)e^{-\frac{2}{h}\Phi(y)}dy\,d\overline{y}$$
$$+\mathcal{O}(1)||\,u\,||_{H_\Phi(V)}e^{\frac{1}{h}(\Phi(x)-\eta)}.$$

Eliminating the y dependence in the amplitude

The representation (3) looks almost like the Bergman projection, but we still need to eliminate the y dependence in a. To this end we show that there exists a symbol $b(x, y, \theta; h)$ with values in $(n-1)$-forms in θ such that for some

$$\widetilde{a}(x, \theta; h) = a(x, x, \theta; h) + \mathcal{O}(h)$$

uniquely determined, we have with $\mathcal{O}(h^\infty)$–errors,

$$e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))}\left(a(x, y, \theta; h) - \widetilde{a}(x, \theta; h)\right)\, d\theta \\ = h d_\theta \left(e^{\frac{2}{h}(\Psi(x,\theta)-\Psi(y,\theta))} b(x, y, \theta; h)\right).$$

This can be accomplished by successive division procedures.

Asymptotic Bergman projection

We finally get an approximate reproducing property in the C^∞ setting,

$$u(x) = \underbrace{\frac{1}{h^n}\int e^{\frac{2}{h}\Psi(x,\overline{y})}\widetilde{a}(x,\overline{y};h)u(y)e^{-\frac{2}{h}\Phi(y)}dy\,d\overline{y}}_{\widetilde{\Pi}u}$$

$$+ Ku, \quad x \in U, \quad u \in H_\Phi(V),$$

where

$$K = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h}L(dx)).$$

To show that the operator $\widetilde{\Pi}$ is an approximation of the honest orthogonal projection Π_Φ, as $h \to 0^+$, we use Hörmander's L^2–estimates for $\overline{\partial}$.

Analytic weights

Let us recall the Catlin-Zelditch expansion,

$$K(x,\overline{y}) = \frac{1}{h^n} e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,\overline{y};h) \\ + \mathcal{O}(h^\infty) e^{\frac{1}{h}(\Phi(x)+\Phi(y))}, \quad x,y \in \mathrm{neigh}(x_0,\mathbb{C}^n). \quad (4)$$

Assume now that the weight Φ is real analytic. We can then choose the polarization Ψ to be holomorphic. We expect the amplitude in (4) to be a classical analytic symbol and the remainder in (4) to be exponentially small.

Classical analytic symbols

Let $U \subset \mathbb{C}^n$ be open. A (formal) classical analytic symbol is given by

$$a(x;h) = \sum_{j=0}^{\infty} a_j(x) h^j,$$

where $a_j \in \mathrm{Hol}(U)$ are such that for each $\widetilde{U} \Subset U$ there exists $C = C_{\widetilde{U}} > 0$ such that

$$|a_j(x)| \leq C^{j+1} j^j, \quad j = 0, 1, 2, \ldots, x \in \widetilde{U}.$$

L. Boutet de Monvel – P. Krée (1967), J. Sjöstrand (1982).

We have a realization of a on $\widetilde{U}$ obtained by performing "la sommation au plus petit terme",

$$a_{\widetilde{U}}(x;h) = \sum_{0 \leq j \leq \left(C_{\widetilde{U}} e h\right)^{-1}} a_j(x) h^j \in \mathrm{Hol}(\widetilde{U}).$$

Theorem (O. Rouby – J. Sjöstrand – S. Vu Ngoc 2018, A. Deleporte 2018)

In the analytic case, there exists a classical analytic symbol $a(x,\widetilde{y};h)$ defined in a neighborhood of $(x_0,\overline{x_0})$, such that, taking a realization of a, we have

$$e^{-\Phi(x)/h}\left(K(x,\overline{y})-\frac{1}{h^n}e^{2\Psi(x,\overline{y})/h}a(x,\overline{y};h)\right)e^{-\Phi(y)/h} = \mathcal{O}(1)e^{-1/Ch}, \quad C>0.$$

There have been more recent alternative proofs by L. Charles (2019) and H. Hezari – H. Xu (2019).

One word about the proof by Rouby – Sjöstrand – Vu Ngoc

- This proof follows the main ideas of the approach by Berman – Berndtsson – Sjöstrand, and in particular, the Kuranishi trick is still an essential ingredient.
- Much more care is needed to be able to pass between the Bergman and the pseudodifferential representations, without dropping out of the class of classical analytic symbols.

Getting rid of the Kuranishi trick

The Kuranishi trick is very nice but may break down in situations when the Levi form $i\partial\overline{\partial}\Phi \geq 0$ of Φ becomes degenerate or nearly degenerate.

