EX 1.1 :

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \, d\xi$$

BY PLANCHEREL THEOREM, USING POLAR COORDINATES:

$$\int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \, d\xi = \int_{S^1}\int_0^\infty |\hat{f}(\sigma\omega)|^2 \sigma \, d\sigma \, d\omega$$

$$= \int_{S^1}\int_0^\infty |\hat{f}(\sigma\omega^\perp)|^2 \sigma \, d\sigma \, d\omega .$$

THEN, BY FOURIER SLICE THEOREM

$$\int_{S^1}\int_0^\infty |\hat{f}(\sigma\omega^\perp)|^2 \sigma \, d\sigma \, d\omega$$

$$= \int_{S^1}\int_0^\infty |(Rf)^\vee(\sigma,\omega)|^2 \sigma \, d\sigma \, d\omega$$

$$= \frac{1}{2}\int_{S^1}\int_{-\infty}^\infty |(Rf)^\vee(\sigma,\omega)|^2 |\sigma| \, d\sigma \, d\omega$$

$$\leq \frac{1}{2}\int_{S^1}\int_{-\infty}^\infty (1+\sigma^2)^{1/2} |(Rf)^\vee(\sigma,\omega)|^2 \, d\sigma \, d\omega$$

$$= \frac{1}{2}\|Rf\|^2_{H^{1/2}_T(\mathbb{R}\times S^1)} \qquad \square$$

EX 2.1:

$$p(x,\xi,\xi,\tau) = -\tau^2 + |\xi|^2$$

A BICHARACTERISTIC CURVE

$$\gamma(s) = (x(s), \xi(s), \xi(s), \tau(s))$$

$$\begin{cases} (\dot{x}(s), \dot{\xi}(s)) = (\nabla_\xi p, \nabla_\tau p)|_{\gamma(s)} \\ (\dot{\xi}(s), \dot{\tau}(s)) = -(\nabla_x p, \nabla_\xi p)|_{\gamma(s)} \end{cases}$$

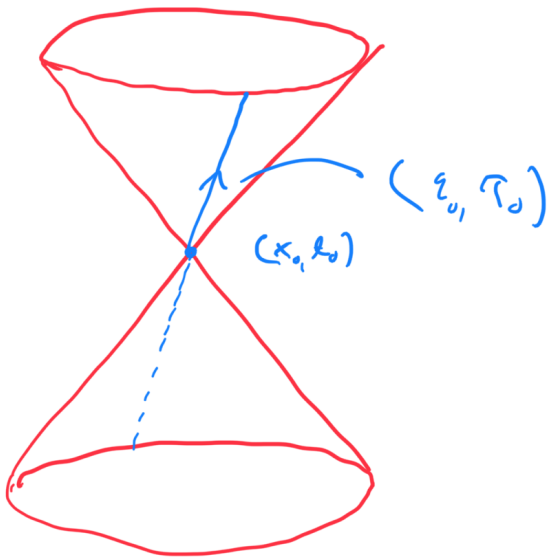
$$\Leftrightarrow \begin{cases} (\dot{x}(s), \dot{\xi}(s)) = (2\xi, -2\tau) \\ (\dot{\xi}(s), \dot{\tau}(s)) = (0, 0) \end{cases}$$

$$\Leftrightarrow \begin{cases} x(s) = 2\xi_0 s + x_0 \\ \xi(s) = -2\tau_0 s + \xi_0 \\ \xi(s) = \xi_0 \\ \tau(s) = \tau_0 \end{cases}$$

THE BICHARACTERISTIC $\gamma(s)$ IS NULL IF

$$p(x(s), \xi(s), \xi(s), \tau(s)) = 0$$

$$\Leftrightarrow |\xi_0|^2 = \tau_0^2$$



Ex 2.2.:

$u(x,t)$ IS OF THE FORM

$u(x,t) = H(t - x\cdot\omega)$, WHERE

$H \in C^\infty(\mathbb{R})$, $H = H(h)$

BY CHAIN RULE

$$\Delta u(x,t) = \nabla\cdot\nabla H(t-x\cdot\omega)$$

$$= \nabla\cdot\left((\partial_h H)(t-x\cdot\omega)\,(-\omega)\right)$$

$$= -\sum_{h=1}^{n} \omega_h \frac{\partial}{\partial x_h}(\partial_h H)(t-x\cdot\omega)$$

$$= -\sum_{h=1}^{n} \omega_h (\partial_h^2 H)(t-x\cdot\omega)(-\omega_h)$$

$$= |\omega|^2 \partial_h^2 H(t-x\cdot\omega).$$

THUS WE HAVE

$$(\partial_t^2 - \Delta)u(x,t) = (\partial_t^2 - \Delta)H(t-x\cdot\omega)$$

$$= (1-|\omega|^2)\,\partial_h^2 H(t-x\cdot\omega) = 0,$$

SINCE $|\omega| = 1$.