Example. (J. Sjöstrand – M.H., work in progress). We develop heat evolution approach to second microlocalization with respect to a real analytic hypersurface of the form

$$\Sigma = p^{-1}(0) \cap \Lambda_{\Phi_0}, \quad dp|_{\Sigma} \neq 0,$$

where Φ_0 is strictly plurisubharmonic real analytic. If $P = p^w(x, hD_x)$, we study the heat evolution semigroup

$$e^{-tP^2/2h} = \mathcal{O}(1) : H_{\Phi_0} \rightarrow H_{\Phi_t}, \quad t \geq 0,$$

for times $t \sim \dfrac{1}{\mu}$, where $\mu \asymp h^{\delta}$, $1/2 < \delta < 1$.

The time evolution of the exponential weight $\Phi_t(x)$ is governed by the real Hamilton-Jacobi equation,

$$\partial_t \Phi_t(x) + \frac{1}{2} p^2 \left(x, \frac{2}{i} \partial_x \Phi_t(x) \right) = 0, \quad \Phi_{t=0} = \Phi_0,$$

and when $t \sim \frac{1}{\mu}$, the solution $\Phi_t(x)$ satisfies

$$\Phi''_{t,\overline{x}x}(x) \zeta \cdot \overline{\zeta} \geq \frac{\mu \left| \zeta \right|^2}{\mathcal{O}(1)}.$$

A model weight :

$$\Phi(x) = \Phi_\mu(x) = \frac{1}{2} (\mathrm{Im}\, x')^2 + \frac{\mu}{2} (\mathrm{Im}\, x_n)^2, \quad x = (x', x_n) \in \mathbb{C}^{n-1} \times \mathbb{C} = \mathbb{C}^n.$$

Φ_μ is strictly plurisubharmonic but not uniformly as $\mu \to 0^+$.

A direct approach to Bergman projections, not relying upon the Kuranishi trick, has recently been developed with A. Deleporte and J. Sjöstrand, in the strictly plurisubharmonic real analytic case. It has then been adapted to the C^{∞} case with M. Stone.

Theorem (A. Deleporte – J. Sjöstrand – M. H. 2020, analytic case)

Let $\Omega \subset \mathbb{C}^n$ be open and let Φ be strictly plurisubharmonic in Ω such that Φ is real analytic near $x_0 \in \Omega$. There exists a unique classical analytic symbol $a(x, \widetilde{y}; h)$ defined near $(x_0, \overline{x}_0)$, solving

$$Aa = 1 + \mathcal{O}(e^{-1/Ch}), \quad C > 0,$$

near $(x_0, \overline{x}_0)$, where A is an explicit elliptic analytic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, such that the operator

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy)$$

satisfies

$$\widetilde{\Pi} - 1 = \mathcal{O}(1) e^{-\frac{1}{Ch}} : H_\Phi(V) \to H_\Phi(U), \quad C > 0.$$

Here $\Psi \in Hol(V \times \overline{V})$ is the polarization of Φ.

Theorem (M. Stone – M. H. 2021, C^∞ case)

Let $\Omega \subset \mathbb{C}^n$ be open and let $\Phi \in C^\infty(\Omega)$ be strictly plurisubharmonic in Ω. Let $x_0 \in \Omega$. There exists a classical C^∞ symbol $a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \widetilde{y}) h^j$, defined near $(x_0, \overline{x}_0)$, with $a_j \in C^\infty$ holomorphic to ∞–order along the anti-diagonal, solving

$$(Aa)(x, \overline{x}; h) = 1 + \mathcal{O}(h^\infty),$$

near x_0, where A is an explicit elliptic Fourier integral operator, and small open neighborhoods $U \Subset V \Subset \Omega$ of x_0, such that the operator

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x, \overline{y}; h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy)$$

satisfies

$$\widetilde{\Pi} - 1 = \mathcal{O}(h^\infty) : H_\Phi(V) \to L^2(U, e^{-2\Phi/h} L(dx)).$$

Here $\Psi \in C^\infty(V \times \overline{V})$ is a polarization of Φ.

A couple of words about the proofs

To fix the ideas, let us discuss the analytic case.

We would like to construct an operator of the form

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \int_V e^{\frac{2}{h}\Psi(x,\overline{y})} a(x,\overline{y};h) u(y) e^{-\frac{2}{h}\Phi(y)} L(dy), \quad u \in H_\Phi(V),$$

where V is a small neighborhood of x_0, enjoying the reproducing property $\widetilde{\Pi} u = u$, at least approximately.