$(x,t) \in \operatorname{supp} u \iff u(x,t) \neq 0$

$\iff a(t-x\cdot\omega) \neq 0$

$\iff |t-x\cdot\omega| \leq \varepsilon$. □

EX 2.3,

a) IN LEMMA 2.4. THE INTEGRAL IDENTITY CHANGES TO

$$(\Lambda_{g_1} - \Lambda_{g_2}) f_1, f_2)_{L^2(\partial M \times \mathbb{R})} \quad (\boxtimes)$$

$$= \int_M \int_{\mathbb{R}} (q_1 - q_2) u_1 \bar{u}_2 \, , \quad \text{WHERE}$$

u_2 SOLVES

$$(\Box + q_2) u_2 = 0 \quad \text{IN } M \times \mathbb{R}$$
$$u_2 = f_2 \quad \text{ON } \partial M \times \mathbb{R}$$
$$u_2 = 0 \quad \text{FOR } t \gg 0.$$

THE REASON IS THAT DERIVATION OF THE IDENTITY JUST USES INTEGRATION BY PARTS.

b) PROP 2.5 MODIFIED:

ASSUME THAT $q \in C_c^\infty(M^{int} \times \mathbb{R})$, AND LET $\gamma : [0, L] \to M$ BE A MAXIMAL GEODESIC... ONE HAS

$$\lim_{\lambda \to \infty} \int_M \int_{\mathbb{R}} \psi |u|^2 \, dt \, dV$$
$$= \int_0^L \psi(\gamma(t), t + s) \, dt \, , \quad \forall s \in \mathbb{R}.$$

MOREOVER, IF $f \in C_c^\infty(M^{int} \times \mathbb{R})$, THERE IS A SOLUTION $\tilde{u} = \tilde{u}_\lambda$ OF $(\Box + q)\tilde{u} = 0$ IN $M \times \mathbb{R}$ WITH $\tilde{u} = 0$ FOR $t \gg 0$, SUCH THAT... ONE HAS

$$\lim_{\lambda \to \infty} \int_M \int_{\mathbb{R}} \psi u \bar{\tilde{u}} \, dt \, dV$$
$$= \int_0^L \psi(\gamma(t), t + s) \, dt \quad (\boxplus)$$

c) BY $(\boxtimes)$ AND $(\boxplus)$ WITH $u_\lambda = u_1$, $\tilde{u}_\lambda = u_2$ WE HAVE

$$\int_0^L f_1(\gamma(t), t + s) \, dt = \int_0^L f_2(\gamma(t), t + s) \, dt.$$

d) FROM c) BY TAKING FOURIER TRANSFORM IN s GIVES

$$\int_0^L e^{2\pi i s \eta} \hat{f}_1(\gamma(t), \eta) \, dt \quad (\diamondsuit)$$
$$= \int_0^L e^{2\pi i s \eta} \hat{f}_2(\gamma(t), \eta) \, dt, \quad \eta \in \mathbb{R},$$

WHERE $\hat{f}_j(x, \eta)$ DENOTES THE PARTIAL FOURIER TRANSFORM OF $f_j(x, s)$ IN s-VARIABLE.

RECALL THAT FOURIER TRASFORM OF COMPACTLY SUPPORTED IS ANALYTIC $\Rightarrow$ $\hat{f}_j(x, \eta)$ ANALYTIC IN η ?

NOW, EVALUATE $(\diamondsuit)$ AT $\eta = 0$:

$$\int_0^L \hat{f}_1(\gamma(t), 0) \, dt = \int_0^L \hat{f}_2(\gamma(t), 0) \, dt$$

FOR ALL MAXIMAL GEODESICS γ. SINCE (M, g) IS SIMPLE, THE X-RAY TRANSFORMATION IS INVERTIBLE $\Rightarrow$ $\hat{f}_1(x, 0) = \hat{f}_2(x, 0)$.

NEXT, DIFFERENTIATE $(\diamondsuit)$ AT $\eta = 0$. THIS GIVES

$$\int_0^L 2\pi i \, \hat{f}_1(\gamma(t), 0) \, dt + \int_0^L \frac{d}{d\eta}\Big|_{\eta=0} \hat{f}_1(\gamma(t), \eta) \, dt$$
$$= \int_0^L 2\pi i \, \hat{f}_2(\gamma(t), 0) \, dt + \int_0^L \frac{d}{d\eta}\Big|_{\eta=0} \hat{f}_2(\gamma(t), \eta) \, dt$$

$$\Rightarrow \frac{d}{d\eta}\Big|_{\eta=0} \hat{f}_1(x, \eta) = \frac{d}{d\eta}\Big|_{\eta=0} \hat{f}_2(x, \eta).$$

X-RAY INV.

CONTINUING IN THIS MANNER RECURSIVELY, WE HAVE THAT

$$\frac{d^k}{d\eta^k}\Big|_{\eta=0} \hat{f}_1(x, \eta) = \frac{d^k}{d\eta^k}\Big|_{\eta=0} \hat{f}_2(x, \eta).$$

BY ANALYTICITY $\Rightarrow$ $\hat{f}_1(x, \eta) = \hat{f}_2(x, \eta)$.

$\Rightarrow$ $f_1(x, s) = f_2(x, s)$, FOR $s \in \mathbb{R}$.