Main idea : demand that the reproducing property should hold in the weak formulation,

$$(\widetilde{\Pi} u, v)_{H_\Phi} = (u,v)_{H_\Phi} + \mathcal{O}(e^{-1/Ch}) \|u\|_{H_\Phi} \|v\|_{H_\Phi},$$

for some $C > 0$ and all $u, v \in H_\Phi(V)$. This should imply that

$$\widetilde{\Pi} - 1 = \mathcal{O}(1) e^{-1/Ch} : H_\Phi(V) \to H_\Phi(V).$$

Switching to the polarized form

Let us write in the polarized form,

$$\widetilde{\Pi} u(x) = \frac{1}{h^n} \iint_{\Gamma} e^{\frac{2}{h}(\Psi(x,\widetilde{y}) - \Psi(y,\widetilde{y}))} a(x,\widetilde{y};h) u(y)\, dy\, d\widetilde{y}, \quad u \in H_\Phi,$$

where $\Gamma = \{(y,\widetilde{y}); \widetilde{y} = \overline{y}\} \subset \mathbb{C}^{2n}_{y,\widetilde{y}}$, and similarly for the scalar product in H_Φ,

$$(u,v)_{H_\Phi} = \int u(x)\overline{v(x)} e^{-\frac{2}{h}\Phi(x)}\, dx\, d\overline{x} = \iint_{\Gamma} u(x) v^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})} dx\, d\widetilde{x}.$$

Here $\Gamma = \{(x,\widetilde{x}); \widetilde{x} = \overline{x}\} \subset \mathbb{C}^{2n}_{x,\widetilde{x}}$ and $v^*(\widetilde{x}) = \overline{v(\overline{\widetilde{x}})}$.

Reshuffling the order of integration

Suppressing the contours and applying Fubini formally, we get

$$(\widetilde{\Pi}u, v)_{H_\Phi} =$$

$$\iint \Bigg(\overbrace{\frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y}))} a(x,\widetilde{y};h)u(y)\,dy\,d\widetilde{y}}^{\widetilde{\Pi}u} \Bigg) v^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x,\widetilde{x})} dx\,d\widetilde{x}$$

$$= \frac{1}{h^n} \iint \Bigg(\iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})-\Psi(x,\widetilde{x}))} a(x,\widetilde{y};h)\,dx\,d\widetilde{y} \Bigg) u(y) v^*(\widetilde{x}) dy\,d\widetilde{x}.$$

This expression agrees with $(u, v)_{H_\Phi}$ provided that

$$\frac{1}{h^n} \iint e^{\frac{2}{h}(\Psi(x,\widetilde{y})-\Psi(y,\widetilde{y})-\Psi(x,\widetilde{x}))} a(x,\widetilde{y};h)\,dx\,d\widetilde{y} = e^{-\frac{2}{h}\Psi(y,\widetilde{x})}. \tag{5}$$

A special canonical transformation

Let us rewrite (5) as follows,

$$\frac{1}{h^n} \iint e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x,\widetilde{y};h)\, dx\, d\widetilde{y} = 1, \tag{6}$$

where

$$\varphi(y,\widetilde{x};x,\widetilde{y}) = \Psi(x,\widetilde{y}) - \Psi(y,\widetilde{y}) - \Psi(x,\widetilde{x}) + \Psi(y,\widetilde{x})$$

is holomorphic near $(x_0, \overline{x_0}; x_0, \overline{x_0}) \in \mathbb{C}^{4n}$.

We have

$$\det \varphi''_{(y,\widetilde{x}),(x,\widetilde{y})}(x_0, \overline{x_0}; x_0, \overline{x_0}) \neq 0 \quad \Longrightarrow$$

φ is a generating function for the canonical transformation

$$\kappa : \left(x, \widetilde{y}; -\frac{2}{i}\partial_x \varphi, -\frac{2}{i}\partial_{\widetilde{y}}\varphi\right) \mapsto \left(y, \widetilde{x}; \frac{2}{i}\partial_y \varphi, \frac{2}{i}\partial_{\widetilde{x}}\varphi\right).$$

The operator

$$(Aa)(y,\widetilde{x};h) = \frac{1}{h^n}\iint e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})}a(x,\widetilde{y};h)\,dx\,d\widetilde{y}$$

is therefore a (formal) elliptic analytic Fourier integral operator quantizing κ, which takes functions of $(x,\widetilde{y})$ to functions of $(y,\widetilde{x})$.

Main observation : the canonical transformation κ maps the zero section $\{\eta = 0\} \subset \mathbb{C}^{4n}_{y,\eta}$ to the zero section $\{\xi = 0\} \subset \mathbb{C}^{4n}_{x,\xi}$.

We also have

$$\det \varphi''_{(x,\widetilde{y}),(x,\widetilde{y})}(x_0,\overline{x_0};x_0,\overline{x_0}) \neq 0.$$

It follows that there exists a good contour $\Gamma(y,\widetilde{x}) \subset \mathbb{C}^{2n}_{x,\widetilde{y}}$, for the pluriharmonic function

$$(x,\widetilde{y}) \mapsto \operatorname{Re}\varphi(y,\widetilde{x};x,\widetilde{y}),$$

such that the corresponding realization of A,

$$(A_\Gamma a)(y,\widetilde{x};h) = \frac{1}{h^n}\iint_{\Gamma(y,\widetilde{x})} e^{\frac{2}{h}\varphi(y,\widetilde{x};x,\widetilde{y})} a(x,\widetilde{y};h)\,dx\,d\widetilde{y}$$

is a bijection from the space of classical analytic symbols defined near $(x_0,\overline{x_0})$ to itself. There exists therefore a unique classical analytic symbol $a(x,\widetilde{y};h)$ such that

$$A_\Gamma a = 1 + \mathcal{O}(e^{-1/Ch}), \quad C > 0.$$

Remark. A natural explicit choice of the good contour $\Gamma(y,\widetilde{x})$ was obtained by M. Stone : we can take

$$\Gamma(y,\widetilde{x}) = (y,\widetilde{x}) + i\Lambda,$$

where $\Lambda = \{(z,\overline{z}); z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$ is the anti-diagonal.

Justifying the formal computation

Our sincere hope is that the classical analytic symbol $a(x, \widetilde{y}; h) \sim \sum_{j=0}^{\infty} a_j(x, \widetilde{y}) h^j$ that we have just constructed is the amplitude of the asymptotic Bergman projection, but this has to be justified. Indeed, an application of Fubini's theorem in the beginning of the proof was purely formal, and we still need to justify the change of the order of integration, using good contours.

Basic difficulty : the natural contour $\Gamma = \{(x, \overline{x}); x \in \mathbb{C}^n\}$ in the polarized expression for the scalar product,

$$(u, v)_{H_\Phi} = \iint_\Gamma u(x) v^*(\widetilde{x}) e^{-\frac{2}{h}\Psi(x, \widetilde{x})} dx\, d\widetilde{x},$$

is not quite good :

$$\left(\Phi(x) + \Phi(\overline{\widetilde{x}}) - 2\mathrm{Re}\, \Psi(x, \widetilde{x})\right)|_\Gamma = 0.$$

A resolution of the identity in H_Φ

To overcome this difficulty, in the actual proof we work with a "coherent states decomposition" of $v \in H_\Phi$ of the form

$$v(x) = \int_{\mathrm{neigh}(x_0,\mathbb{C}^n)} v_y(x) L(dy) + \mathcal{O}(1) \|v\|_{H_\Phi} e^{\frac{1}{h}\left(\Phi(x) - \frac{1}{C}\right)},$$

where

$$v_y \in H_{\Phi_y}, \quad \Phi_y(x) \leq \Phi(x) - |x-y|^2/C.$$

Such a decomposition can be obtained by the Fourier inversion formula in H_Φ (cf. with the approach by Berman-Berndtsson–Sjöstrand).

Perspectives

- Attack the Gevrey case : assume that $\Phi \in \mathcal{G}^s(\Omega)$, for some $s > 1$, in the sense that for each $\widetilde{\Omega} \Subset \Omega$ there exists $C = C_{\widetilde{\Omega}} > 0$ such that for all α, $\beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\alpha \partial_{\overline{x}}^\beta \Phi(x)\right| \leq C^{1+|\alpha|+\beta}(\alpha!\beta!)^s, \quad x \in \widetilde{\Omega}.$$

There exists in this case a polarization Ψ of Φ such that

$$\left|\overline{\partial}_{x,y}\Psi(x,y)\right| \leq \mathcal{O}(1)\exp\left(-\frac{1}{C}\left|y-\overline{x}\right|^{-\frac{1}{s-1}}\right).$$

Carleson (1961), Mather (1971), . . . , H. Hezari – H. Xu (2018), A. Deleporte (2020), R. Lascar – J. Sjöstrand – M. Zerzeri – M. H. (2020) (H_Φ–techniques for semiclassical Gevrey operators). Can we show that $a(x, \widetilde{y}; h)$ is a Gevrey symbol in the natural sense ?

- Apply the direct approach to degenerate situations.

THANK YOU VERY MUCH FOR YOUR ATTENTION !